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PHILOSOPHICAL TRANSACTIONS.

I. *On the Bodily Tides of Viscous and Semi-elastic Spheroids, and on the Ocean Tides upon a Yielding Nucleus.*

By G. H. DARWIN, *M.A., Fellow of Trinity College, Cambridge.*
Communicated by J. W. L. GLAISHER, *M.A., F.R.S.*

Received May 14,—Read May 23, 1878.

IN a well-known investigation Sir WILLIAM THOMSON has discussed the problem of the bodily tides of a homogeneous elastic sphere, and has drawn therefrom very important conclusions as to the great rigidity of the earth.*

Now it appears improbable that the earth should be perfectly elastic; for the contortions of geological strata show that the matter constituting the earth is somewhat plastic, at least near the surface. We know also that even the most refractory metals can be made to flow under the action of sufficiently great forces.

Although Sir W. THOMSON's investigation has gone far to overthrow the old idea of a semi-fluid interior to the earth, yet geologists are so strongly impressed by the fact that enormous masses of rock are being, and have been, poured out of volcanic vents in the earth's surface, that the belief is not yet extinct that we live on a thin shell over a sea of molten lava. Under these circumstances it appears to be of interest to investigate the consequences which would arise from the supposition that the matter constituting the earth is of a viscous or imperfectly elastic nature; for if the interior *is* constituted in this way, then the solid crust, unless very thick, cannot possess rigidity enough to repress the tidal surgings, and these hypotheses must give results fairly conformable to the reality. The hypothesis of imperfect elasticity will be principally interesting as showing how far Sir W. THOMSON'S results are modified by the supposition that the elasticity breaks down under continued stress.

---

* Sir WILLIAM states that M. LAMÉ had treated the subject at an earlier date, but in an entirely different manner. I am not aware, however, that M. LAMÉ had fully discussed the subject in its physical aspect.

In this paper, then, I follow out these hypotheses, and it will be seen that the results are fully as hostile to the idea of any great mobility of the interior of the earth as is that of Sir W. THOMSON.

The only terrestrial evidence of the existence of a bodily tide in the earth would be that the ocean tides would be less in height than is indicated by theory. The subject of this paper is therefore intimately connected with the theory of the ocean tides.

In the first part the equilibrium tide-theory is applied to estimate the reduction and alteration of phase of ocean tides as due to bodily tides, but that theory is acknowledged on all hands to be quite fallacious in its explanation of tides of short period.

In the second part of this paper, therefore, I have considered the dynamical theory of tides in an equatorial canal running round a tidally-distorted nucleus, and the results are almost the same as those given by the equilibrium theory.

The first two sections of the paper are occupied with the adaptation of Sir W. THOMSON'S work* to the present hypotheses; as, of course, it was impossible to reproduce the whole of his argument, I fear that the investigation will only be intelligible to those who are either already acquainted with that work, or who are willing to accept my quotations therefrom as established.

As some readers may like to know the results of this inquiry without going into the mathematics by which they are established, I have given in Part III. a summary of the whole, and have as far as possible relegated to that part of the paper the comments and conclusions to be drawn. I have tried, however, to give so much explanation in the body of the paper as will make it clear whither the argument is tending.

The case of pure viscosity is considered first, because the analysis is somewhat simpler, and because the results will afterwards admit of an easy extension to the case of elastico-viscosity.

I.

THE BODILY TIDES OF VISCOUS AND ELASTICO-VISCOUS SPHEROIDS.

1. *Analogy between the flow of a viscous body and the strain of an elastic one.*

The general equations of flow of a viscous fluid, *when the effects of inertia are neglected,* are

---

* His paper will be found in Phil. Trans., 1863, p. 573, and §§ 733–737 and 834–846 of THOMSON and TAIT'S 'Natural Philosophy,' edit. of 1867.

$$\left.\begin{aligned}-\frac{dp}{dx}+\upsilon\nabla^2\alpha+\mathrm{X}=0\\-\frac{dp}{dy}+\upsilon\nabla^2\beta+\mathrm{Y}=0\\-\frac{dp}{dz}+\upsilon\nabla^2\gamma+\mathrm{Z}=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(1)$$

where x, y, z are the rectangular coordinates of a point of the fluid; α, β, γ are the component velocities parallel to the axes; p is the mean of the three pressures across planes perpendicular to the three axes respectively; X, Y, Z are the component forces acting on the fluid, estimated per unit volume; υ is the coefficient of viscosity; and ∇^2 is the Laplacian operation $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$

Besides these we have the equation of continuity $\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0$

Also if P, Q, R, S, T, U are the normal and tangential stresses estimated in the usual way across three planes perpendicular of the axes

$$\left.\begin{aligned}&\mathrm{P}=-p+2\upsilon\frac{d\alpha}{dx},\quad \mathrm{Q}=-p+2\upsilon\frac{d\beta}{dy},\quad \mathrm{R}=-p+2\upsilon\frac{d\gamma}{dz}\\&\mathrm{S}=\upsilon\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right),\quad \mathrm{T}=\upsilon\left(\frac{d\gamma}{dx}+\frac{d\alpha}{dz}\right),\quad \mathrm{U}=\upsilon\left(\frac{d\alpha}{dy}+\frac{d\beta}{dx}\right)\end{aligned}\right\}\quad\ldots\ldots\quad(2)$$

Now in an elastic solid, if α, β, γ be the displacements, $m-\frac{1}{3}n$ be the coefficient of dilatation, and n that of rigidity, and if $\delta=\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}$; the equations of equilibrium are

$$\left.\begin{aligned}m\frac{d\delta}{dx}+n\nabla^2\alpha+\mathrm{X}=0\\m\frac{d\delta}{dy}+n\nabla^2\beta+\mathrm{Y}=0\\m\frac{d\delta}{dz}+n\nabla^2\gamma+\mathrm{Z}=0\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(3)^*$$

Also

$$\mathrm{P}=(m-n)\delta+2n\frac{d\alpha}{dx},\quad \mathrm{Q}=(m-n)\delta+2n\frac{d\beta}{dy},\quad \mathrm{R}=(m-n)\delta+2n\frac{d\gamma}{dz}\quad\ldots\quad(4)$$

and S, T, U have the same forms as in (2), with n written instead of υ.

Therefore if we put $-p=\frac{1}{3}(\mathrm{P}+\mathrm{Q}+\mathrm{R})$, we have $p=-(m-\frac{n}{3})\delta$, so that (3) may be written

---

* THOMSON and TAIT'S 'Nat. Phil.,' § 698, eq. (7) and (8).

$$-\frac{m}{m-\frac{n}{3}}\frac{dp}{dx}+n\nabla^2\alpha+X=0, \text{ \&c., \&c.}$$

Also

$$P=-\frac{m-n}{m-\frac{n}{3}}p+2n\frac{d\alpha}{dx}, \quad Q=\text{ \&c., } R=\text{ \&c.}$$

Now if we suppose the elastic solid to be incompressible, so that m is infinitely large compared to n, then it is clear that the equations of equilibrium of the incompressible elastic solid assume exactly the same form as those of flow of the viscous fluid, n merely taking the place of ν.

Thus every problem in the equilibrium of an incompressible elastic solid has its counterpart in a problem touching the state of flow of an incompressible viscous fluid, when the effects of inertia are neglected; and the solution of the one may be made applicable to the other by merely reading for "displacements" "velocities," and for the coefficient of "rigidity" that of "viscosity."

2. *A sphere under influence of bodily force.*

Sir W. Thomson has solved the following problem:—

To find the displacement of every point of the substance of an elastic sphere exposed to no surface traction, but deformed infinitesimally by an equilibrating system of forces acting *bodily* through the interior.

If for "displacement" we read velocity, and for "elastic" viscous, we have the corresponding problem with respect to a viscous fluid, and *mutatis mutandis* the solution is the same.

But we cannot find the tides of a viscous sphere by merely making the equilibrating system of forces equal to the tide-generating influence of the sun or moon, because the substance of the sphere must be supposed to have the power of gravitation.

For suppose that at any time the equation to the free surface of the earth (as the viscous sphere may be called for brevity) is $r=a+\sum_{2}^{\infty}\sigma_i$, where σ_i is a surface harmonic. Then the matter, positive or negative, filling the space represented by $\Sigma\sigma_i$ exercises an attraction on every point of the interior; and this attraction, together with that of a homogeneous sphere of radius a, must be added to the tide-generating influence to form the whole force in the interior of the sphere. Also it is a spheroid, and no longer a true sphere with which we have to deal. If, however, we cut a true sphere of radius a out of the spheroid (leaving out $\Sigma\sigma_i$), then by a proper choice of surface actions, the tidal problem may be reduced to finding the state of flow in a true sphere under the action of (i) an external tide-generating influence, (ii) the attraction of the true sphere, and of the positive and negative matter filling the space $\Sigma\sigma_i$, but (iii) subject to certain surface forces.

Since (i) and (ii) together constitute a bodily force, the problem only differs from that of Sir W. THOMSON in the fact that there are forces acting on the surface of the sphere.

Now as we are only going to consider small deviations from sphericity, these surface actions will be of small amount, and an approximation will be permissible.

It is clear that rigorously there is tangential action* between the layer of matter $\Sigma\sigma_i$ and the true sphere, but by far the larger part of the action is normal, and is simply the weight (either positive or negative) of the matter which lies above or below any point on the surface of the true sphere.

Thus, in order to reduce the earth to sphericity, the appropriate surface action is a normal traction equal to $-gw\Sigma\sigma_i$, where g is gravity at the surface, and w is the mass per unit volume of the matter constituting the earth.

In order to show what alteration this normal surface traction will make in Sir W. THOMSON'S solution, I must now give a short account of his method of attacking the problem.

He first shows that, where there is a potential function, the solution of the problem may be subdivided, and that the complete values of α, β, γ consist of the sums of two parts which are to be found in different ways. The first part consists of *any* values of α, β, γ, which satisfy the equations throughout the sphere, without reference to surface conditions. As far as regards the second part, the bodily force is deemed to be non-existent and is replaced by certain surface actions, so calculated as to counteract the surface actions which correspond to the values of α, β, γ found in the first part of the solution. Thus the first part satisfies the condition that there is a bodily force, and the second adds the condition that the surface forces are zero. The first part of the solution is easily found, and for the second part Sir W. THOMSON discusses the case of an elastic sphere under the action of any surface tractions, but without any bodily force acting on it. The component surface tractions parallel to the three axes, in this problem, are supposed to be expanded in a series of surface harmonics; and the harmonic terms of any order are shown to have an effect on the displacements independent of those of every other order. Thus it is only necessary to consider the typical component surface tractions A_i, B_i, C_i of the order i.

He proves that (for an incompressible elastic solid for which m is infinite) this one surface traction A_i, B_i, C_i produces a displacement throughout the sphere given by

$$\alpha=\frac{1}{na^{i-1}}\left\{\frac{a^2-r^2}{2(2i^2+1)}\frac{d\Psi_{i-1}}{dx}+\frac{1}{i-1}\left[\frac{i+2}{(2i^2+1)(2i+1)}r^{2i+1}\frac{d}{dx}(\Psi_{i-1}r^{-2i+1})+\frac{1}{2i(2i+1)}\frac{d\Phi_{i+1}}{dx}+A_ir^i\right]\right\}(5)\dagger$$

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* I shall consider some of the effects of this tangential action in a future paper, viz.: "Problems connected with the Tides of a Viscous Spheroid," read before the Royal Society on December 19th, 1878.

† THOMSON and TAIT'S 'Nat. Phil.,' 1867, § 737, equation (52).

with symmetrical expressions for β and γ; where Ψ and Φ are auxiliary functions defined by

$$\left.\begin{aligned}\Psi_{i-1}&=\frac{d}{dx}(\mathrm{A}_i r^i)+\frac{d}{dy}(\mathrm{B}_i r^i)+\frac{d}{dz}(\mathrm{C}_i r^i)\\ \Phi_{i+1}&=r^{2i+3}\left\{\frac{d}{dx}(\mathrm{A}_i r^{-i-1})+\frac{d}{dy}(\mathrm{B}_i r^{-i-1})+\frac{d}{dz}(\mathrm{C}_i r^{-i-1})\right\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (6)$$

In the case considered by Sir W. Thomson of an elastic sphere deformed by bodily stress and subject to no surface action, we have to substitute in (5) and (6) only those surface actions which are equal and opposite to the surface forces corresponding to the first part of the solution;[*] but in the case which we now wish to consider, we must add to these latter the components of the normal traction $-gw\Sigma\sigma_i$, and besides must include in the bodily force both the external disturbing force, and the attraction of the matter of the spheroid on itself.

Now from the forms of (5) and (6) it is obvious that the tractions which correspond to the first part of the solution, and the traction $-gw\Sigma\sigma_i$ produce quite independent effects, and therefore we need only add to the complete solution of Sir W. Thomson's problem of the elastic sphere, the terms which arise from the normal traction $-gw\Sigma\sigma_i$. Finally we must pass from the elastic problem to the viscous one, by reading $\mathfrak{v}$ for n, and velocities for displacements.

I proceed then to find the state of internal flow in the viscous sphere, which results from a normal traction at every point of the surface of the sphere, given by the surface harmonic S_i.

In order to use the formulæ (5) and (6), it is first necessary to express the component tractions $\frac{x}{a}\mathrm{S}_i$, $\frac{y}{a}\mathrm{S}_i$, $\frac{z}{a}\mathrm{S}_i$ as surface harmonics.

Now if V_i be a solid harmonic,

$$\frac{d}{dx}(r^{-2i-1}\mathrm{V}_i)=-(2i+1)r^{-(2i+3)}x\mathrm{V}_i+r^{-(2i+1)}\frac{d\mathrm{V}_i}{dx}$$

So that

$$x\mathrm{V}_i=\frac{1}{2i+1}\left\{r^2\frac{d\mathrm{V}_i}{dx}-r^{2i+3}\frac{d}{dx}(r^{-2i-1}\mathrm{V}_i)\right\}$$

Therefore

$$\frac{x}{a}\mathrm{S}_i=\frac{1}{2i+1}\left\{\left[r^{-i+1}\frac{d}{dx}(r^i\mathrm{S}_i)\right]-\left[r^{i+2}\frac{d}{dx}(r^{-i-1}\mathrm{S}_i)\right]\right\}$$

The quantities within the brackets [ ] being independent of r, and being surface harmonics of orders $i-1$ and $i+1$ respectively, we have $\frac{x}{a}\mathrm{S}_i$ expressed as the sum of two surface harmonics A_{i-1}, A_{i+1}, where

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[*] Where the solid is incompressible, this surface traction is normal to the sphere at every point, provided that the potential of the bodily force is expressible in a series of solid harmonics.

$$\mathrm{A}_{i-1}=\frac{1}{2i+1}r^{-i+1}\frac{d}{dx}(r^i\mathrm{S}_i),\ \mathrm{A}_{i+1}=-\frac{1}{2i+1}r^{i+2}\frac{d}{dx}(r^{-i-1}\mathrm{S}_i)$$

Similarly $\frac{y}{a}\mathrm{S}_i$, $\frac{z}{a}\mathrm{S}_i$ may be expressed as $\mathrm{B}_{i-1}+\mathrm{B}_{i+1}$ and $\mathrm{C}_{i-1}+\mathrm{C}_{i+1}$, where the B's and C's only differ from the A's in having y, z written for x.

We have now to form the auxiliary functions Ψ_{i-1}, Φ_i corresponding to A_{i-1}, B_{i-1}, C_{i-1} and Ψ_i, Φ_{i+2} corresponding to A_{i+1}, B_{i+1}, C_{i+1}.

Then by the formulæ (6)

$$(2i+1)\Psi_{i-2}=\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)r^i\mathrm{S}_i=0$$

$$\frac{2i+1}{r^{2i+1}}\Phi_i=\frac{d}{dx}\left[r^{-2i+1}\frac{d}{dx}(r^i\mathrm{S}_i)\right]+\frac{d}{dy}\left[\quad\right]+\frac{d}{dz}\left[\quad\right]=-\frac{i(2i-1)}{r^{2i+1}}r^i\mathrm{S}_i$$

$$-(2i+1)\Psi_i=\frac{d}{dx}\left[r^{2i+3}\frac{d}{dx}(r^{-i-1}\mathrm{S}_i)\right]+\frac{d}{dy}\left[\quad\right]+\frac{d}{dz}\left[\quad\right]=-(i+1)(2i+3)r^i\mathrm{S}_i$$

$$-\frac{2i+1}{r^{2i+5}}\Phi_{i+2}=\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)r^{-i-1}\mathrm{S}_i=0$$

Thus

$$\Psi_{i-2}=0,\ \Phi_i=-\frac{i(2i-1)}{2i+1}r^i\mathrm{S}_i,\ \Psi_i=\frac{(i+1)(2i+3)}{2i+1}r^i\mathrm{S}_i,\ \Phi_{i+2}=0$$

Then by (5) we form α corresponding to A_{i-1}, B_{i-1}, C_{i-1}, and also to A_{i+1}, B_{i+1}, C_{i+1}, and add them together. The final result is that a normal traction S_i gives,

$$\alpha'=\frac{1}{\upsilon a^i}\left[\left\{\frac{i(i+2)}{2(i-1)[2(i+1)^2+1]}a^2-\frac{(i+1)(2i+3)}{2(2i+1)[2(i+1)^2+1]}r^2\right\}\frac{d}{dx}(r^i\mathrm{S}_i)\right.$$
$$\left.-\frac{i}{(2i+1)[2(i+1)^2+1]}r^{2i+3}\frac{d}{dx}(r^{-i-1}\mathrm{S}_i)\right]\quad\ldots\ldots\quad(7)$$

and symmetrical expressions for β' and γ'.

α', β', γ' are here written for α, β, γ to show that this is only a partial solution, and υ is written for n to show that it corresponds to the viscous problem. If we now put $\mathrm{S}_i=-gw\sigma_i$, we get the state of flow of the fluid due to the transmitted pressure of the deficiencies and excesses of matter below and above the true spherical surface. This constitutes the solution as far as it depends on (iii).

There remain the parts dependent on (i) and (ii), which may for the present be classified together; and for this part Sir W. THOMSON's solution is directly applicable. The state of internal strain of an elastic sphere, subject to no surface action, but under the influence of a bodily force of which the potential is W_i, may be at once adapted to give the state of flow of a viscous sphere under like conditions. The solution is—

$$\alpha''=\frac{1}{\upsilon}\left[\left\{\frac{i(i+2)}{2(i-1)[2(i+1)^2+1]}a^2-\frac{(i+1)(2i+3)}{2(2i+1)[2(i+1)^2+1]}r^2\right\}\frac{dWi}{dx}\right.$$
$$\left.-\frac{i}{(2i+1)[2(i+1)^2+1]}r^{2i+3}\frac{d}{dx}(r^{-2i-1}W_i)\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)^*$$

with symmetrical expressions for β'' and γ''.

I will first consider (ii); *i.e.*, the matter of the earth is now supposed to possess the power of gravitation.

The gravitation potential of the spheroid $r=a+\sigma_i$ (taking only a typical term of σ) at a point in the interior, estimated per unit volume, is

$$\frac{gw}{2a}(3a^2-r^2)+\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i$$

according to the usual formula in the theory of the potential.

Now the first term, being symmetrical round the centre of the sphere, can clearly cause no flow in the incompressible viscous sphere. We are therefore left with $\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i$.

Now if $\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i$ be substituted for W_i in (8), and if the resulting expression be compared with (7) when $-gw\sigma_i$ is written for S_i, it will be seen that $-\alpha''=\frac{3}{2i+1}\alpha'$. Thus

$$\alpha'+\alpha''=\alpha''\left(1-\frac{2i+1}{3}\right)^{\dagger}=-\frac{2}{3}(i-1)\alpha''.$$

And if $V_i=\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i$,

$$\alpha'+\alpha''=-\frac{1}{\upsilon}\left[\left\{\frac{i(i+2)}{2(i-1)[2(i+1)^2+1]}a^2-\frac{(i+1)(2i+3)}{2(2i+1)[2(i+1)^2+1]}r^2\right\}\frac{d}{dx}\left(\frac{2}{3}(i-1)V_i\right)\right.$$
$$\left.-\frac{i}{(2i+1)[2(i+1)^2+1]}r^{2i+3}\frac{d}{dx}\left(r^{-2i-1}\frac{2}{3}(i-1)V_i\right)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

with symmetrical expressions for $\beta'+\beta''$ and $\gamma'+\gamma''$.

Equation (9) then embodies the solution as far as it depends on (ii) and (iii). And since (9) is the same as (8) when $-\frac{2}{3}(i-1)V_i$ is written for W_i, we may include all the effects of mutual gravitation in producing a state of flow in the viscous sphere, by adopting THOMSON's solution (8), and taking instead of the true potential of the layer

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[*] 'Nat. Phil.', § 834, equation (8) when m is infinite compared with n, and $i-1$ written for i, and υ replaces n.

[†] The case of § 815 in THOMSON and TAIT's 'Nat. Phil.' is a special case of this.

of matter σ_i, $-\frac{2}{3}(i-1)$ times that potential, and by adding to it the external disturbing potential.

We have now learnt how to include the surface action in the potential; and if W_i be the potential of the external disturbing influence, the *effective* potential per unit volume at a point within the sphere, now free of surface action and of mutual gravitation, is $W_i-\frac{2gw(i-1)}{2i+1}\left(\frac{r}{a}\right)^i\sigma_i=r^iT_i$ suppose.

The complete solution of our problem is then found by writing r^iT_i in place of W_i in THOMSON's solution (8).*

In order however to apply the solution to the case of the earth, it will be convenient to use polar coordinates. For this purpose, write wr^iS_i for W_i, and let r be the radius vector; θ the colatitude; ϕ the longitude. Let ρ, ϖ, ν be the velocities radially, and along and perpendicular to the meridian respectively. Then the expressions for ρ, ϖ, ν will be precisely the same as those for α, β, γ in (8), save that for $\frac{d}{dx}$ we must put $\frac{d}{dr}$; for $\frac{d}{dy}$, $\frac{d}{r\sin\theta d\phi}$; and for $\frac{d}{dz}$, $\frac{d}{rd\theta}$.

Then after some reductions we have

$$\left.\begin{aligned}\rho&=\frac{i^2(i+2)a^2-i(i^2-1)r^2}{2(i-1)[2(i+1)^2+1]\upsilon}r^{i-1}T_i\\ \varpi&=\frac{i(i+2)a^2-(i-1)(i+3)r^2}{2(i-1)[2(i+1)^2+1]\upsilon}r^{i-1}\frac{dT_i}{d\theta}\\ \nu&=\frac{i(i+2)a^2-(i-1)(i+3)r^2}{2(i-1)[2(i+1)^2+1]\upsilon}\frac{r^{i-1}}{\sin\theta}\frac{dT_i}{d\phi}\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(10)\dagger$$

where $T_i=w\left(S_i-2g\frac{i-1}{2i+1}\frac{\sigma_i}{a^i}\right)$.

These equations for ρ, ϖ, ν give us the state of internal flow corresponding to the external disturbing potential r^iS_i, including the effects of the mutual gravitation of the matter constituting the spheroid.

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[*] The introduction of the effects of gravitation may be also carried out synthetically, as is done by Sir W. THOMSON (§ 840, 'Nat. Phil.'); but the effects of the lagging of the tide-wave render this method somewhat artificial, and I prefer to exhibit the proof in the manner here given. Conversely, the elastic problem may be solved as in the text.

[†] There seems to be a misprint as to the signs of the $\mathfrak{S}$'s in the second and third of equations (13) of § 834 of the 'Nat. Phil.' (1867). When this is corrected μ and ν admit of reduction to tolerably simple forms. It appears to me also that the differentiation of ρ in (15) is incorrect; and this falsifies the argument in three following lines. The correction is not, however, in any way important.

3. *The form of the free surface at any time.*

If ρ' be the surface value of ρ, then

$$\rho' = \frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\,\frac{a^{i+1}}{v}\,\mathrm{T}_i$$

Hence after a short interval of time δt, the equation to the bounding surface of the spheroid becomes $r=a+\sigma_i+\rho'\delta t$; but during this same interval, σ_i has become $\frac{d\sigma_i}{dt}\delta t$, whence

$$\frac{d\sigma_i}{dt}=\rho'=\frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\,\frac{wa^{i+1}}{v}\,\mathrm{S}_i-\frac{i}{2(i+1)^2+1}\,\frac{gwa}{v}\,\sigma_i$$

or

$$\frac{d\sigma_i}{dt}+\frac{i}{2(i+1)^2+1}\,\frac{gwa}{v}\,\sigma_i=\frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\,\frac{wa^{i+1}}{v}\,\mathrm{S}_i \quad . \quad . \quad . \quad . \quad . \quad (11)$$

This differential equation gives the manner in which the surface changes, under the influence of the external potential $r^i\mathrm{S}_i$.

If S_i be not a function of the time, and if s_i be the value of σ_i when $t=0$,

$$\sigma_i=\frac{2i+1}{2(i-1)}\,\frac{a^i\mathrm{S}_i}{g}\left[1-exp\left(\frac{-gwait}{[2(i+1)^2+1]v}\right)\right]+s_i\,exp\left(\frac{-gwait}{[2(i+1)^2+1]v}\right) \quad . \quad . \quad (12)^*$$

When t is infinite

$$\sigma_i=\frac{2i+1}{2(i-1)}\,\frac{a^i\mathrm{S}_i}{g} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and there is no further state of flow, for the fluid has assumed the form which it would have done if it had not been viscous. This result is of course in accordance with the equilibrium theory of tides.

If S_i be zero, the equation shows how the inequalities on the surface of a viscous globe would gradually subside under the influence of simple gravity. We see how much more slowly the change takes place if i be large; that is to say, inequalities of small extent die out much more slowly than wide-spread inequalities. Is it not possible that this solution may throw some light on the laws of geological subsidence and upheaval?

4. *Digression on the adjustments of the earth to a form of equilibrium.*

In a former paper I had occasion to refer to some points touching the precession of a viscous spheroid, and to consider its rate of adjustment to a new form of equilibrium,

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[*] I write "exp." for "e to the power of."

when its axis of rotation had come to depart from its axis of symmetry.* I propose then to discuss the subject shortly, and to establish the law which was there assumed.

Suppose that the earth is rotating with an angular velocity ω about the axis of z, but that at the instant at which we commence our consideration the axis of symmetry is inclined to the axis of z at an angle α in the plane of xy. and that at that instant the equation to the free surface is

$$r=a\left\{1+\frac{5m}{4}\left(\frac{1}{3}-[\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos\phi]^2\right)\right\}$$

where m is the ratio of centrifugal force at the equator to pure gravity, and therefore equal to $\frac{\omega^2 a}{g}$.

Then putting $i=2$ in (12), and dropping the suffixes of S, s, σ, $s=\frac{5ma}{4}\left(\frac{1}{3}-[\quad]^2\right)$.

We may conceive the earth to be at rest, if we apply a potential

$$wr^2\text{S}=\frac{1}{2}\omega^2 wr^2\left(\frac{1}{3}-\cos^2\theta\right)$$

so that

$$\text{S}=\frac{1}{2}\omega^2\left(\frac{1}{3}-\cos^2\theta\right)$$

By (12) we have

$$\sigma=\frac{5a^2\text{S}}{2g}\left[1-exp\left(-\frac{2wgat}{19\upsilon}\right)\right]+s\;exp\left(-\frac{2wgat}{19\upsilon}\right)$$

Then, substituting for S and s, and putting $\kappa=\frac{2wga}{19\upsilon}$

$$\sigma=\frac{5ma}{4}\left\{\left(\frac{1}{3}-\cos^2\theta\right)[1-exp(-\kappa t)]+\left(\frac{1}{3}-[\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos\phi]^2\right)exp(-\kappa t)\right\}$$

Now

$$\begin{aligned}&[1-exp(-\kappa t)]\cos^2\theta+exp(-\kappa t)(\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos\phi)^2\\&\quad=\cos^2\theta\,[1-\sin^2\alpha\;exp(-\kappa t)]+\sin^2\alpha\sin^2\theta\cos^2\phi\;exp(-\kappa t)\\&\quad+2\sin\alpha\cos\alpha\sin\theta\cos\theta\cos\phi\;exp(-\kappa t).\end{aligned}$$

Therefore the Cartesian equation to the spheroid at the time t is,

$$\frac{x^2+y^2+z^2}{1+\frac{5m}{6}}=a^2-\frac{5m}{2}\{z^2(1-\sin^2\alpha\;exp(-\kappa t))+x^2\sin^2\alpha\;exp(-\kappa t)+2xz\sin\alpha\cos\alpha\,exp(-\kappa t)\}$$

or

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* "On the Influence of Geological Changes on the Earth's Axis of Rotation," Phil. Trans., Vol. 167, Part I., sec. 5.

$$x^2\left\{1+\frac{5m}{2}\sin^2\alpha\, exp(-\kappa t)\right\}+y^2+z^2\left\{1+\frac{5m}{2}(1-\sin^2\alpha\, exp(-\kappa t))\right\}$$
$$+5m\sin\alpha\cos\alpha\, xz\, exp(-\kappa t)=a^2\left(1+\frac{5m}{6}\right)$$

Let α' be the inclination of the principal axis at this time to the axis of z, then

$$\tan 2\alpha'=\frac{\sin 2\alpha\, exp(-\kappa t)}{1-2\sin^2\alpha\, exp(-\kappa t)}$$

If α be small, as it was in the case I considered in my former paper, then $\alpha'=\alpha\, exp(-\kappa t)$ and $\frac{d\alpha'}{dt}=-\kappa\alpha'$.

Therefore the velocity of approach of the principal axis to the axis of rotation varies as the angle between them, which is the law assumed.

Also $\kappa=\frac{2wga}{19\upsilon}$, so that κ (the ν of my former paper) varies inversely as the coefficient of viscosity,—as was also assumed.

5. *Bodily tides in a viscous earth.*[*]

The only case of interest in which S_i of equation (11) is a function of the time, is where it is a surface harmonic of the second order, and is periodic in time; for this will give the solution of the tidal problem. Since, moreover, we are only interested in the case where the motion has attained a permanently periodic character, the exponential terms in the solution of (11) may be set aside.

Let $S_2=S\cos(vt+\eta)$, and in accordance with THOMSON's notation,[†] let $\frac{2g}{5a}=\mathfrak{g}$, and $\frac{19\upsilon}{5wa^2}=\mathfrak{r}$; and therefore $\frac{2gwa}{19\upsilon}=\frac{\mathfrak{g}}{\mathfrak{r}}$.

Then putting $i=2$ in (11), and omitting the suffix of σ for brevity, we have

$$\frac{d\sigma}{dt}+\frac{\mathfrak{g}}{\mathfrak{r}}\sigma=\frac{a}{\mathfrak{r}}S\cos(vt+\eta)\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

It is evident that σ must be of the form $A\cos(vt+B)$, and therefore

$$A\{-v\mathfrak{r}\sin(vt+B)+\mathfrak{g}\cos(vt+B)\}=aS\cos(vt+\eta)$$

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[*] In certain cases the forces do not form a rigorously equilibrating system, but there is a very small couple tending to turn the earth. The effects of this unbalanced couple, which varies as the square of $\frac{3}{2}\frac{m}{c^3}$, will be considered in a succeeding paper on the "Precession of a Viscous Spheroid." (Read before the Royal Society, December 19th, 1878.)

[†] 'Nat. Phil.', § 840, eq. (27).

or if we put $\tan\epsilon=\frac{v\mathfrak{r}}{\mathfrak{g}}$,

$$A\mathfrak{g}\sec\epsilon\cos(vt+B+\epsilon)=a\mathrm{S}\cos(vt+\eta).$$

Hence $A=\frac{a}{\mathfrak{g}}\mathrm{S}\cos\epsilon$, and $B=\eta-\epsilon$.

Therefore the solution of (14) is,

$$\sigma=\frac{a}{\mathfrak{g}}\mathrm{S}\cos\epsilon\cos(vt+\eta-\epsilon)\quad\ldots\ldots\ldots\quad(15)$$

Where $\tan\epsilon=\frac{v\mathfrak{r}}{\mathfrak{g}}=\frac{19\upsilon v}{2gaw}$

But if the globe were a perfect fluid, and if the equilibrium theory of tides were true, we should have by (13),

$$\sigma=\frac{5a}{2g}\cdot a\mathrm{S}\cos(vt+\eta)=\frac{\mathrm{S}a}{\mathfrak{g}}\cos(vt+\eta).$$

Thus we see that the tides of the viscous sphere are to the equilibrium tides of a fluid sphere as $\cos\epsilon:1$, and that there is a retardation in time of $\frac{\epsilon}{v}$.

A parallel investigation will be applicable to the general case where the disturbing potential is $wr^i\mathrm{S}_i\cos(vt+\eta)$; and the same solution will be found to hold save that we now have $\tan\epsilon=\frac{2(i+1)^2+1}{i}\cdot\frac{\upsilon v}{gaw}$, and that in place of $\mathfrak{g}$ we have $\frac{2(i-1)g}{(2i+1)a}$.

6. *Diminution of ocean tides on equilibrium theory.*

Suppose now that there is a shallow ocean on the viscous nucleus, and let us find the effects on the ocean tides of the motion of the nucleus according to the equilibrium theory, neglecting the gravitation of the water.

The potential at a point outside the nucleus is

$$g\frac{a^2}{r}+\frac{3}{5}g\left(\frac{a}{r}\right)^2\sigma+r^2\mathrm{S}\cos(vt+\eta),$$

and if this be put equal to a constant, we get the form which the ocean must assume. Let $r=a+u$ be the equation to the surface of the ocean. Then substituting for r in the potential, and neglecting u in the small terms, and equating the whole to a constant, we find,

$$-gu+\frac{3}{5}g\sigma+a^2\mathrm{S}\cos(vt+\eta)=0$$

or

$$u=\frac{3}{5}\sigma+\frac{a^2}{g}\mathrm{S}\cos(vt+\eta).$$

But the rise and fall of the tide relative to the nucleus is given by $u-\sigma$, and

$$\begin{aligned} u-\sigma &= \frac{a^2 \mathrm{S}}{g}\cos(vt+\eta)-\frac{2}{5}\sigma \\ &= \frac{2}{5}\frac{a\mathrm{S}}{\mathfrak{g}}\left[\cos(vt+\eta)-\cos\epsilon\cos(vt+\eta-\epsilon)\right] \\ &= -\frac{2}{5}\frac{a\mathrm{S}}{\mathfrak{g}}\sin\epsilon\sin(vt+\eta-\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17) \end{aligned}$$

Now if the nucleus had been rigid, the rise and fall would have been given by

$$\frac{2}{5}\frac{a\mathrm{S}}{\mathfrak{g}}\cos(vt+\eta)=\mathrm{H}\cos(vt+\eta) \text{ suppose.}$$

Therefore

$$u-\sigma=-\mathrm{H}\sin\epsilon\sin(vt+\eta-\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Hence the apparent tides on the yielding nucleus are equal to the tides on a rigid nucleus reduced in the proportion $\sin\epsilon : 1$; and since $-\sin(vt+\eta-\epsilon) = \cos\left(vt+\eta+\frac{\pi}{2}-\epsilon\right)$ they are retarded by $\frac{1}{v}\left(\epsilon-\frac{\pi}{2}\right)$. As ϵ is necessarily less than $\frac{\pi}{2}$, this is equivalent to an acceleration of the time of high water equal to $\frac{1}{v}\left(\frac{\pi}{2}-\epsilon\right)$.

It is, however, worthy of notice that this is only an acceleration of phase relatively to the nucleus, and there is an absolute retardation of phase equal to arc-tan $\frac{3\sin\epsilon\cos\epsilon}{3+2\cos^2\epsilon}$.

7. *Semidiurnal and fortnightly tides.*

Let the axis of z be the earth's axis of rotation, and let the plane of xz be fixed in the earth; let c be the moon's distance, and m its mass.

Suppose the moon to move in the equator with an angular velocity ω relatively to the earth, and let the moon's terrestrial longitude, measured from the plane of xz, at the time t be ωt.

Then at the time t, the gravitation potential of the tide generating force, estimated per unit volume of the earth's mass is

$$-\frac{3}{2}\frac{m}{c^3}wr^2\left\{\frac{1}{3}-\sin^2\theta\cos^2(\phi-\omega t)\right\}$$

which is equal to

$$\frac{3}{4}\frac{m}{c^3}wr^2\left(\frac{1}{3}-\cos^2\theta\right)+\frac{3}{4}\frac{m}{c^3}wr^2\{\sin^2\theta\cos 2\phi\cos 2\omega t+\sin^2\theta\sin 2\phi\sin 2\omega t\}.$$

The first term of this expression is independent of the time, and therefore produces an effect on the viscous earth, which will have died out when the motion has become steady; its only effect is slightly to increase the ellipticity of the earth's surface.

The two latter terms give rise to two tides, in one of which (according to previous notation)

$$S\cos(vt+\eta)=\frac{3}{4}\frac{m}{c^3}\sin^2\theta\cos 2\phi\cos 2\omega t,$$

and in the second of which

$$S\cos(vt+\eta)=-\frac{3}{4}\frac{m}{c^3}\sin^2\theta\sin 2\phi\cos\left(2\omega t+\frac{\pi}{2}\right).$$

Now ϵ, which depends on the frequency of the tide generating potential, will clearly be the same for both these tides; and therefore they will each be equal to the corresponding tides of a fluid spheroid, reduced by the same amount and subject to the same retardation. They may therefore be recompounded into a single tide; and since v will here be equal to 2ω, it follows that the retardation of the bodily semidiurnal tide is $\frac{\epsilon}{2\omega}$, where $\tan\epsilon=\frac{2\omega\mathfrak{r}}{\mathfrak{g}}=\frac{19v\omega}{gaw}$. Also the height of the tide is less than the corresponding equilibrium tide of a fluid spheroid in the proportion of $\cos\epsilon$ to unity.

Similarly by section (6) the height of the ocean tide on the yielding nucleus is given by the corresponding tide on a rigid nucleus multiplied by $\sin\epsilon$, and there is an acceleration of relative high water equal to $\frac{\pi}{4\omega}-\frac{\epsilon}{2\omega}$.

The case of the fortnightly tide is somewhat simpler.

If Ω be the moon's orbital angular velocity, and I the inclination of the plane of the orbit to the earth's equator, then the part of the tide generating potential, on which the fortnightly tide depends, is—

$$\frac{9}{8}\frac{m}{c^3}wr^2\sin^2 \mathrm{I}\left(\tfrac{1}{3}-\cos^2\theta\right)\cos 2\Omega t$$

and we see at once by sections (5) and (6) that $\tan\epsilon=\frac{19\,v\Omega}{gaw}$. The bodily tide is the tide of a fluid spheroid multiplied by $\cos\epsilon$; the reduction of ocean tide is given by $\sin\epsilon$; and there is a time-acceleration of relative high water of $\frac{\pi}{4\Omega}-\frac{\epsilon}{2\Omega}$ or $\frac{1}{2}-\frac{\epsilon}{\pi}$ of a week.

In order to make the meaning of the previous analytical results clearer, I have formed the following numerical tables, to show the effects of this hypothesis on the semidiurnal and fortnightly tides. The coefficient of viscosity is usually expressed in gravitation units of force so that the formula for ϵ becomes, $\tan\epsilon=\frac{19\,v\omega}{wa}$. In the tables v is expressed in the centimetre-gramme-second system, and in gravitation units of force; a is taken as $6{\cdot}37\times 10^8$, and w as $5{\cdot}5$, and the angular velocity ω of the moon relatively to the earth as ·00007025 radians per second.

With these data I find $v=10^{12}\times 2{\cdot}625\tan\epsilon$. As a standard of comparison with the coefficients of viscosity given in the tables, I may mention that, according to some rough experiments of my own, the viscosity of British pitch at near the freezing temperature (34° Fahr.), when it is hard and brittle, is about $10^8 \times 1{\cdot}3$ when measured in the same units.

Lunar Semidiurnal Tide.						
Coefficient of viscosity $\times 10^{-10}$ $(v \times 10^{-10})$	Retardation of bodily tide $\left(\frac{\epsilon}{2\omega}\right)$		Height of bodily tide is tide of fluid spheroid multiplied by $(\cos \epsilon)$.	Height of ocean tide is tide on rigid nucleus multiplied by $(\sin \epsilon)$.	High tide relatively to viscous nucleus accelerated by $\frac{1}{\omega}\left(\frac{\pi}{4}-\frac{\epsilon}{2}\right)$	
	Hrs.	min.			Hrs.	min.
Fluid 0	0	0	1·000	·000	3	6
46	0	21	·985	·174	2	46
96	0	41	·940	·342	2	25
152	1	2	·866	·500	2	4
220	1	23	·766	·643	1	44
313	1	44	·643	·766	1	23
455	2	4	·500	·866	1	2
721	2	25	·342	·940	0	41
1,488	2	46	·174	·985	0	21
Rigid ∞	3	6	·000	1·000	0	0
Fortnightly Tide.						
	Days.	hrs.			Days.	hrs.
Fluid 0	0	0	1·000	·000	3	10
1,200	0	9	·985	·174	3	1
2,500	0	18	·940	·342	2	16
4,000	1	3	·866	·500	2	6
5,800	1	12	·766	·643	1	21
8,300	1	21	·643	·766	1	12
12,000	2	6	·500	·866	1	3
19,000	2	16	·342	·940	0	18
39,300	3	1	·174	·985	0	9
Rigid ∞	3	10	·000	1·000	0	0

I now pass on to a case which is intermediate between the hypothesis of Sir W. Thomson and that just treated.

8. *The tides of an elastico-viscous spheroid.*

The term elastico-viscous is used to denote that the stresses requisite to maintain the body in a given strained configuration decrease the longer the body is thus constrained, and this is undoubtedly the case with many solids. In the particular case which is here treated, it is assumed that the stresses diminish in geometrical progression, as the time increases in arithmetical progression. If, for example, a cubical block of the substance be strained to a given amount by a shearing stress T, and maintained in that position, then after a time t, the shearing stress, is $\mathrm{T}\, exp\left(-\frac{t}{\mathfrak{t}}\right)$.

The time $\mathbf{t}$ measures the rate at which the stress falls off, and is called (I believe by Professor MAXWELL) "the modulus of the time of relaxation of rigidity;" it is the time in which the initial stress has been reduced to e^{-1} or $\cdot 3679$ of its initial value. I do not suppose, however, that any solid conforms exactly to this law; but I conceive that it is often useful in physical problems to discuss mathematically an ideal case, which presents a sufficiently marked likeness to the reality, where we are unable to determine exactly what that reality may be.

Mr. J. G. BUTCHER has found the equations of motion of such an ideal substance from the consideration that the elasticity of groups of molecules is continually breaking down, and that the groups rearrange themselves afterwards.* These considerations lead him to the following results for the stresses across rectangular planes at any point in the interior, viz. (with the notation of § 1):—

$$\mathrm{P}=(m-n)\delta+2n\left(\frac{1}{\mathrm{t}}+\frac{d}{dt}\right)^{-1}\left(\frac{d\alpha}{dx}+\frac{1}{3\mathrm{t}}\alpha\right),\ \mathrm{S}=n\left(\frac{1}{\mathrm{t}}+\frac{d}{dt}\right)^{-1}\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right)$$

and similar expressions for Q, R, T, U; where $m-\frac{n}{3}$ is the coefficient of dilatation, n that of rigidity, δ the dilatation, and α, β, γ, the components of flow.

These expressions are clearly in accordance with the above definition of elastico-viscosity, for $\frac{d\mathrm{S}}{dt}+\frac{\mathrm{S}}{\mathrm{t}}=n\left(\frac{d\beta}{dz}+\frac{d\gamma}{dy}\right)$.

If the expressions for P, S, &c., be substituted in the equations of equilibrium of the elementary parallelopiped, it is found by aid of the equation of continuity $\frac{d\delta}{dt}=\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}$, that when inertia is neglected

$$\left(\frac{1}{\mathrm{t}}+\frac{d}{dt}\right)^{-1}\left\{\left[\left(m-\tfrac{1}{3}n\right)\frac{1}{\mathrm{t}}+m\frac{d}{dt}\right]\frac{d\delta}{dx}+n\nabla^2\alpha\right\}+\mathrm{X}=0$$

and two similar equations.

By the same reasoning as in § 1, we may put, $\delta=\frac{-p}{m-\frac{1}{3}n}$, and the equations become

$$-\left(\frac{1}{\mathrm{t}}+\frac{d}{dt}\right)^{-1}\left\{\left[\frac{1}{\mathrm{t}}+\frac{m}{m-\frac{1}{3}n}\frac{d}{dt}\right]\frac{dp}{dx}-n\nabla^2\alpha\right\}+\mathrm{X}=0.$$

Then supposing the substance to be incompressible, so that m is infinitely large compared to n, and therefore $m\div m-\frac{1}{3}n$ is unity, the equations become

$$-\frac{dp}{dx}+n\left(\frac{1}{\mathrm{t}}+\frac{d}{dt}\right)^{-1}\nabla^2\alpha+\mathrm{X}=0$$

and two similar equations.

---

* Proc. Lond. Math. Soc., Dec. 14, 1876, p. 107–9. It seems to me that the hypothesis ought to represent the elastico-viscosity of ice very closely.

Now these equations have exactly the same form as those for the motion of a viscous fluid, save that the coefficient of viscosity v is replaced by $n\left(\frac{1}{\mathfrak{t}}+\frac{d}{dt}\right)^{-1}$. We may therefore at once pass to the differential equation (11) which gives the form of the surface of the spheroid at any time.

Substituting, therefore, in (11) for $\frac{1}{\nu}, \frac{1}{n}\left(\frac{1}{\mathfrak{t}}+\frac{d}{dt}\right)$, we get

$$\left[1+\frac{i}{2(i+1)^2+1}\frac{gwa}{n}\right]\frac{d\sigma_i}{dt}+\frac{i}{2(i+1)^2+1}\frac{gwa}{n\mathfrak{t}}\sigma_i=\frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\frac{wa^{i+1}}{n}\left(\frac{1}{\mathfrak{t}}+\frac{d}{dt}\right)\mathrm{S}_i.$$

This equation admits of solution just in the same way that equation (11) was solved; but I shall confine myself to the case of the tidal problem, where $i=2$ and $\mathrm{S}_2=\mathrm{S}\cos(vt+\eta)$. In this special case the equation becomes

$$\left(1+\frac{2gwa}{19n}\right)\frac{d\sigma}{dt}+\frac{2gwa}{19n\mathfrak{t}}\sigma=\frac{5wa^3}{19n}\left[\frac{1}{\mathfrak{t}}\cos(vt+\eta)-v\sin(vt+\eta)\right]\mathrm{S}.$$

And if we put $\frac{19n}{2gwa}+1=\frac{1}{k}$, $\tan\psi=v\mathfrak{t}$, and $\mathfrak{g}=\frac{2g}{5a}$

This may be written

$$\frac{d\sigma}{dt}+\frac{k}{\mathfrak{t}}\sigma=\frac{vak}{\mathfrak{g}\sin\psi}\mathrm{S}\cos(vt+\eta+\psi).$$

In the solution appropriate to the tidal problem, we may omit the exponential term, and assume $\sigma=\mathrm{A}\cos(vt+\mathrm{B})$. Then if we put $\tan\chi=\frac{v\mathfrak{t}}{k}$

$$\frac{d\sigma}{dt}+\frac{k}{\mathfrak{t}}\sigma=\frac{\mathrm{A}v}{\sin\chi}\cos(vt+\mathrm{B}+\chi).$$

Whence it follows that $\mathrm{B}=\eta+\psi-\chi$, and

$$\mathrm{A}=\frac{a}{\mathfrak{g}}k\frac{\sin\chi}{\sin\psi}=\frac{a}{\mathfrak{g}}\frac{\cos\chi}{\cos\psi},$$

so that

$$\sigma=\frac{a\mathrm{S}}{\mathfrak{g}}\frac{\cos\chi}{\cos\psi}\cos(vt+\eta+\psi-\chi).$$

Hence the bodily tide of the elastico-viscous spheroid is equal to the equilibrium tide of a fluid spheroid multiplied by $\frac{\cos\chi}{\cos\psi}$, and high tide is retarded by $\chi-\psi\div v$.

The formula for $\tan\chi$ may be expressed in a somewhat more convenient form; we have $\tan\psi=v\mathfrak{t}$, and therefore $\tan\chi=\tan\psi+\frac{19nv\mathfrak{t}}{2gwa}$

But $n\mathfrak{t}$ is the coefficient of viscosity, and in treating the tides of the purely viscous spheroid we put $\tan\epsilon=\frac{19v}{2gwa}\times$ coefficient of viscosity; therefore adopting the same notation here, we have $\tan\chi=\tan\psi+\tan\epsilon$.

If the modulus of relaxation $\mathfrak{t}$ be zero, whilst the coefficient of rigidity n becomes infinite, but $n\mathfrak{t}$ finite, the substance is purely viscous, and we have $\psi=0$ and $\chi=\epsilon$, so that the solution reduces to the case already considered. If $\mathfrak{t}$ be infinite, the substance is purely elastic, and we have $\psi=\frac{\pi}{2}$, $\chi=\frac{\pi}{2}$ and since $\frac{\cos\chi}{\cos\psi}=k\frac{\sin\chi}{\sin\psi}$, therefore

$$\sigma=\frac{ak}{\mathfrak{g}}\mathrm{S}\cos(vt+\eta).$$

But according to THOMSON's notation* $\frac{19n}{2gwa}=\frac{\mathfrak{r}}{\mathfrak{g}}$, so that $\sigma=\frac{a}{\mathfrak{r}+\mathfrak{g}}\mathrm{S}\cos(vt+\eta)$, which is the solution of THOMSON's problem of the purely elastic spheroid.

The present solution embraces, therefore, both the case considered by him, and that of the viscous spheroid.

9. *Ocean tides on an elastico-viscous nucleus.*

If $r=a+u$ be the equation to the ocean spheroid, we have, as in sec. (6), that the height of tide relatively to the nucleus is given by

$$u-\sigma=\frac{a^2}{g}\mathrm{S}\cos(vt+\eta)-\frac{2}{5}\sigma,$$

and substituting the present value of σ,

$$\begin{aligned}u-\sigma&=\frac{2}{5}\frac{a}{\mathfrak{g}}\mathrm{S}\left[\cos(vt+\eta)-\frac{\cos\chi}{\cos\psi}\cos(vt+\eta+\psi-\chi)\right]\\&=-\frac{2}{5}\frac{a}{\mathfrak{g}}\mathrm{S}\frac{\sin(\chi-\psi)}{\cos\psi}\sin(vt+\eta-\chi).\end{aligned}$$

If the nucleus had been rigid the rise and fall would have been given by $\mathrm{H}\cos(vt+\eta)$, where $\mathrm{H}=\frac{2}{5}\frac{a}{\mathfrak{g}}\mathrm{S}$; therefore on the yielding nucleus it is given by

$$\begin{aligned}u-\sigma&=-\mathrm{H}\frac{\sin(\chi-\psi)}{\cos\psi}\sin(vt+\eta-\chi)\\&=-\mathrm{H}\cos\chi(\tan\chi-\tan\psi)\sin(vt+\eta-\chi)\\&=-\mathrm{H}\cos\chi\tan\epsilon\sin(vt+\eta-\chi).\end{aligned}$$

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* 'Nat. Phil.', § 840.

Hence the apparent tides on the yielding nucleus are equal to the corresponding tides on a rigid nucleus reduced in the proportion of $\cos\chi\tan\epsilon$ to unity, and there is an acceleration of the time of high water equal to $\frac{1}{v}\left(\frac{\pi}{2}-\chi\right)$.

As these analytical results present no clear meaning to the mind, I have compiled the following tables. I take the two cases considered by Sir W. Thomson, where the spheroid has the rigidity of glass, and that of iron, and I work out the results for various times of relaxation of rigidity, for the semidiurnal and fortnightly tides. The last line in each division of each table is Thomson's result.

I may remind the reader that the modulus of relaxation of rigidity is the time in which the stress requisite to retain the body in its strained configuration falls to ·368 of its initial value.

Spheroid with Rigidity of Glass ($2·44\times10^{8}$).

Lunar Semidiurnal Tide.

Modulus of relaxation of rigidity (t).			Coefficient of viscosity ($nt\times10^{-10}$).	Ocean tide is tide on rigid nucleus multiplied by ($\cos\chi\tan\epsilon$).	High tide relatively to nucleus is accelerated by $\left(\frac{\pi}{2}-\chi\right)\frac{1}{v}$	
	Hrs.				Hrs.	min.
Fluid	0		0	·000	3	6
	1		88	·256	1	44
	2		176	·342	1	3
	3		264	·370	0	45
	4		351	·382	0	34
	5		439	·388	0	28
Elastic	∞		∞	·398	0	0
Fortnightly Tide.						
	Days.	hrs.			Days.	hrs.
Fluid	0	0	0	·000	3	10
	0	6	500	·099	2	21
	0	12	1,100	·181	2	9
	1	0	2,100	·285	1	16
	2	0	4,200	·357	1	0
	3	0	6,300	·379	0	16
Elastic	∞		∞	·398	0	0

Spheroid with Rigidity of Iron ($7\cdot8\times10^8$).

Lunar Semidiurnal Tide.

Modulus of relaxation.			Viscosity.	Reduction of ocean tide.	Acceleration of high water.	
	Hrs.	min.			Hrs.	min.
Fluid	0	0	0	·000	3	6
	0	30	140	·420	1	47
	1	0	280	·573	1	7
	2	0	560	·647	0	36
	3	0	840	·665	0	25
Elastic	∞		∞	·679	0	0

Fortnightly Tide.

Modulus of relaxation.			Viscosity.	Reduction of ocean tide.	Acceleration of high water.	
	Days	hrs.			Days.	hrs.
Fluid	0	0	0	·000	3	10
	0	6	1,700	·294	2	11
	0	12	3,400	·470	1	18
	1	0	6,700	·602	1	1
	2	0	13,500	·657	0	13
	3	0	20,200	·669	0	9
Elastic	∞		∞	·679	0	0

10. *The influence of inertia.*

In establishing these results inertia has been neglected, and I will now show that this neglect is not such as to materially vitiate my results.*

Suppose that the spheroid is constrained to execute such a vibration as it would do if it were a perfect fluid, and if the equilibrium theory of tides were true. Then the effective forces which are, according to D'Alembert's principle, the equivalent of inertia, are found by multiplying the acceleration of each particle by its mass.

Inertia may then be safely neglected if the effective force on that particle which has the greatest amplitude of vibration is small compared with the tide-generating force on it. In the case of a viscous spheroid, the inertia will have considerably less effect than it would have in the supposed constrained oscillation.

Now suppose we have a tide-generating potential $wr^2\,\mathrm{S}\cos(vt+\eta)$, then, according to the equilibrium theory of tides, the form of the surface is given by

$$\sigma=\frac{5a^2}{2g}\,\mathrm{S}\cos(vt+\eta)\,;$$

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* In a future paper (read on December 19th, 1878) I shall give an approximate solution of the problem, inclusive of the effects of inertia.

and this function gives the proposed constrained oscillation. It is clear that it is the particles at the surface which have the widest amplitude of oscillation. The effective force on a unit element at the surface is

$$-w\frac{d^2\sigma}{dt^2}=\frac{5a^2}{2g}wv^2\,\mathrm{S}\cos(vt+\eta).$$

But the normal disturbing force at the surface is $2wa\,\mathrm{S}\cos(vt+\eta)$. Therefore inertia may be neglected if $\frac{5a^2}{2g}wv^2$ is small compared with $2wa$, or if $\frac{5a}{4g}v^2$ is a small fraction. The tide of the shortest period with which we have to deal is that in which $v=2\omega$, so that we must consider the magnitude of the fraction $4\times\frac{5a\omega^2}{4g}$. If ω were the earth's true angular velocity, instead of its angular velocity relatively to the moon, then $\frac{5a\omega^2}{4g}$ would be the ellipticity of its surface if it were homogeneous. This ellipticity is, as is well known, $\frac{1}{232}$. Hence the fraction, which is the criterion of the negligeability of inertia, is about $\frac{1}{58}$.

If, then, it be considered that this way of looking at the subject certainly exaggerates the influence of inertia, it is clear that the neglect of inertia is not such as to materially vitiate the results given above.

II.

A TIDAL YIELDING OF THE EARTH'S MASS, AND THE CANAL-THEORY OF TIDES.

In the first part of this paper the equilibrium theory has been used for the determination of the reduction of the height of tide, and the alteration of phase, due to bodily tides in the earth.

Sir W. THOMSON remarks, with reference to a supposed elastic yielding of the earth's body: "Imperfect as the comparisons between theory and observation as to the actual height of the tides has been hitherto, it is scarcely possible to believe that the height is in reality only two-fifths of what it would be if, as has been universally assumed in tidal theories, the earth were perfectly rigid. It seems, therefore, nearly certain, with no other evidence than is afforded by the tides, that the tidal effective rigidity of the earth must be greater than that of glass."*

The equilibrium theory is quite fallacious in its explanation of the semidiurnal tide, but Sir W. THOMSON is of opinion that it must give approximately correct results for tides of considerable period. It is therefore on the observed amount of the fortnightly tide that he places reliance in drawing the above conclusion. Under these circumstances, a dynamical investigation of the effects of a tidal yielding of the earth on a tide of short period, according to the canal theory, is likely to be interesting.

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* 'Nat. Phil.', § 843.

The following investigation will be applicable either to the case of the earth's mass yielding through elasticity, plasticity, or viscosity; it thus embraces Sir W. Thomson's hypothesis of elasticity, as well as mine of viscosity and elastico-viscosity.

11. *Semidiurnal tide in an equatorial canal on a yielding nucleus.*

I shall only consider the simple case of the moon moving uniformly in the equator, and raising tide waves in a narrow shallow equatorial canal of depth h.

The potential of the tide-generating force, as far as concerns the present inquiry, is, with the old notation, $\left(\frac{r}{a}\right)^2\frac{\tau}{2}\sin^2\theta\cos 2(\phi-\omega t)$, where $\tau=\frac{3}{2}\frac{ma^2}{c^3}$. This force will raise a bodily tide in the earth, whether it be elastic, plastic, or viscous. Suppose, then, that the greatest range of the bodily tide at the equator is $2\mathrm{E}$, and that it is retarded after the passage of the moon over the meridian by an angle $\frac{\epsilon}{2}$. Then the equation to the bounding surface of the solid earth, at the time t, is $r=a+\mathrm{E}\sin^2\theta\cos[2(\phi-\omega t)+\epsilon]$; or with former notation $\sigma=\mathrm{E}\sin^2\theta\cos[2(\phi-\omega t)+\epsilon]$.

The whole potential V, at a point outside the nucleus, is the sum of the potential of the earth's attraction, and of the potential of the tide-generating force. Therefore

$$\mathrm{V}=g\frac{a^2}{r}+\frac{3}{5}g\left(\frac{r}{a}\right)^2\mathrm{E}\sin^2\theta\cos[2(\phi-\omega t)+\epsilon]+\frac{\tau}{2}\left(\frac{r}{a}\right)^2\sin^2\theta\cos 2(\phi-\omega t)$$

$$=g\frac{a^2}{r}+\{\mathrm{F}\cos[2(\phi-\omega t)+\epsilon]+\mathrm{G}\sin[2(\phi-\omega t)+\epsilon]\}\left(\frac{r}{a}\right)^2\sin^2\theta$$

where $\mathrm{F}=\frac{3}{5}g\mathrm{E}+\frac{\tau}{2}\cos\epsilon$, $\mathrm{G}=\frac{\tau}{2}\sin\epsilon$.

Sir George Airy shows, in his article on "Tides and Waves" in the 'Encyclopædia Metropolitana,' that the motion of the tide-wave in a canal running round the earth is the same as though the canal were straight, and the earth at rest, whilst the disturbing body rotates round it. This simplification will be applicable here also.

As before stated, the canal is supposed to be equatorial, and of depth h.

After the canal has been developed, take the origin of rectangular coordinates in the undisturbed surface of the water, and measure x along the canal in the direction of the moon's motion, and y vertically downwards.

We have now to transform the potential V, and the equation to the surface of the solid earth, so as to make them applicable to the supposed development. If v be the velocity of the tide-wave, then $\omega a=v$; also the wave length is half the circumference of the earth's equator, or πa; and let $m=\frac{2}{a}$. Then we have the following transformations:—

$$\theta=\frac{\pi}{2},\ \phi=\frac{mx}{2},\ r=a+h-y.$$

Also in the small terms we may put $r=a$. Thus the potential becomes

$$\mathrm{V}=\text{const.}+gy+\mathrm{F}\cos\left[m(x-vt)+\epsilon\right]+\mathrm{G}\sin\left[m(x-vt)+\epsilon\right].$$

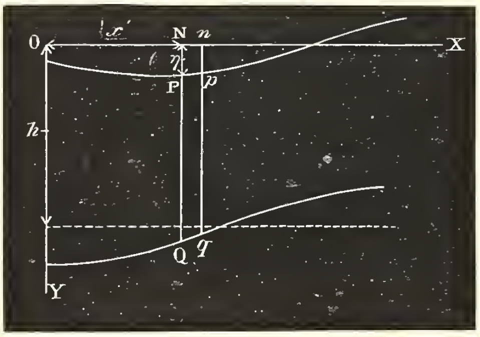
Again, to find the equation to the bottom of the canal, we have to transform the equation

$$r=a+\mathrm{E}\sin^2\theta\cos\left[2(\phi-\omega t)+\epsilon\right].$$

If y' be the ordinate of the bottom of the canal, corresponding to the abscissa x, this equation becomes after development

$$y'=h-\mathrm{E}\cos\left[m(x-vt)+\epsilon\right].$$

We now have to find the forced waves in a horizontal shallow canal, under the action of a potential V, whilst the bottom executes a simple harmonic motion. As the canal is shallow, the motion may be treated in the same way as Professor STOKES has treated the long waves in a shallow canal, of which the bottom is stationary. In this method it appears that the particles of water, which are at any time in a vertical column, remain so throughout the whole motion.



Suppose, then, that $x+\xi=x'$ is the abscissa of a vertical line of particles PQ, which, when undisturbed, had an abscissa x.

Let η be the ordinate of the surface corresponding to the abscissa x'.

Let pq be a neighbouring line of particles, which when undisturbed were distant from PQ a small length k.

Conceive a slice of water cut off by planes through PQ, pq perpendicular to the length of the canal, of which the breadth is b. Then the volume of this slice is $b\times\mathrm{PQ}\times\mathrm{N}n$.

$$\text{Now } \mathrm{PQ}=h-\mathrm{E}\cos\left[m(x'-vt)+\epsilon\right]-\eta,$$

$$\text{and } \mathrm{N}n=k\left(1+\frac{d\xi}{dx}\right).$$

Hence treating E and η as small compared with h, the volume of the slice is—

$$bhk\left\{1+\frac{d\xi}{dx}-\frac{\mathrm{E}}{h}\cos\left[m(x'-vt)+\epsilon\right]-\frac{\eta}{h}\right\}.$$

But this same slice, in its undisturbed condition, had a volume bhk. Therefore the equation of continuity is

$$\eta=h\frac{d\xi}{dx}-\mathrm{E}\cos\left[m(x'-vt)+\epsilon\right].$$

Now the hydrodynamical equation of motion is approximately

$$\frac{dp}{dx'}=\frac{d\mathrm{V}}{dx'}-\frac{d^2\xi}{dt^2}$$

The difference of the pressures on the two sides of the slice $PQqp$ at any depth is $\mathrm{N}n\times\frac{dp}{dx'}$; and this only depends on the difference of the depressions of the wave-surface below the axis of x on the two sides of the slice, viz. at P and p. Thus $\frac{dp}{dx'}=-g\frac{d\eta}{dx'}$

Substituting then for η from the equation of continuity, and observing that $\frac{d^2\xi}{dxdx'}$ is very nearly the same as $\frac{d^2\xi}{dx^2}$, we have as the equation of wave motion,

$$gh\frac{d^2\xi}{dx^2}+mg\,\mathrm{E}\sin\left[m(x'-vt)+\epsilon\right]=-\frac{d\mathrm{V}}{dx'}+\frac{d^2\xi}{dt^2}$$

But

$$\frac{d\mathrm{V}}{dx'}=-m\,\mathrm{F}\sin\left[m(x'-vt)+\epsilon\right]+m\,\mathrm{G}\cos\left[m(x'-vt)+\epsilon\right].$$

So that

$$\frac{d^2\xi}{dt^2}=gh\frac{d^2\xi}{dx^2}+m\{\mathrm{G}\cos\left[m(x'-vt)+\epsilon\right]-(\mathrm{F}-\mathrm{E}g)\sin\left[m(x'-vt)+\epsilon\right]\}.$$

In obtaining the integral of this equation, we may omit the terms which are independent of G, F, E, because they only indicate free waves, which may be supposed not to exist.

The approximation will also be sufficiently close, if x be written for x' on the right hand side.

Assume, then, that

$$\xi=\mathrm{A}\cos\left[m(x-vt)+\epsilon\right]+\mathrm{B}\sin\left[m(x-vt)+\epsilon\right].$$

By substitution in the equation of motion we find

$$-m^2(v^2-gh)\{\mathrm{A}\cos+\mathrm{B}\sin\}=m\{\mathrm{G}\cos-(\mathrm{F}-\mathrm{E}g)\sin\}.$$

And as this must hold for all times and places,

$$\mathrm{A}=-\frac{\mathrm{G}}{m(v^2-gh)}=\frac{-\frac{1}{2}a\tau\sin\epsilon}{2(a^2\omega^2-gh)}$$

$$\mathrm{B}=\frac{\mathrm{F}-\mathrm{E}g}{m(v^2-gh)}=\frac{a(\frac{\tau}{2}\cos\epsilon-\frac{2}{5}g\mathrm{E})}{2(a^2\omega^2-gh)}$$

In the case of such seas as exist in the earth, the tide-wave travels faster than the free-wave, so that $a^2\omega^2$ is greater than gh; and the denominators of A and B are positive.

We have then—

$$\xi=\frac{a}{2(a^2\omega^2-gh)}\left\{\left(\frac{\tau}{2}\cos\epsilon-\frac{2}{5}g\mathrm{E}\right)\sin-\frac{\tau}{2}\sin\epsilon\cos\right\}$$

But the present object is to find the motion of the wave-surface relatively to the bottom of the canal, for this will give the tide relatively to the dry land. Now the height of the wave relatively to the bottom is

$$\begin{aligned}\mathrm{PQ}&=h-\mathrm{E}\cos[m(x-vt)+\epsilon]-\eta\\&=h-h\frac{d\xi}{dx}\end{aligned}$$

And

$$\frac{d\xi}{dx}=\frac{1}{a^2\omega^2-gh}\left\{\left(\frac{\tau}{2}\cos\epsilon-\frac{2}{5}g\mathrm{E}\right)\cos+\frac{\tau}{2}\sin\epsilon\sin\right\}$$

Hence reverting to the sphere, and putting a for $a+h$, we get as the equation to the relative spheroid of which the wave-surface in the equatorial canal forms part—

$$r=a-\frac{h\sin^2\theta}{a^2\omega^2-gh}\left\{\frac{\tau}{2}\cos 2(\phi-\omega t)-\frac{2}{5}g\mathrm{E}\cos[2(\phi-\omega t)+\epsilon]\right\}$$

But according to the equilibrium theory, if V has the same form as above, viz.—

$$g\frac{a^2}{r}+\frac{3}{5}g\left(\frac{r}{a}\right)^2\mathrm{E}\sin^2\theta\cos[2(\phi-\omega t)+\epsilon]+\frac{\tau}{2}\left(\frac{r}{a}\right)^2\sin^2\theta\cos 2(\phi-\omega t)$$

and if $r=a+u$ be the equation to the tidal spheroid, we have, as in Part I.,

$$u=\frac{\sin^2\theta}{g}\left\{\frac{\tau}{2}\cos 2(\phi-\omega t)+\frac{3}{5}g\mathrm{E}\cos[2(\phi-\omega t)+\epsilon]\right.$$

and the equation to the relative tidal spheroid is

$$\begin{aligned}r&=a+u-\sigma\\&=a+\frac{\sin^2\theta}{g}\left\{\frac{\tau}{2}\cos 2(\phi-\omega t)-\frac{2}{5}g\mathrm{E}\cos[2(\phi-\omega t)+\epsilon]\right\}\end{aligned}$$

Now in either the case of the dynamical theory or of the equilibrium theory, if E be put equal to zero, we get the equations to the tidal spheroid on a rigid nucleus. A comparison, then, of the above equations shows at once that both the reduction of tide and the acceleration of phase are the same in one theory as in the other. But where the one gives high water, the other gives low water. The result is applicable to any kind of supposed yielding of the earth's mass; and in the special case of viscosity, the table of results for the fortnightly tide at the end of Part I. is applicable.

III.

SUMMARY AND CONCLUSIONS.

In § 1 an analogy is shown between problems about the state of strain of incompressible elastic solids, and the flow of incompressible viscous fluids, when inertia is neglected; so that the solutions of the one class of problems may be made applicable to the other. Sir W. THOMSON's problem of the bodily tides of an elastic sphere is then adapted so as to give the bodily tides of a viscous spheroid. The adaptation is rendered somewhat complex by the necessity of introducing the effects of the mutual gravitation of the parts of the spheroid.

The solution is only applicable where the disturbing potential is capable of expansion as a series of solid harmonics, and it appears that each harmonic term in the potential then acts as though all the others did not exist; in consequence of this it is only necessary to consider a typical term in the potential.

In § 3 an equation is found which gives the form of the free surface of the spheroid at any time, under the action of any disturbing potential, which satisfies the condition of expansibility. By putting the disturbing potential equal to zero, the law is found which governs the subsidence of inequalities on the surface of the spheroid, under the influence of mutual gravitation alone. If the form of the surface be expressed as a series of surface harmonics, it appears that any harmonic diminishes in geometrical progression as the time increases in arithmetical progression, and harmonics of higher orders subside much more slowly than those of lower orders. Common sense, indeed, would tell us that wide-spread inequalities must subside much more quickly than wrinkles, but only analysis could give the law connecting the rapidity of the subsidence with the magnitude of the inequality.*

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* On this Lord RAYLEIGH remarks, that if we consider the problem in two dimensions, and imagine a number of parallel ridges, the distance between which is λ, then inertia being neglected, the elements on which the time of subsidence depends are gw (force per unit mass due to weight), v the coefficient of viscosity, and λ. Thus the time T must have the form

$$T=(gw)^x v^y \lambda^z.$$

The dimensions of gw, v, λ are respectively $ML^{-2}T^{-2}$, $ML^{-1}T^{-1}$, L; hence

$$\begin{aligned} x+y&=0 \\ -2x-y+z&=0 \\ -2x-y&=1, \end{aligned}$$

And $x=-1$, $y=1$, $z=-1$, so that T varies as $\dfrac{v}{gw\lambda}$.

If we take the case on the sphere, then when i, the order of harmonics, is great, λ compares with $\dfrac{a}{i}$; so that T varies as $\dfrac{vi}{gwa}$.

I hope at some future time to try whether it will not be possible to throw some light on the formation of parallel mountain chains and the direction of faults, by means of this equation. Probably the best way of doing this will be to transform the surface harmonics, which occur here, into BESSEL'S functions.

In § 4 the rate is considered at which a spheroid would adjust itself to a new form of equilibrium, when its axis of rotation had separated from that of figure; and the law is established which was assumed in a previous paper.*

In § 5 I pass to the case where the disturbing potential is a solid harmonic of the second degree, multiplied by a simple time harmonic. This is the case to be considered for the problem of a tidally distorted spheroid. A remarkably simple law is found connecting the viscosity, the height of tide, and the amount of lagging of tide; it is shown that if v be the speed of the tide, and if $\tan\epsilon$ varies jointly as the coefficient of viscosity and v, then the height of bodily tide is equal to that of the equilibrium tide of a perfectly fluid spheroid multiplied by $\cos\epsilon$, and the tide lags by a time equal to $\frac{\epsilon}{v}$.

It is then shown (§ 6) that in the equilibrium theory the *ocean* tides on the yielding nucleus will be equal in height to the ocean tides on a rigid nucleus multiplied by $\sin\epsilon$, and that there will be an acceleration of the time of high water equal to $\frac{\pi}{2v}-\frac{\epsilon}{v}$.

The tables in § 7 give the results of the application of the preceding theories to the lunar semidiurnal and fortnightly tides for various degrees of viscosity. A comparison of the numbers in the first columns with the viscosity of pitch at near the freezing temperature (viz., about $1{\cdot}3\times 10^8$, as found by me), when it is hard, apparently solid and brittle, shows how enormously stiff the earth must be to resist the tidally deforming influence of the moon. For unless the viscosity were very much larger than that of pitch, the viscous sphere would comport itself sensibly like a perfect fluid, and the ocean tides would be quite insignificant. It follows, therefore, that no very considerable portion of the interior of the earth can even distantly approach the fluid state.

This does not, however, seem to be conclusive against the existence of bodily tides in the earth of the kind here considered; for although (as remarked by Sir W. THOMSON) a very great hydrostatic pressure probably has a tendency to impart rigidity to a substance, yet the very high temperature which must exist in the earth at a small depth would tend to induce a sort of viscosity—at least if we judge by the behaviour of materials at the earth's surface.

In § 8 the theory of the tides of an imperfectly elastic spheroid is developed. The kind of imperfection of elasticity considered is where the forces requisite to maintain the body in any strained configuration diminish in geometrical progression as the time increases in arithmetical progression. There can be no doubt that all bodies *do* possess an imperfection in their elasticity of this general nature, but the exact law here assumed has not, as far as I am aware, any experimental justification; its adoption was rather due to mathematical necessities than to any other reason.

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* Phil. Trans., Vol. 167, Part I., sec. 5 of my paper.

It would, of course, have been much more interesting if it had been possible to represent more exactly the mechanical properties of solid matter. One of the most important of these is that form of resistance to relative displacement, to which the term "plasticity" has been specially appropriated. This form of resistance is such that there is a change in the law of resistance to the relative motion of the parts, when the forces tending to cause flow have reached a certain definite intensity. This idea was founded, I believe, by MM. TRESCA and ST. VÉNANT on a long course of experiments on the punching and squeezing of metals;[*] and they speak of a solid being reduced to the state of fluidity by stresses of a given magnitude. This theory introduces a discontinuity, since it has to be determined what parts of the body are reduced to the state of fluidity and what are not. But apart from this difficulty, there is another one which is almost insuperable, in the fact that the differential equations of flow are non-linear.

The hope of introducing this form of resistance must be abandoned, and the investigation must be confined to the inclusion of those two other continuous laws of resistance to relative displacement—elasticity and viscosity.

As above stated, the law of elastico-viscosity assumed in this paper has not got an experimental foundation. Indeed, KOHLRAUSCH's experiments on glass[†] show that the elasticity degrades rapidly at first, and that it tends to attain a final condition, from which it does not seem to vary for an almost indefinite time. But glass is one of the most perfectly elastic substances known, and, by the light of TRESCA's experiments, it seems probable that experiments with lead would have brought out very different results. It seems, moreover, hardly reasonable to suppose that the materials of the earth possess much mechanical similarity with glass. Notwithstanding all these objections, I think, for my part, that the results of this investigation of the tides of an ideal elastico-viscous sphere are worthy of attention.

There are two constants which determine the nature of this ideal solid: first, the coefficient of rigidity, at the instant immediately after the body has been placed in its strained configuration; and secondly, "the modulus of the time of relaxation of rigidity," which is the time in which the force requisite to retain the body in its strained configuration has fallen away to ·368 of its initial value.

In this section it is shown that the equations of flow of this incompressible elastico-viscous body have the same mathematical form as those for a purely viscous body; so that the solutions already attained are easily adapted to the new hypothesis.

The only case where the problem is completely worked out, is when the disturbing potential has the form appropriate to the tidal problem. The laws of reduction of bodily tide, of its lagging, of the reduction of ocean tide, and of its acceleration, are somewhat more complex than in the case of pure viscosity; and the reader is referred to § 8 for the statement of those laws. It is also shown that by appropriate choice of the values of the two constants, the solutions may be either made to give the results of the problem for a purely viscous sphere, or for a purely elastic one.

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[*] "Sur l'écoulement des Corps Solides," Mém. des Savants Étrangers, tom. xviii. and tom. xx., p. 75 and p. 137. See also 'Comptes Rendus,' tom. 66, 68, and Liouville's Journ., 2^{me} série, xiii., p. 379, and xvi., p. 308, for papers on this subject.

[†] POGGENDORFF Ann., vol. 119, p. 337.

The tables give the results, for the semidiurnal and fortnightly tides, of this theory for spheroids which have the rigidity of glass or of iron—the two cases considered by Sir W. Thomson. As it is only possible to judge of the amount of bodily tide by the reduction of the ocean tide, I have not given the heights and retardations of the bodily tide.

It appears that if the time of relaxation of rigidity is about one quarter of the tidal period, then the reduction of ocean tide does not differ much from what it would be if the spheroid were perfectly elastic. The amount of tidal acceleration still, however, remains considerable. A like observation may be made with respect to the acceleration of tide in the case of pure viscosity approaching rigidity: and this leads me to think that one of the most promising ways of detecting such tides in the earth would be by the determination of the periods of maximum and minimum in a tide of long period, such as the fortnightly in a high latitude.

In § 10 it is shown that the effects of inertia, which had been neglected in finding the laws of the tidal movements, cannot be such as to materially affect the accuracy of the results.

[[*] The hypothesis of a viscous or imperfectly elastic nature for the matter of the earth would be rendered extremely improbable, if the ellipticity of an equatorial section of the earth were not very small. An ellipsoidal figure with three unequal axes, even if theoretically one of equilibrium, could not continue to subsist very long, because it is a form of greater potential energy than the oblate spheroidal form, which is also a figure of equilibrium.

Now, according to the results of geodesy, which until very recently have been generally accepted as the most accurate—namely, those of Colonel A. R. Clarke[†]—there is a difference of 6,378 feet between the major and minor equatorial radii, and the meridian of the major axis is 15° 34′ E. of Greenwich.

The heterogeneity of the earth would have to be very great to permit so large a deviation from the oblate spheroidal shape to be either permanent, or to subside with extreme slowness. But since this paper was read, Colonel Clarke has published a revision of his results, founded on new data;[‡] and he now finds the difference between the equatorial radii to be only 1,524 feet, whilst the meridian of the greatest axis is 8° 15′ west. This exhibits a change of meridian of 24°, and a reduction of equatorial ellipticity to about one quarter of the formerly-received value. Moreover, the new value of the polar axis is about 1,000 feet larger than the old one.

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[*] The part within brackets [ ] was added in November, 1878, in consequence of a conversation with Sir W. Thomson.

[†] Quoted in Thomson and Tait, Nat. Phil., sec. 797.

[‡] Phil. Mag., August, 1878.

Colonel CLARKE himself obviously regards the ellipsoidal form of the equator as doubtful. Thus there is at all events no proved result of geodesy opposed to the present hypothesis concerning the constitution of the earth. Sir W. THOMSON remarks in a letter to me that "we may look to further geodetic observations and revisals of such calculations as those of Colonel CLARKE for verification or disproof of your viscous theory."]

In the first part of the paper the equilibrium theory is used in discussing the question of ocean tides; in the second part I consider what would be the tides in a shallow equatorial canal running round the equator, if the nucleus yielded tidally at the same time. The reasons for undertaking this investigation are given at the beginning of that part. In § 11 it is shown that the height of tide relatively to the nucleus bears the same proportion to the height of tide on a rigid nucleus as in the equilibrium theory, and the alteration of phase is also the same; but where the one theory gives high water the other gives low water.

The chief practical result of this paper may be summed up by saying that it is strongly confirmatory of the view that the earth has a very great effective rigidity. But its chief value is that it forms a necessary first chapter to the investigation of the precession of imperfectly elastic spheroids, which will be considered in a future paper.* I shall there, as I believe, be able to show, by an entirely different argument, that the bodily tides in the earth are probably exceedingly small at the present time.

APPENDIX.

November 7, 1878.

On the observed height and phase of the fortnightly oceanic tide.

In the following note I attempt to carry out the suggestion concerning the fortnightly tide made in the preceding paper.

The reports of the Tidal Committee of the British Association for 1872 and 1876 contain the reductions of the tidal observations at a number of stations, into a series of harmonic tides, corresponding to the theoretical harmonic constituents of the tide-generating forces of the moon and sun. The tide with which we are here concerned is the fortnightly declinational tide.

The heights of the tides at various times are all expressed in the form R cos $(nt-\epsilon)$, where R is half the range of the tide in English feet, n the "speed" of the tide, and ϵ the retardation of phase, so that $\epsilon \div n$ is the "lag" of the tide.

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* Read before the Royal Society on December 19th, 1878.

With the notation of the present paper $n=2\Omega$ for the fortnightly tide, and Ωt is the "mean moon's" longitude from her node.

The following are the results, giving the place of observation, its N. latitude, and the years of observations. With respect to Brest and Toulon, R is reduced to feet from centimètres, so as to be made comparable with the other results :—

	Ramsgate, about 51° 21′.	Liverpool, 53° 40′.				Hartlepool, 54° 41′.		
	1864.	1857–58.	1858-59.	1859–60.	1866–67.	1858-59.	1859–60.	1860–61.
R	·0331	·093	·037	·024	·036	·052	·053	·073
ϵ	268°·29	170°·7	148°·8	72°·9	340°·6	190°·34	222°·34	158°·62

	Brest, 48° 23′.	Toulon, 43° 7′.	Kurrachee, 24° 53′.			Cat Island, Gulf of Mexico, 30° 23′.
	1875.	1853.	1868–69.	1869–70.	1870 71.	—
R	·099	·051	·038	·064	·035	·043
ϵ	80°·65	139°·50	335°·40	333°·91	283°·22	136°·69

In their present form the observations do not appear to present any semblance of law, but when they are rearranged we shall be able to form some idea as to whether they are really quite valueless or not for the point under consideration.

The theoretical expression for the fortnightly tide of an ocean covering the whole earth, according to the equilibrium theory, is

$$\frac{3}{10}\frac{\tau}{\mathfrak{g}}a\sin^2 i(\tfrac{1}{3}-\cos^2\theta)\cos 2\Omega t.$$

Where $\tau=\frac{3m}{2c^3}$, $\mathfrak{g}=\frac{2g}{5a}$, $a=$ earth's radius, i the average obliquity of the earth's axis to the normal to the plane of the lunar orbit during the fortnight in question, θ the colatitude of the place of observation.

If we take $i=23°\ 28'$ the obliquity of the ecliptic, $a=20\cdot9$ million feet, we find

$$\frac{3}{10}\frac{\tau}{\mathfrak{g}}a\sin^2 i=\cdot207 \text{ foot.}$$

So that the fortnightly tide should be expressible by

$$\cdot207\ (\tfrac{1}{3}-\sin^2(\text{lat.}))\cos 2\Omega t.$$

In THOMSON's corrected equilibrium theory the second factor should be

$$\frac{1+\mathfrak{E}}{3}-\sin^2(\text{lat.})$$

where $\mathfrak{C}$ is a certain definite integral, depending on the distribution of land and water, but which has not yet been evaluated.

The latitude of evanescent fortnightly tide is 36° 15′ if $\mathfrak{C}$ is zero; and if we bear in mind that $\mathfrak{C}$ may be negative, it is clear that the observations at Cat Island (lat. 30° 23′) are made too near the critical latitude to be trustworthy for determining the true fortnightly tide. It is also hardly possible to believe that the observations at Toulon should show a true tide of this denomination, because the Mediterranean must be regarded as a virtually closed sea. The observations at Cat Island, and at Toulon, will therefore be set aside.

The first process to be applied to the above observations is obviously to divide each value of R by $\frac{1}{3}-\sin^2$ (lat.); the following are the factors for reducing the values of R:—

$[\frac{1}{3}-\sin^2(\text{lat.})]^{-1}$

Ramsgate.	3·62
Liverpool.	3·17
Hartlepool.	3·01
Brest.	3·07
Kurrachee.	6·40

These factors will be applied to the values of R in the table first given.

The next point to consider is the phase of the tide. The formula we have given shows that the fortnightly tide consists in an alternate deformation of the ocean level into an oblate and prolate spheroid of revolution, when the tide is deemed to be superposed on a true sphere, instead of on an oblate nucleus.

When t is zero the spheroid is oblate, and this may be called high-tide; when $t=\frac{\pi}{2\Omega}$ it may be called low-tide. It follows, therefore, that N. of lat. 36° 15′ high-*tide* is low-*water*, and *vice-versâ;* but S. of this latitude the tide and water agree. But the formulæ in the tidal reductions always refer to high-*water*, hence to find the retardation of the *tide* we must subtract 180° from all the ϵ's for places N. of 36° 15′—that is to say (Cat Island being rejected) for all except Kurrachee.

For Kurrachee, we may observe that any retardation ϵ may be regarded as a retardation $\epsilon-2\pi$, which, if negative, is an acceleration of tide. If $2\pi-\epsilon$ be less than 180°, this appears to be the more correct light in which to look at it.

Now if we reduce all the observations in the way indicated, so that the fortnightly tide is given by $R'(\frac{1}{3}-\cos^2\theta)\cos(2\Omega t-\eta)$, we find the following results:—

	Ramsgate.	Liverpool.				Hartlepool.		
R'	·120	·295	·117	·076	·114	·157	·160	·220
η	+88°·29	−9°·3	−31°·2	−107°·1	+160°·6	+10°·34	+42°·34	−21°·38

	Brest.	Kurrachee.		
R'	·304	·243	·410	·224
η	−99°·35	−24°·60	−26°·09	−76°·78

We will consider R' first.

From these twelve values we find $R' = {\cdot}203$, with a probable error $\pm{\cdot}068$.

The value of R' is almost exactly that indicated by theory (viz., ·207), but the very large probable error renders the result so uncertain, that it can only be asserted that the results do not disprove a diminution of fortnightly tide.

With regard to phase, it will be observed that there are eight cases of accelerated tide to four of retarded. Two of the retarded tides refer to Hartlepool, and concerning this station Sir W. Thomson says in the report: "There is scarcely sufficient agreement between the results deduced from the long-period tides to be satisfactory, although the quantities of some are within reasonable limits."

It may be remarked, in passing, that Cat Island gives a retarded tide, and Toulon an accelerated one.

If we treat these alterations of phase in the same way as R' was treated, we find a mean acceleration of phase of 7°·85, but with a *p.e.* several times larger than the result itself. But, in fact, with so few and such irregular observations the method of least squares is useless.

The cases of retarded phase certainly show considerably more irregularity than those of accelerated phase. If we take the mean of the cases with accelerated phase, we shall find an acceleration of 48°, which corresponds in time to an acceleration of 1 day 20 hours.

Now three out of four years of observations show an accelerated tide at Liverpool; all three years show an acceleration at Kurrachee; the Hartlepool observations are not of very much value; while the single year of acceleration at Brest may be set off against the single year of retardation at Ramsgate. If then we ask ourselves whether acceleration or retardation is the more probable, I think it must be answered in favour of acceleration; and if so there seem to be some indications of a viscous yielding of the earth's mass. It must be admitted, however, that the evidence is exceedingly uncertain.

It does not seem to be noticed in the tidal reports, that amongst the "Helmholtz compound shallow-water tides" there will be found several which have the same period, or very nearly the same period, as the true fortnightly declinational tide. If we write (as in the report) γ, σ for earth's rotaton and moon's mean motion, then we shall find the following speeds will combine so as to give shallow-water tides indistinguishable from the true fortnightly tide, namely, $2(\gamma-\sigma)$ and 2γ, also $\gamma-2\sigma$ and γ; and besides there are four combinations of the elliptic tides which give the same period of compound tide, if we neglect the motion of the moon's perigee. It therefore seems quite possible that in certain stations the true fortnightly tide may be masked, or have its phase largely affected by these compound tides, and this is, perhaps, the explanation of the great irregularity in the phases.

A series of observations in some oceanic island near the Equator, or better still far north or south, would be of immense value to decide this point.

II. *On the Spectra of Metalloids—Spectrum of Oxygen.*

By ARTHUR SCHUSTER, *Ph.D., F.R.A.S.*

Communicated by J. CLERK MAXWELL, *F.R.S., Professor of Experimental Physics in the University of Cambridge.*

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[PLATE 1.]

1. *Introductory.*

THE many unexplained phenomena attending the passage of electricity through gases will probably for some time to come occupy the attention of experimental physicists. It is desirable that the subject should be approached from as many different sides as possible. One of our most powerful instruments of research is the spectroscope; but before it can be applied to the study in question, we have to settle the chemical origin of the different spectra which we observe in tubes, and to discuss in what way such spectra are liable to change under different circumstances. A special investigation has to be made for each gas; we have to study the effect of various impurities, the influence of the electrodes and that of the glass, which in the tubes generally used is considerably heated up by the spark. To make the investigation complete we have to vary as much as possible the pressure, the bore of the vacuum tube, and the strength of the spark.

I have chosen Oxygen as a first subject of investigation. Though PLÜCKER and WÜLLNER have, as far as their experiments went, accurately described the phenomena seen in oxygen tubes, the following paper contains much that is new, and will put some of the older facts on a firmer basis. When I first began to work, it was my intention to take the gases in groups, and to study their mixture; but as the following investigation has taken me a year's nearly continuous work, and is complete in itself, I trust it will not be found unworthy of publication. I must, of course, at present confine myself to the purely spectroscopic point of view. As several of the observations which I shall have to record bear directly on the general theory of double spectra, I must briefly refer to our knowledge on that point.

2. *Multiple Spectra.*

We may roughly divide all known spectra into three kinds or orders: continuous spectra, line spectra, and, standing between them, spectra of fluted bands or channelled spaces, as they are commonly called.

Continuous spectra.—During the first years of spectrum analysis it was generally supposed that a continuous spectrum was characteristic either of a solid or of a liquid body; but as our experimental knowledge gradually increased, it was found that the fact of a body being in its solid, liquid, or gaseous state only indirectly influenced the order of the spectrum. It was believed at first that a chemical compound always showed the spectrum of its individual molecules; in other words, that the vibrations of the molecules were made up of the vibration of the component atoms. If we accept this theory, then, of course, we have only one explanation of the different orders of spectra, viz., the influence of one molecule on another. According to this view, the state of aggregation of the body must be the direct cause of the change in the spectrum. Since the work of MITCHERLICH, however, it has gradually been recognised that a combination of atoms changes entirely the spectrum. This change is not slight, but fundamental. The whole vibrating system seems to have been altered; and there is at present, in the vast majority of cases, no hope that we may consider the vibrations of a molecule to be only the slightly disturbed vibrations of the atoms. We have now a second cause for a change in the different orders of spectra. We need no more look on a continuous spectrum as a line spectrum, disturbed by the shocks of other molecules; but we may explain it by the changes which have taken place in the individual molecule.

According to this view, liquid and solid bodies give generally continuous spectra, not because they are liquid or solid, but because in these states the molecules have a more complicated structure than in the gaseous state. Experiment has to decide between the two theories: the theory of molecular disturbance, and the theory of molecular structure. I think that the facts are decidedly in favour of the latter theory. Mr. LOCKYER's investigations have shown that most bodies give us a continuous spectrum, as a gas, before they condense, and many at a considerable temperature above the boiling point. Mr. LOCKYER has rightly drawn the conclusion from these facts, that the atomic aggregation of the molecules is the cause of the different orders of spectra. If we observe the changes in a spectrum which gradually take place on heating or cooling a vapour, we find that the continuous spectrum is produced, not by a widening of the bands, but by a direct replacement, which is sometimes sudden and sometimes gradual, and which leaves no doubt in the observer's mind that he has to deal with two vibrating systems, and not simply with a disturbed one. I do not, of course, mean to say that the impacts of other molecules have no observable influence. If the hydrogen lines widen through increased pressure, it is very likely that the alteration is produced by impacts; but the change from a line spectrum to a continuous spectrum, as a rule, is quite different from the change which takes place with hydrogen. According to the theory of molecular aggregation, it seems quite possible that a liquid should give the same spectrum as its vapour, and this, indeed, seems to be true in some cases.

I shall show that we can obtain a continuous spectrum of oxygen, not by an increase, but by a decrease, in the strength of the spark. I shall prove that, at the lowest temperature at which oxygen is luminous, it gives a continuous spectrum, which disappears as the temperature is increased. It is not easy to see how this can be due to any other cause than the one which I have advocated.*

Channelled space spectra.—Standing between line spectra and continuous spectra, we have, as I already mentioned, spectra of fluted bands or channelled spaces. Such spectra, when seen in spectroscopes of small dispersive powers, appear, as a rule, made up of bands, which have a sharp boundary on one side, and gradually fade away on the other. When seen with a more perfect instrument, each band seems to be made up of a number of lines of nearly equal intensity, which gradually come nearer and nearer together as we approach the sharp edge, and finally fuse together. The sharp edge is the limit toward which the lines gradually approach. This edge is sometimes towards the red and sometimes towards the blue end of the spectrum.

There is another class of channelled space spectra, which, though never accurately described or measured, is, as I have reason to believe, by no means uncommon. These spectra appear, in instruments of low dispersive power, as made up of bands covering a greater or smaller space, and do not gradually fade away on either side, but seem to have the same intensity throughout, with a sudden fall at both edges. Such are the bands which appear at the negative pole of oxygen tubes. With high dispersion I was able to resolve these bands into a series of lines. Their appearance, under these circumstances, is represented in Plate 1, figs. 1 and 2.

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* Since the above was written, Professor STOKES has suggested to me a way by which the continuous spectrum might be produced, and which differs from the one described above as that of molecular impacts.

We have reason to believe that the mere motion of matter through the ether is insufficient to produce vibrations. There must be two portions of matter exerting forces on each other in order that the ether should be thrown into agitation. In ordinary line spectra we consider that the two portions of matter form part of the same molecule. Now, it seems possible that also two portions of different molecules should in their rapid approach towards each other, or recession from each other, cause forces in the ether which produce vibration. These latter vibrations we might expect not to take place in fixed periods, but to produce what we call a continuous spectrum. We may suppose that at the lowest temperature at which, for instance, oxygen is luminous, the vibrations in the ether are chiefly produced by this rapid relative motion of different molecules, while at higher temperatures the relative motions of different portions of one molecule might have the upper hand; the continuous spectrum in one case, and the line spectrum in the other, might thus be explained.

I admit that there is much to be said in favour of this view, and must leave to further experimentation to decide whether it is the one most accordant with facts or not. That the discontinuous spectra of different orders (line and band spectra) are due to different molecular combination, I consider to be pretty well established, and analogy has led me (and Mr. LOCKYER before me) to explain the continuous spectra by the same cause; for the change of the continuous spectrum to the line or band spectrum takes place in exactly the same way as the change of spectra of different orders into each other. Analogy is not a strong guide, yet some weight may be given to it in a case like the one under discussion, where experiment hitherto has failed to give a decided answer. Professor STOKES has also suggested that the continuous spectrum which comes out under high pressures may really be only the continuous spectrum of low temperatures. At such pressures the discharge of an induction coil consists of a bright and sensibly instantaneous spark, followed by an apparently continuous discharge, for which the spark appears to open a path; and it is possible that the line spectrum may really belong to the spark, and the continuous background to the continuous discharge. Professor STOKES has suggested an experiment with a rotating mirror to decide the point: an experiment which I hope to be able to make before long.—Note added December 1, 1878.

It seems remarkable that most bodies should, under certain conditions, give spectra which in character resemble each other so nearly as these channelled space spectra; and the idea seems natural, and almost necessary, that such a similarity of spectra should be produced by a similarity in molecular structure. We know that certain bodies giving line spectra show, when they combine with others, spectra of fluted bands; and this seems to me to lead to a necessary explanation of double spectra—the one which indeed has been adopted by most experimentalists. They hold that when a spectrum suddenly changes from a line spectrum to a band spectrum, we have to deal with a molecular combination either of the molecules of the same body or of different bodies.

Line spectra.—The appearance of these spectra is too well known to need any description. Until lately, line spectra have kept out of the way of the many difficulties which beset their brethren of different orders, but at last they could no more resist the invasion. It gradually appeared that lines which are prominent at one temperature may be comparatively weak at another. This fact is sometimes exaggerated to such a degree that all the lines which form the spectrum of lower temperature disappear without a remnant at higher temperatures, while the lines of higher temperature are not to be seen at lower temperatures. Oxygen, which is a typical gas for every difficulty in spectroscopy, is a case in point. Oxygen seems to have two perfectly distinct line spectra, each of which generally appears by itself and unmixed with a trace of the other spectrum.

Let us for a moment consider how we may explain these facts without having recourse to molecular combinations. We have to assume that, when the vibrations of a molecule are small, they take place in a series of periods, and consequently show lines in the spectroscope at places corresponding to these periods. Now imagine the temperature to be raised. We must assume that, as the intensity of vibration increases, other periods come into play which rapidly increase in intensity, while those which were prominent at first gradually disappear, so that, after a while, a different set of vibrations is formed. I do not know that there is any fact which could be adduced against such an explanation, nor do I think that the explanation is theoretically impossible. Yet the passage of one line spectrum into the other resembles so very closely the passage of a band spectrum into that of a line spectrum, that one feels inclined to refer both phenomena to the same cause. In the case of oxygen, for instance, one line spectrum appears under precisely the same circumstances as the band spectrum of nitrogen, and it changes into the other line spectrum almost identically in the same way as the band spectrum of nitrogen changes into the line spectrum.

The homogeneousness of the phenomena seen in vacuum tubes is greatly increased, if we consider the one line spectrum of oxygen as the representative of the band spectrum of nitrogen, and I shall therefore consider it as such. I have called the spectrum the compound line spectrum of oxygen, because, according to the theory of molecular combination, it would be due to a more complicated molecular structure than the other (the elementary) line spectrum. I must, however, add that I do not at all feel certain that the two line spectra are necessarily to be explained the same way as the line and band spectra. I have found it necessary, for the sake of clearness, to explain the results of the investigation in the language of a definite theory. It will be easy for any one to separate the two. The fact that nearly all lines which make their appearance at low temperatures only are lines which widen easily seems to be suggestive.

Spectra of the negative pole.—It has long been known that in many gases the glow surrounding the negative electrode has a different colour from the remainder of the discharge. This colour appears generally more blue to the eye than that of the positive discharge, and this fact was the cause that a certain blueness was often associated with the idea of negative electricity. When examined with the spectroscope, it is found that the negative glow has a spectrum of its own, different, of course, in different gases. Here, also, I think, it will be found simpler to assume that a definite molecular combination is formed at the negative pole than to suppose that exterior forces peculiar to the pole modify the period of vibration. The following experiments seem to me to support strongly, if not to prove, this view. When the pressure is very small, the spectrum of the negative pole is seen throughout the tube.* In vacuum tubes of the form usually employed for purposes of spectrum analysis, it is easy to obtain a state of exhaustion in which they show the spectrum of the negative pole throughout that half of the tube which contains the negative electrode (which I shall call the negative half), and also in the capillary part, but not in that half of the tube which contains the positive electrode (which I shall call the positive half). The experiments are supposed to be made at that stage.

Experiment 1.—Allow the spark to pass some time in one direction; the spectrum of the negative pole is seen in the negative half and in the capillary part only. Now reverse the current suddenly, and the spectrum will now be seen at first throughout the tube, but will gradually disappear in the positive half, so that finally the same permanent state as before is established, and the spectrum is only seen in the negative half and capillary part.

Experiment 2.—In order to show that it is the reversal, and not the interruption of the current which produces the appearance of the spectrum in the positive half, break the current and make contact the same way: no change can be observed.

Experiment 3.—But if, after the spark has passed some time, you break the contact and wait a little time, say one minute, so as to allow any compounds which might have been formed in the negative half to diffuse into the other half, and then make contact either the same way as before, or the opposite way, the spectrum of the negative pole will be seen throughout the tube at first, and only gradually disappear in the positive half.

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* GOLDSTEIN, 'Berliner Monatsberichte,' May, 1876.

Experiment 4.—Make and break contact very rapidly, always changing the direction of the current; after a little time, when the effect of the first make has passed away, the permanent state will at once be established for each current in either direction, so that in this case the spectrum characteristic of the negative glow is always only seen surrounding the negative electrode; but I believe that the glow itself in this case is not so much developed.

Experiment 5.—After alternating the current quickly, as in Experiment 4, you wait a little while, and then make contact either way; the spectrum of the negative pole will be at first seen throughout the tube, and gradually die away in the positive half.

These experiments succeed as well in nitrogen as in oxygen. Experiments 4 and 5 weaken somewhat the argument. I have obtained, however, other results on suddenly reversing the current, which require further investigation. Without pretending to be able to give a complete explanation of the experiments, it seems to be proved by them that the spectrum of the negative pole may appear away from the pole, and must, therefore, be due to secondary causes. I have proved in another place that a mere difference in temperature cannot account for the production of the spectrum at the negative pole.* I cannot help quoting, in this place, a passage taken out of FARADAY'S 'Experimental Researches.' After having referred to an explanation which makes the so-called unipolarity in the conduction of soap and similar bodies depend on secondary chemical phenomena, which take place at the poles, he says :—

"Probably the effects which have been called effects of unipolarity, and the peculiar differences of the positive and negative surface when discharging into air, gases or other dielectrics, which have been already referred to, may have considerable relation to each other."†

3. *Description of Apparatus.*

In an investigation of the spectra of pure gases it is essentially necessary to avoid all traces of carbon impurities. Indiarubber tubings, as well as any joint involving the introduction of grease, paraffin, &c., cannot be used. I do not say that such joints are always detrimental to the correct experiment, but if the question has to be decided whether a certain spectrum is due to an impurity or not, all excuse for the presence of the impurity must be avoided. These precautions are absolutely necessary if a pure spectrum of oxygen is desired. I have examined a great number of tubes supposed to be filled with pure oxygen by various makers, but I have only seen one tube which did not show the spectrum of carbonic oxide. This spectrum has been described by various observers as a spectrum of oxygen ; but Ångström, Wüllner, and Salet have recognised it as identical with the spectrum of carbonic oxide. By taking the necessary precautions I have succeeded in filling a great number of tubes, which show no trace of the presence of any carbon compound. I have used a Sprengel pump, with air trap, such as is sold by Mr. J. J. Hicks. The connexion between the pump and the vacuum tube is made by means of a ball and socket-joint. The fall tube of the pump ends at its upper extremity in a hemispherical socket. A hollow sphere is ground together with this socket, so as to fit perfectly into it. The inner part of the sphere communicates with the pump by means of a small hole, and ends at the other side in a glass tube. I have always fused my vacuum tubes directly to this tube. For ordinary purposes the sphere is slightly covered with grease and fitted into the socket. A small quantity of mercury poured on the top will then make the joint perfectly tight. I have nearly throughout the investigation avoided even this trace of grease. Mercury alone will run through the joint as the air is exhausted, but I find that strong sulphuric acid, although running through the joint, will do so only very slowly ; so that if care be taken to fill up the sulphuric acid from time to time, the joint will keep perfectly tight. Such a joint allows great freedom of motion, which is of great convenience, as the spectroscopic examination is always best carried on while the tube is attached to the pump.

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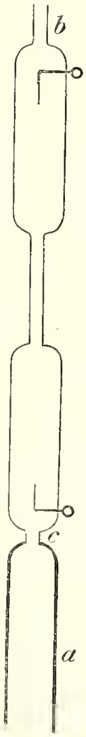
* Proceedings of the Cambridge Phil. Soc., vol. iii., p. 57.

† 'Experimental Researches,' Series XIII., 1636.

Fig. 1.

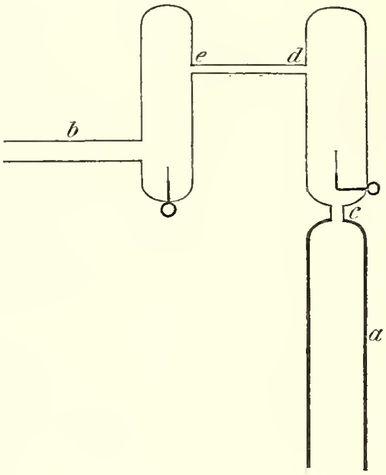


The vacuum tubes used were, as a rule, of the form given in the accompanying woodcut (fig. 1); that is, they had the shape of an ordinary Plücker tube, to one end of which a tube of hard glass (*a*) was fused. On the other side it ended in a tube (*b*), which was fused to the air pump. The hard-glass tube was filled with some substance which gave off oxygen on heating. Permanganate of potash, chlorate of potash, and oxide of mercury were used, so as to eliminate any lines which might be due to the substance from which the oxygen was prepared. It is necessary to subject all these substances to a preliminary heating, in order to free them beforehand, as much as possible, from moisture, which always is attached to them. I find permanganate of potash to be the most convenient substance, and I have only used the others in test experiments. I have used capillary tubes of different bores, and I think that some of the contradictory statements made by different observers are simply due to the fact that the term "capillary tube" is rather vague, and comprises tubes of diameters sufficiently different to affect the spectrum. The dimensions of two kinds of tubes chiefly used are as follows :—

	A	B
Distance between the poles	14	8
Length of capillary tube	4	3·5
Diameter of capillary tube	0·07	0·14

The measures are given in centimetres. Tubes with the dimensions given under the heading A were generally employed. I have sometimes suppressed the capillary tube altogether, and I think I may say I have observed the spectrum of oxygen under as many different conditions as it is in our power to produce, at least for all pressures less than one atmosphere. When the light is weak, fig. 2 is a convenient form of tube; *e d* is the capillary part, and the collimator of the spectroscope is placed parallel to it, so as to receive the light from the whole length of the tube. These tubes were first used by M. VAN MONCKHOVEN.

FIG. 2.



The following was the usual way of preparing tubes for experiment:—A plug of clean asbestos was introduced into the tube *c*, separating the hard-glass tube *a* from the PLÜCKER'S tube. This plug prevents any solid substance from being bodily carried into the PLÜCKER'S tube. As the asbestos had been subjected beforehand to a considerable temperature, no gas could be given out by it; but to prevent any possibility of error, some tubes were filled without the asbestos, and showed not the least difference which could be traced to the asbestos. Permanganate of potash, which had been heated in a test tube until the crystals were disintegrated and fell into powder, was next introduced into the tube *a*, which was then drawn out at the blow-pipe. The tube *b* was then fused to the sphere of the ball and socket-joint. The joint was rendered air-tight by means of sulphuric acid, and the tube was ready for exhaustion. There was now no danger of any carbon compounds, except such as were unavoidably attached to the glass tube; and I found, indeed, that the tube had sometimes to be filled and exhausted ten times before the spectrum of pure oxygen appeared. It is not quite easy to get rid of all traces of moisture, but by continued exhaustion it is not impossible.

One of the difficulties in the investigation of the spectrum of oxygen consists in the oxidation of the electrodes. It is necessary to work while the tube is attached to the pump, so that it can always be refilled. If this is not done, the oxygen combines with the electrodes, and a vacuum is soon established which does not conduct electricity.

I have eliminated the effects of the electrodes by using different metals, such as aluminium, platinum, silver, brass, and iridium. Only in case of aluminium did I observe lines which could be traced to the electrodes. When the vacuum is very good a spectrum is sometimes faintly seen, which can be easily obtained from aluminium, and which I believe to be due to the oxide of aluminium. The spectrum, however, is capricious in its appearance, and as aluminium electrodes were used in the great majority of experiments, it is quite possible that an effect of the electrodes of other metals exists, but escaped my notice.

Any possible effect of the glass was eliminated by using tubes of different width. I have finally seen all the lines or bands which I ascribe to oxygen in a glass receiver six inches in diameter. The electrodes were about one inch apart, and no part of the spark was within two and a half inches from the glass.

4. *Measurements.*

It is important to vary both the dispersive and magnifying power of the instrument. Some of the phenomena are best seen under small dispersion, others under large dispersion. I have examined the various spectra of oxygen with flint-glass prisms of 55° refracting angle, using almost any number from one up to seven. Whenever measurements are to be made it is best to use as large a dispersion as possible. Some of the measurements which I shall give are made with a dispersion of seven prisms. During the investigation, however, I procured from Mr. HILGER, in London, two very heavy flint-glass prisms (spec. grav. 5), of 62° each. These prisms contain much lead: they are yellow, and therefore can only be conveniently used to about the solar line G. The dispersive power of each is fully equal to that of two ordinary flint-glass prisms of the same refracting angle. Each prism under minimum deviation separated the D lines through an angular distance of 52 seconds. The focal length of the observing telescope was 17·4 inches. The micrometer screw of the eye-piece with which the measurements were made had 100 turns to the inch. The distance between the two D lines when both prisms were used was 0·88, taking as unit one turn of the screw. Most of the measurements were made with these prisms.

During the spring of 1872 I published an investigation on the spectrum of nitrogen (Proc. Roy. Soc., xx., 484), in which I showed that when nitrogen is heated up with sodium in a vacuum tube the nitrogen bands disappear, and a line spectrum appeared in my tubes, which, when measured with the instrumental means at my disposal, was found to agree fairly well with the line spectrum of nitrogen. SALET has since shown that the cause why the bands disappeared in my experiments must be looked for in the absorption of nitrogen by sodium, under the influence of the electric spark. The lines which I saw were really due to the metallic sodium. In order to avoid any similar error, I have carried my measurements to a degree of accuracy which renders any confusion between lines of different bodies quite impossible.

No measurements are of any value unless they are given in wave-lengths. It has been suggested to use the air lines as reference lines, between which we might interpolate any known line. It seems to me that a much better choice might be made. The air lines are not spread evenly through the spectrum ; they are by no means sharp under high dispersion, and their wave-lengths have only been obtained by interpolation. I do not think that these possess the requisite degree of accuracy to be used again as reference lines. It should be our endeavour to make our measurements as homogeneous as possible, and as long as ÅNGSTRÖM'S solar maps is the standard work for wave-length measurements we should refer all our measurements to that map. THALÈN has adopted this plan in his excellent series of measurements, but the method which he used presents serious difficulties when the light to be examined is weak.

I have used as reference lines the bright lines of such metals as are present in the sun, the wave-lengths of which can therefore be directly found on ÅNGSTRÖM'S map. The lines of iron are very useful in this respect.

The interpolation formula which is usually employed is based on the fact that the difference in refractive indices of two lines varies nearly as the difference between the inverse squares of the wave-lengths. If reference lines can be found sufficiently close, the formula does pretty well ; but I find that for the dispersion which I have used, the formula is no longer applicable when the distance between the reference lines is about ten times the distance between the two sodium lines. As it is not always possible to find good reference lines within that distance, I have generally used three reference lines, and taken the inverse fourth powers of the wave-lengths into account. I am not quite certain that even then the errors of interpolation did not exceed, in some cases, slightly the errors of observation. The reason why the ordinary interpolation formula cannot be used, does not lie so much in the fact that the law of the inverse square does not hold accurately for refractive indices (although this, of course, is also true), but that, except in automatic spectroscopes, our measurements are not proportional to refractive indices.

It is difficult to say to what degree of accuracy measurements are made, yet I trust that, except for the weakest lines, the error of my measurements will be seldom found to exceed 0·5 x^{th} metre.

After these remarks I pass to the description of the various spectra of oxygen. I distinguish four such spectra. At the lowest temperature at which oxygen becomes luminous it gives a continuous spectrum. As the temperature is raised, the continuous spectrum changes successively into two line spectra, which, as I have already mentioned, I call the compound line spectrum and the elementary line spectrum. It is one of the principal objects of this paper to show that these two line spectra, which have been much confounded with each other, have a separate existence ; and that the generation of one involves the destruction of the other. The fourth spectrum is that which is seen at the negative pole.

5. *The Continuous Spectrum of Oxygen.*

Wüllner (Pogg. Ann., 137, p. 350) has shown that when the pressure in his vacuum tube is gradually increased, the background begins to be illuminated at a pressure of 280 m.m. This illumination, according to him, first extends from the solar line E to a point about half way between F and G. As the pressure was increased, the continuous spectrum extended more towards both sides, but chiefly towards the blue. I do not know whether this continuous illumination of the background is due to the same cause as the continuous spectrum, which is seen under the circumstances presently to be described. There is nothing antagonistic to known facts in the assumption that the vibration of the gas, though chiefly taking place in certain definite periods, can be disturbed by mutual impacts in such a way that a slight admixture of vibrations of all periods is produced which increases as the pressure is increased. At any rate, if this is the cause of the continuous spectrum which is seen under great pressure, it has nothing to do with the continuous spectrum which appears by a reduction of temperature, and which is not the background to bright lines, but forms the whole visible spectrum. The following facts seem to me to prove that oxygen at the lowest temperature at which it is luminous gives a continuous spectrum :—

1. The wide part of a Geissler tube filled with pure oxygen generally shines with a faint yellow light; when analysed by the prism, this yellow light gives a continuous spectrum.

2. *Experiment* 6.—A spark was passed through oxygen gas at the atmospheric pressure. The maximum striking distance of the coil was about one inch. No jar was included in the circuit. The poles were about 4 m.m. apart. A lens concentrated the image of the spark on the slit on the spectroscope. Under these circumstances the elementary line spectrum of oxygen, which is the spectrum commonly observed at atmospheric pressure, is seen. But when the break of the induction coil was put out of adjustment, so that the spark decreased considerably in strength, it took a yellow colour, and the spectrum was perfectly continuous. The phenomena were very striking when I kept the wheel of the break in my hand, while looking through the spectroscope. A slight turn would change the line spectrum into a continuous spectrum, and *vice versâ*. When the line spectrum was seen, the spark was considerably stronger and the poles became red hot. When the continuous spectrum was seen, the poles were not sufficiently heated to become luminuous. The maximum intensity of the light was in the yellowish green, and decreased on either side.

I attach some importance to this experiment, as the sudden conversion of the line spectrum into a continuous spectrum cannot I think, be explained on any other hypothesis than of molecular combination.

3. An interesting observation of Becquerel, which he describes in a foot-note ('La Lumière,' vol. i., page 196), bears on this point. He says that an excess of oxygen in the oxyhydrogen flame makes itself visible by a yellow streak of light

at the point of the flame. He compares the colour to that of the phosphorescence seen in tubes containing a mixture of oxygen with a small quantity of nitrogen (pure oxygen does not show the phosphorescence). Now the colour of this phosphorescence is exactly the same as that of the spark in oxygen which gives the continuous spectrum.

It is reasonable to suppose that M. BECQUEREL heated up the oxygen to a sufficiently high temperature in the oxyhydrogen flame to render it luminous. That this is possible appears from an experiment made by PLÜCKER, who, working with an excess of hydrogen in the oxyhydrogen flame, saw the lines of hydrogen in his spectroscope.

I have already stated that I do not know whether the continuous background which is seen at higher pressure when the line spectrum appears has its origin in a separate cause, or whether it has to be brought into connexion with the continuous spectrum of low temperature. The following experiment may assist those who like to form an opinion on the matter :—

Mr. LOCKYER has made use of the fact that when a spark is taken between metallic poles, and a lens is employed to form an image of the spark on the slit of the spectroscope, the lines of the metals do not stretch across the field, but are confined to the neighbourhood of the poles. As the pressure is decreased, some of the lines decrease rapidly in length, and finally disappear, while the effect on others is little perceptible. I find that when oxygen is the gas through which the spark strikes, the lines of the gas as well as the metallic lines seem to stretch away from the poles and fade away towards the middle. The centre of the field is taken up by the second line spectrum, which I have called the compound line spectrum. Here is what happens :—

Experiment 7.—At atmospheric pressure the lines of the elementary line spectrum were seen to stretch across the field; the continuous spectrum illuminating the background was strongest in the green. As the pressure was decreased the compound lines appeared much broadened, and at the same time a dark streak was seen to pass through the continuous spectrum in the centre of the spark. This means that at that pressure the continuous spectrum is chiefly confined to the electrodes. As the pressure was decreased to about half an atmosphere, the continuous spectrum only occupied a narrow band adjoining the electrodes. The compound lines came out strong in the centre of the field and were better defined. At still lower pressures the lines of oxygen separated in the middle of the field, as the continuous spectrum had done before; and the continuous spectrum now seemed to be gradually absorbed into the electrodes, and finally disappeared altogether. Gradually the oxygen lines became shorter and shorter until they could be no more distinguished from the metallic lines. The centre of the field was now entirely occupied by the compound line spectrum. It is necessary to add that it appears conclusively from the experiments of WÜLLNER, as well as my own, that the continuous spectrum is really due to oxygen and not to the metallic poles. The reason why there should be a difference in the spectrum of the gas in different cross sections of the spark seems to me to be that the lines of flow spread out more midway between the poles than they do at the poles, so that the temperature is not so high some distance away from the poles.

6. *The Elementary Line Spectrum.*

This is the spectrum which is seen when a strong spark passes through oxygen at atmospheric pressure. It can be seen at all pressures in vacuum tubes, when a Leyden jar and air break are introduced into the circuit. PLÜCKER gives a drawing of the lines in his paper on "Double Spectra" (Phil. Trans., 1865); he does not give, however, his measurements. HUGGINS gives a measurement of the oxygen lines as seen at atmospheric pressure; but these are referred to an arbitrary scale, and some of the weaker lines are invisible at atmospheric pressure owing to the continuous background. THALÈN gives the wave-lengths of the air lines.

When the spark is taken in air, however, the oxygen lines are weak compared to the nitrogen lines, so that even some strong oxygen lines do not appear in THALÈN'S list. In their last paper on the spectra of metalloids, Messrs. ÅNGSTRÖM and THALÈN gave a drawing of the lines of oxygen.

This drawing is pretty complete, but the numbers which have been obtained by measurement are not given separately, so that we are dependent on the accuracy of the engraver, which cannot be relied on to any great extent. As far as can be judged from the map, THALÈN'S measurements agree fairly well with mine. SALET also has given some measurements, but the small dispersion which he used did not enable him to obtain great accuracy. There are two weak lines on ÅNGSTRÖM and THALÈN'S map ($\lambda=5163$ and $\lambda=4964$) which I have never been able to see, although I have been constantly on the look-out for them. On the other hand I see a number of weak lines not included in that map. Most of them are given by SALET. Some of these lines have a different appearance from the oxygen lines; they are not so well defined, and are sometimes bands. This may lead some observers to think that they are due to impurities. Greatly struck with the difference in appearance of these lines, and the ordinary oxygen lines, I have paid some attention to them, but I find they appear with electrodes of very different metals, and cannot be due to the glass, as I have seen them in the receiver, six inches in diameter, which I have mentioned. I have finally come to the conclusion that we have no evidence for rejecting them. I have tried to represent in the spectrum in Plate 1, marked A, the relative intensities of the oxygen lines. The intensities were carefully estimated, but cannot, of course, be greatly relied on except as far as the relative intensity of a group is concerned. It seems hardly possible to compare the intensity of a yellow and a blue line. The following table contains my measurements. I have not carried them beyond the group of lines which is near the solar line G. Photography will do the work much better in the extreme violet than I could have done.

Elementary Line Spectrum of Oxygen.

Wave-length.	Intensity.	Remarks.
5205·4	6	This is the strongest line of Oxygen in the green group and well visible in the atmospheric lines. It is, however, omitted in THALÈN's map of the atmospheric lines.
5189·6	3	
5175·4	3	
5159·3	5	
4954·4	3	
4942·2	8	
4940·2	3	
4923·7	6	
4906·1	5	
4890·1	3	
4871·0	4	
4864·0	3	
4860·2	3	
4856·2	4	
4850·0 4841·6	1	This is a very weak band, but it is always present.
4750·1	1	
4740·9	1	
4709·0	6	
4704·6	10	
4698·5	8	
4695·5	1	
4675·4	8	
4673·1	1	
4660·7	8	
4649·3	4	
4648·0	12	
4640·6	7	
4637·4	6	
4608·0	2	A weak band.
4605·7	1	
4595·1	6	
4589·9	6	
4469·2	4	The space between these two lines is filled up with weak light.
4465·3	3	
4452·7	2	
4448·3	2	
4443·0	1	
4416·8	7	
4414·5	8	
4395·6	1	
4366·2	4	This line must not be mistaken for the line 0δ of the compound spectrum, of which faint traces are sometimes seen at its less refrangible edge.
4353·5	5	
4349·0	6	
4346·9	5	
4345·0	1	
4341·4	1	
4336·6	1	
4319·2	3	
4316·5	3	

7. *The Compound Line—Spectrum of Oxygen.*

In one of his early papers (Pogg. Ann., 107), Plücker gives the spectrum of oxygen as consisting of four lines, with the following wave-lengths.

6150
5328
5185
4367

In his later drawing of the spectrum of oxygen (Phil. Trans., 1865) he gives a great many lines of which these four form part. Wüllner says that these four lines are always the first to appear in oxygen tubes. Ångström and Thalèn do not give these lines; Huggins does not give them; Salet does not give them. Plücker and Wüllner are the only observers who experimented under the circumstances under which the lines come out, and they seem to assume that they form part of the ordinary line spectrum of oxygen; yet when the ordinary line spectrum is brightest, these lines are quite invisible, and when these lines are brightest, no trace of the ordinary line spectrum can be detected. I must repeat that these lines come out equally well, whatever way the oxygen is prepared. Wüllner prepared his gas by electrolysis and saw the lines; I prepared my gas, as already mentioned, in three different ways. When I began work I was under the impression that these lines must be due to impurities; but after having spent a great deal of time in trying to trace their origin, I feel certain that they are due to pure oxygen. I have seen the lines almost daily for many months, and in at least 30 different tubes.

The appearance of an oxygen tube when a spark passes while it undergoes gradual exhaustion is as follows:—

Experiment 8.—At first the spark has a yellow colour, and the spectrum is perfectly continuous. Almost immediately, however, four lines are seen in the capillary part above the continuous spectrum. One of these lines is in the red, two are in the green, and one is in the blue. The discharge still passes as a narrow spark throughout the length of the tube. In the wide part the spectrum remains continuous, and it extends more towards the red and blue than in the capillary part. It seems as if the four lines had taken away part of the energy of the continuous spectrum. As the pressure diminishes, these lines increase considerably in strength; the spark spreads out in the wide part of the tube, and the intensity of the continuous spectrum is therefore considerably diminished, while it still forms a prominent part in the spectrum of the capillary part. When the pressure is small the continuous spectrum decreases in intensity. At the same time the negative glow with its own characteristic spectrum, gradually extends through the negative half of the tube into the capillary part. The continuous spectrum has now entirely disappeared; the bands of the negative pole and the four lines stand out on a perfectly black background. It is under these conditions that the change from the compound line spectrum to the elementary line spectrum is best studied. The mere insertion of a Leyden jar, I find makes hardly a difference ; the jar does not seem to be charged at all. If in addition to the jar we insert a movable air break, which can be opened or closed at will, while we look through the spectroscope, we shall be able to see alternately two perfectly distinct spectra. If the air break is closed the four lines of the compound spectrum only are seen ; if the air break is opened these four lines will disappear entirely, and the elementary line spectrum will come out. We have here as complete a transformation as we have from the band to the line spectrum of nitrogen taking place under exactly the same circumstances. We have, therefore, the same right to consider the two line spectra of oxygen as two distinct spectra, as we have in the case of the two spectra of nitrogen.

There are two reasons why the existence of a separate line spectrum of oxygen might have escaped the notice of previous observers. The blue line of the compound line spectrum is very near a bright line of the elementary line spectrum, and cannot be distinguished from it unless a high dispersive power is used, and even then only when special attention is paid to the line.

To an observer who looks at this line with small power, it would seem as if on the introduction of the air break other lines spring up to its right and left, which are brighter than the blue line he is looking at ; but he would not notice that the blue line itself has disappeared, and has been replaced by another line a little more refrangible and of about the same brightness.

Again, in tubes of rather wide capillary bore, as that of which the dimensions have been given on page 43, under the heading B, it is impossible to get rid altogether of the lines of the compound line spectrum. The effect of the jar and air break is only to widen the lines considerably ; not to extinguish them. If the pressure in a tube is large, these lines widen to a considerable extent, and finally fuse together so as to form a continuous spectrum. This has been well described by WÜLLNER, who likened the lines in this respect to the hydrogen lines. The observations which refer to the disappearence of these lines must therefore be made under low pressure when the continuous spectrum has disappeared, and the phenomenon then is very striking. If a spark is taken through oxygen at atmospheric pressure, a strong red but flushy line is seen very nearly coincident with the red line of the compound spectrum. I confess I have not been able to find out with certainty whether the two lines are identical or not. The compound line spectrum widens considerably with increased pressure, and chiefly towards the less refrangible side. At first sight the atmospheric line is decidedly less refrangible than the line of the compound line spectrum, and the measurements by THALÈN and HUGGINS of the line in the atmosphere give a wavelength sensibly larger than that which I have obtained for the line of the compound spectrum. Yet if account be taken of a one-sided widening, so that the line has to be compared with the edge of the atmospheric band, not with its middle, I am inclined to think that the two may prove identical. It appeared to me sometimes, it is true, that the line of the compound spectrum is more refrangible even than the edge of the atmospheric band, but I am not quite certain of this. I have left out therefore for the present the line from the list of the elementary line spectrum of oxygen.

In a letter to 'Nature' (Dec. 20, 1877) I have drawn attention to the fact that the lower temperature spectrum of oxygen seems to be reversed in the sun, and I have tried to bring this fact into connexion with the statement made by Professor Draper, that the higher temperature lines of oxygen appear bright on the solar disc. I have now nothing to add to what I have stated in my letter to 'Nature,' and only give the list of the exact wave-lengths of the compound line spectrum, together with the wave-lengths of the nearest solar lines.

Oxygen.	Width.	Solar Lines.	
		(A.)	(S.)
α 6156·86	±0·3	6156·70	6156·69
β 5435·55	±0·3	5435·44	5435·56
γ 5329·41	±0·6	5329·30	5329·10
δ 4367·62		4367·58	

The first column contains the wave-lengths of the compound line spectrum of oxygen; the second column contains the number which has to be added to or subtracted from the wave-length, in order to get the edge of the lines, as it is their centres which are given in the first column; the third and fourth columns give the wave-lengths of the corresponding solar lines as observed by Ångström (A.) or myself (S.). In Plate 1, I have given a drawing of this spectrum, marked B.

8. *The Spectrum of the Negative Pole.*

This spectrum was first accurately described by Wüllner.* When looked at, as he did, with low dispersive power, it seems to consist of four distinct, not shaded bands; but, as I have already said, I was able to resolve the bands into separate lines. I have also in this case spared no trouble to show that the spectrum is really due to pure oxygen, and not either to impurities in the gas, to the nature of the electrode, or influence of the glass. It seems interesting to notice that in Experiment 6 (page 47), when the spectrum of the spark at atmospheric pressure was continuous, a purple point was seen on the negative electrode which gave the spectrum of the negative pole. The peculiar circumstances which favour the production of this spectrum always exist therefore at the negative pole, and are independent of the mode of discharge.

When examined with low power, the bands of the negative pole seem to occupy the following wave-lengths.

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* Pogg. Ann., 147, page 357.

$$\left.\begin{array}{r}6010\\5960\end{array}\right\}$$

$$\left.\begin{array}{r}5900\\5840\end{array}\right\}$$

$$\left.\begin{array}{r}5630\\5553\end{array}\right\}$$

$$\left.\begin{array}{r}5292\\5205\end{array}\right\}$$

There is another band still more towards the red end than the least refrangible of these bands, but it was too weak to be measured. Unless care be taken, the two first bands might easily be regarded as one; but even with weak instruments a dark space is seen to separate them. The following are the more detailed measures of the centres of each line making up the two most refrangible bands. The second column gives half the width of each line. The maximum of light in most of the lines is, however, not in the middle but towards the least refrangible edge of each line. Plate 1 gives a drawing of the bands when seen with large magnifying powers. The excessive weakness of the bands is an insurmountable obstacle to any very great accuracy. Thus it is possible that the five most refrangible lines of the most refrangible band (Plate 1, fig. 1), which seem somewhat irregularly placed, are really at equal intervals, but I do not consider this likely. If the light is strong, the eye can readily detect an unsymmetrical arrangement without measurement; but, with the narrow slit necessary for measurement, the light was so weak that nothing but the line, which was specially fixed by the eye, could be seen, and no general view of the band could therefore be obtained.

Least refrangible of the two green bands.			Most refrangible of the two green bands.		
5552·8	±0·5		5205·0	±0·6	
5558·4	±0·5		5213·3	±0·6	
5564·5	±0·5		5216·9	±0·6	
5570·1	±0·5		5225·3	±0·6	
5575·8	0·5		5231·2	±1·7	
5576·5	..	Limits of the maximum of light.	5239·0	±0·6	
5586·0			5247·7	±1·9	
5591·4	0·7		5255·0	±1·9	Brightest part.
5601·2	0·5		5262·7	±1·7	
5611·2	0·5		5269·5	±1·6	
5618·8	0·5		5276·9	±1·0	
5629·6	0·5		5284·4	±1·2	
			5292·5	±0·6	

The experiments were all made in the Cavendish Laboratory, Cambridge, and I am much obliged to Professor CLERK MAXWELL for the kindness with which he has placed the resources of the laboratory at my disposal.

III. *Electrodynamic Qualities of Metals.*—Part VII. *Effects of Stress on the Magnetization of Iron, Nickel, and Cobalt.*

By Professor Sir William Thomson, *F.R.S.*

Received May 9,—Read May 23, 1878.

[Plates 2–13.]

§§ 198 and 199.

198. In a preliminary notice of investigations regarding the effects of stress on inductive magnetization in soft iron, communicated to the Royal Society on the 10th of June, 1875, I described experiments which afforded a complete explanation of the seeming anomalies referred to in §§ 194 and 195,* which had at first been so perplexing. These experiments showed that the diminution of magnetism in a soft iron wire, which I had found to be produced by pull, while the wire was under the influence of a constant magnetizing force, was to be observed only when the magnetizing force exceeded a certain critical value, and that when the magnetizing force was below that critical value the effect of pull was to increase the magnetism—a result which I afterwards found had been previously obtained by Villari.† The critical value of the magnetizing force I found to be about twenty-four times the vertical component of the terrestrial magnetic force at Glasgow. Hence the magnetizing force which I had used in my first experiment, which (§ 183) was nearly 300 times the vertical component of the terrestrial force, must have been about twelve times as great as the critical value. Further (which was most puzzling), I found the absolute amount of the effects of pull to be actually greater with the small magnetizing force of the earth than that of the opposite effects of the 300-fold greater magnetizing force of my early experiments. Thus the effect of the terrestrial force was not only in the right direction, but was of amply sufficient amount to account for the seeming anomalies which had at first been so perplexing; and in going over the details of the old observations I find all the anomalies quite explained. One of them, that particularly referred to in § 195, is still interesting. The alternate augmentation of the residual magnetism by "on" and diminution of it by "off," with the weight of 14 lbs., corresponded to the normal effect on residual magnetism in soft iron. The elongation of 8 per cent. produced when the 28 lbs. was hung on, was no doubt accompanied by a shaking out of nearly all the residual magnetism, and an inductive magnetization in the opposite direction by the vertical component of

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* Phil. Trans. for 1876, p. 710 (Read May 27, 1875).

† Poggendorff's 'Annalen,' 1868; also Wiedemann's 'Galvanismus,' vol. ii., § 499.

the earth's magnetic force. The reversed effects of the "ons" and "offs," observed after this change, were really augmentations and diminutions of magnetism induced by the earth's vertical force, and were therefore the proper effects for soft iron when subject to a magnetizing force of less than the VILLARI critical value. Further experimental investigation is necessary to explain the *greater amount* of effect, the same in kind as those observed before the stretching by 28 lbs., which the wire showed after it had been stretched by this weight.

199. The experiments indicated in my preliminary notice of June 10, 1875, were the commencement of an elaborate series of investigations by Mr. ANDREW GRAY and Mr. THOMAS GRAY, which have been continued with little intermission from that time until now, and which are still in progress, with the general object of investigating the effects of longitudinal and transverse stress upon the magnetization of different qualities of iron and steel, and of nickel and cobalt. A separate series of investigations was made nearly two years ago by Mr. DONALD MACFARLANE on the effects of torsion on the magnetization of soft iron, bringing out some very remarkable results, also included in this paper (§§ 223–229, below).

§§ 200–212. *Investigation by the Ballistic* method, of the change of Magnetization produced in a specimen of exceedingly soft Iron Wire, by the application and removal of pulling force.*

200. The wire used in these experiments was specially prepared for this investigation by Messrs. RICHARD JOHNSON and Nephew, Manchester. It was of No. 22 Birmingham wire gauge; its weight per metre was 3·47 grammes, and its diameter was ·075 of a centimetre. A steel pianoforte wire of the same gauge would bear about 230 lbs. on and off, without experiencing in consequence any permanent change of quality. This iron wire was so soft that, after it was stretched by a scale-pan weighing 1 lb., which thenceforth was kept always hanging on it, an additional weight of 14 lbs. on and off gave a permanent elongation of ·4 per cent., and 4 lbs. more gave it a further permanent elongation of 1·6 per cent. (making in all 2 per cent. of permanent elongation produced by 18 lbs. on and off, the permanent weight of 1 lb. being always on). The weight of 18 lbs. was applied and removed several times on the 14th of May, without producing any more permanent elongation than the 1·6 per cent. which was observed after the first on and off. During three-quarters of a year after that day no weight of more than 14 lbs. (in addition to the permanent 1 lb.) was ever applied to the wire, and electro-magnetic experiments were made upon it from day to day, with little intermission, with "ons" and "offs" of 14 lbs., but sometimes with 7 lbs. During all this time the length of the wire (about 4 metres) remained sensibly constant for the same weight, and the wire experienced regular elastic elongations

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* Compare §§ 178, 179. Phil. Trans. for 1876, p. 693.

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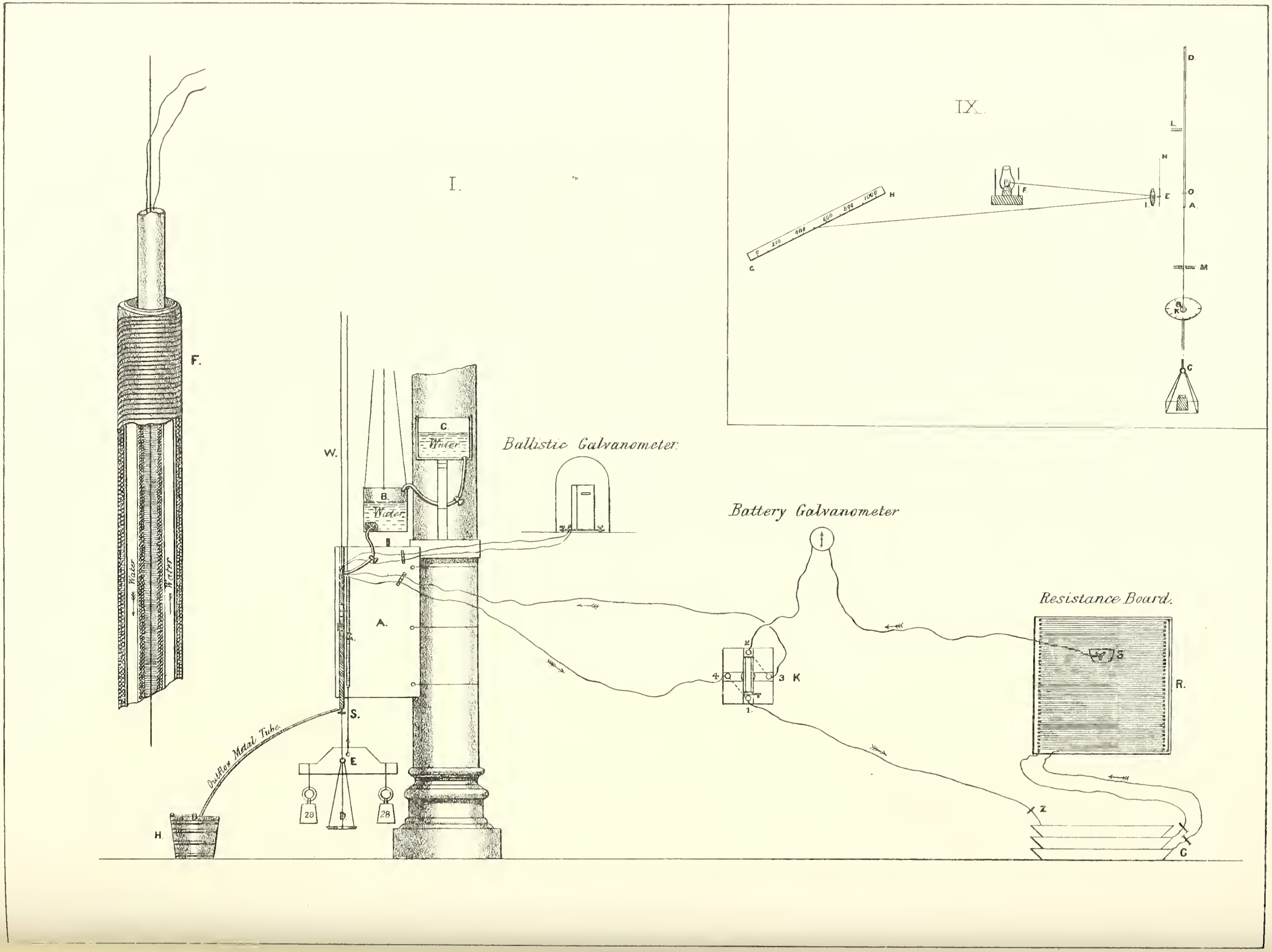

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and contractions when the weights were applied and removed. In some of the experiments with "ons" and "offs" of 14 lbs., about 90 centims. of the wire were heated to 100° C. by a stream of hot water (as described in § 201), but this left no permanent change in the wire.

201. The accompanying diagram (I.) shows the arrangement of the apparatus by means of which the results were obtained. The wire, W, experimented on was attached to a fixed support near the ceiling of the laboratory, and hung vertically downward, passing along the common axis of the magnetizing and induction coils, F, and was kept stretched by the scale-pan, D, which, as stated above, weighed exactly 1 lb. The coils, F, were supported on a clamp attached to the wire at their lower end, as shown in the drawing; and thus, as the length of the induction coil was small compared with the total length of the wire, motion of the wire relatively to the induction coil due to the stretching produced by the applied stress, was in great measure avoided. The magnetizing coil was 86 centims. long, and was composed of two layers of silk-covered copper wire. The inner layer contained 26·7 metres of wire, of No. 23 B.W.G., arranged in a solenoid* of 960 turns, the outer layer, 24·3 metres of No. 20 wire, arranged in a solenoid of 728 turns. The resistance per metre of these wires was ·0673 and ·0522 ohm respectively, and the total resistance of the coil after it was wound was 3·134 ohms. The total length of the induction coil, which was contained within the magnetizing coil, was 31·5 centims. It was made up of two layers of silk-covered copper wire, of No. 29 B.W.G., laid on in 1439 turns. The wire thus coiled on was 10 metres long, and had a resistance of ·184 ohm per metre, and the total resistance of the coil, including 1 metre of electrodes, was 2·204 ohms.

The magnetizing coil was wound on the outside of a compound tube, made up of two tubes of thin brass, of different diameters, placed one within the other with their axes coincident, the external diameter of the inner tube being less than the internal diameter of the outer by about 3 millims. The induction coil, which was wound on a thin copper tube just fitting the wire experimented on, was enclosed within the inner brass tube in such a position that its ends were at equal distances from the extremities of the magnetizing coil. The space between the two brass tubes formed a channel through which water could be made to flow from the cistern B, and thus, by regulation of the temperature of the water in B, the part of the wire within the magnetizing coil could be kept at any temperature from about 10° to nearly 100° C.

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* The common use of the word "helix" in this sense is utterly illogical. The idea of helix is not essential, but accidental, and in no practical case is it of any consequence whether it is a right-handed or a left-handed helix. There is nothing of helical quality in a cylindrical tube composed of two metals in two parts of its circumference, with the junctions of these metals kept at unequal temperatures. The thermo-electric current round the circumference of the solenoid produces the kind of magnetizing influence which is commonly produced by a helix, and constitutes precisely the arrangement which AMPÈRE called a solenoid. It is only because the ordinary helix, with electric current flowing through it, produces more or less approximately (very approximately indeed in the case of a helix with many turns) the same effect, that it is available for the electro-magnetic uses. It seems desirable, therefore, to take advantage of AMPÈRE'S original word "solenoid," and, except in cases in which the helical quality is taken into account, to give up the name "helix." The electro-magnetic solenoid may also be called a bar electro-magnet without soft iron core.

202. In order that the elongation and contractions produced in the wire by the applications and removals of the pulling stress might be observed, a second wire was hung from the same support, and kept stretched by two 28 lb. weights, hung from the ends of the cross-bar, E. To this wire a scale of half a millim. was vertically attached, in such a position that a pointer fixed to the magnetizing coil moved along it as the coil moved downward or upward with the application or removal of the weight.

203. The electrodes of the magnetizing coil were connected with the studs 3 and 4 of the commutator, K. One of the other pair of studs was connected with the zinc pole of a battery of my tray DANIELLS, the other stud with the sliding piece of a resistance-slide, R. This slide was designed for the purpose of allowing the battery strength to be raised continuously from 0 to nearly 1 cell, and from 1 cell to nearly 2 cells, and so on. It consisted of a contact-making slider, S, movable along a bare copper wire connecting the two poles of the cell to be sub-divided. This wire, which was 64 metres in length, and had a resistance of 0·67 ohm, was stretched for convenience alternately from one side to the other of a large board, in the manner represented in the diagram. Thus, with the number of cells and arrangement of connexions figured in the diagram, when the slider was brought up as nearly as possible to C, the current flowing was very nearly that due to 3 cells, and when the slider made contact at any other point of the wire, the current flowing through the magnetizing coil was less than that due to 3 cells by an amount depending on the distance along the wire of the slider from C.

204. A galvanometer, the resistance of which was only a very small fraction of an ohm, was used to measure the strength of the magnetizing current, and was so placed in the circuit that the current always flowed through it in the same direction: thus the whole range of the galvanometer scale was available for measuring the deflections produced by the stronger currents used, without the necessity for shifting the zero of the scale by means of a magnet. In these experiments at first the battery galvanometer was placed for convenience in the circuit between the commutator K and the coils F, and a reversing key used to keep the current through it always in the same direction, but it was afterwards transferred to the position in the circuit shown in the diagram and the reversing key dispensed with. The deflections of the needle of this galvanometer were read on a scale of half millimetres placed at a distance of 75 centims. from the mirror, and were used as the values of the magnetizing forces for the abscissas of the curves below. The total resistance in the circuit of the magnetizing coil, which was measured from time to time in the course of the experiments, and showed little or no variation, was 3·828 ohms. The induction coil was placed in circuit with an astatic galvanometer (described in Part VI., § 181, and called in that

paper the ballistic galvanometer), the "throw" of which, as observed on a scale placed 120 centims., or 2400 of its own divisions, from the mirror, measured the strength of the induced current. This galvanometer was placed within a case of thin sheet copper, and the whole enclosed within a glass bell jar to guard it from the effects of currents of air.

205. The procedure in experimenting was similar to that described in Part VI., § 186 (Trans. for 1876, p. 697). One observer took the readings of the ballistic galvanometer, and made and broke the circuit of the magnetizing current; while a second, on word of command from the first, applied or removed the weight, and noted the elongations or contractions of the wire, as shown by the pointer and scale described above. The results were entered in the register on the system followed in my previous experiments, according to which +M denoted that the current was made in such a direction as to cause the image on the scale of the ballistic galvanometer to move towards the right or towards increasing numbers; −M, that the current was made in the contrary direction; B, that the current was stopped; Z, the zero of the ballistic galvanometer scale; "On," the application of the weight; and "Off," its removal. The polarity of the magnetization produced in the wire by +M was the same as that shown by the wire when under the influence alone of the vertical component of the earth's magnetic force; and consequently a deflection of the image on the scale of the ballistic galvanometer towards the right, produced by a change in the magnetic condition of the wire after it had been thus magnetized, indicated an increase of its magnetization, and a deflection to the left a diminution of its magnetization. This magnetization produced by +M, I shall call positive magnetization, and that produced by −M negative magnetization. Hence a deflection of the ballistic image towards the right indicates an increase of positive magnetization, and a deflection towards the left, a diminution of positive, or an increase of negative, magnetization.

206. The conclusions of the preliminary notice of June 17, 1876, and the statements of § 198, are proved by the following table of results obtained by a very careful repetition of the experiments referred to in that notice. The first column of results in the table gives the operations in the order in which they were performed; the second, the deflections of the image on the scale of the ballistic galvanometer obtained after each operation; and the third column gives the battery strengths. During the whole of the experiments cold water was kept flowing from the cistern B through the channel between the coils to the vessel H, to prevent any heating of the coils and wire by the passage of the magnetizing current. The amount of the elongation and contraction of the wire by the application and removal of the weight of 14 lbs. was constant throughout the experiments. The experiment was begun with zero current in each case, and the weight of 14 lbs. applied and removed until equal and opposite effects were obtained by on and off; and the same process was followed after each augmentation of the magnetizing current. The table contains for each step only the results obtained after this state had been reached.

Table I.—Temperature of Cold Water.

Results for +M.			Results for −M.		
Operations.	Deflections.	Current strengths.	Operations.	Deflections.	Current strengths.
14 on	+26	0	14 on	0	11
,, off	−26		,, off	0	
,, on	+33	10	,, on	−27	22
,, off	−33		,, off	+27	
,, on	+36	22	,, on	−36	46
,, off	−36		,, off	+36	
,, on	+36·5	44	,, on	−36·5	66
,, off	−36·5		,, off	+36·5	
,, on	+33·5	67	,, on	−33·5	96
,, off	−33·5		,, off	+33·5	
,, on	+19	143	,, on	−25	145
,, off	−19		,, off	+25	
,, on	+12	180	,, on	−19	169
,, off	−12		,, off	+19	
,, on	+ 6	220	,, on	−15	195
,, off	− 6		,, off	+15	
,, on	+ 2	246	,, on	−11	223
,, off	− 2		,, off	+11	
,, on	0	267	,, on	− 4	267
,, off	0		,, off	+ 4	
,, on	− 4	310	,, on	0	290
,, off	+ 4		,, off	0	
,, on	− 7·5	352	,, on	+ 5	355
,, off	+ 7·5		,, off	− 5	
,, on	− 9	403	,, on	+ 7·5	394
,, off	+ 9		,, off	− 7·5	
,, on	−11	447	,, on	+ 9	430
,, off	+11		,, off	− 9	
,, on	−11·5	503	,, on	+13	830
,, off	+11·5		,, off	−13·	
,, on	−12	600			
,, off	+12				
,, on	−13	830			
,, off	+13				

207. We see from the above table that the vertical component of the magnetizing force of the earth is balanced by an opposite magnetizing force due to a current measured by 11 divisions of the battery galvanometer, or about $\frac{1}{25}$ of the Villari critical value of the magnetizing force, with 14 lbs. "off" and "on."

If now a Daniell's cell be taken as one volt, or 10^8 on the C. G. S. system of units, we have for the magnetizing force of 1 cell estimated in the manner explained above (Part VI., § 183), the value $4\pi\frac{1688\times10^8}{86\times3{\cdot}828\times10^9}=6{\cdot}443$. But the strength of the magnetizing current of 1 cell was measured by a deflection on the scale of the battery galvanometer of about 130 divisions, or a little more than twelve times the amount of the magnetizing force of the earth. Hence we have in absolute measure, ·5 as a rough

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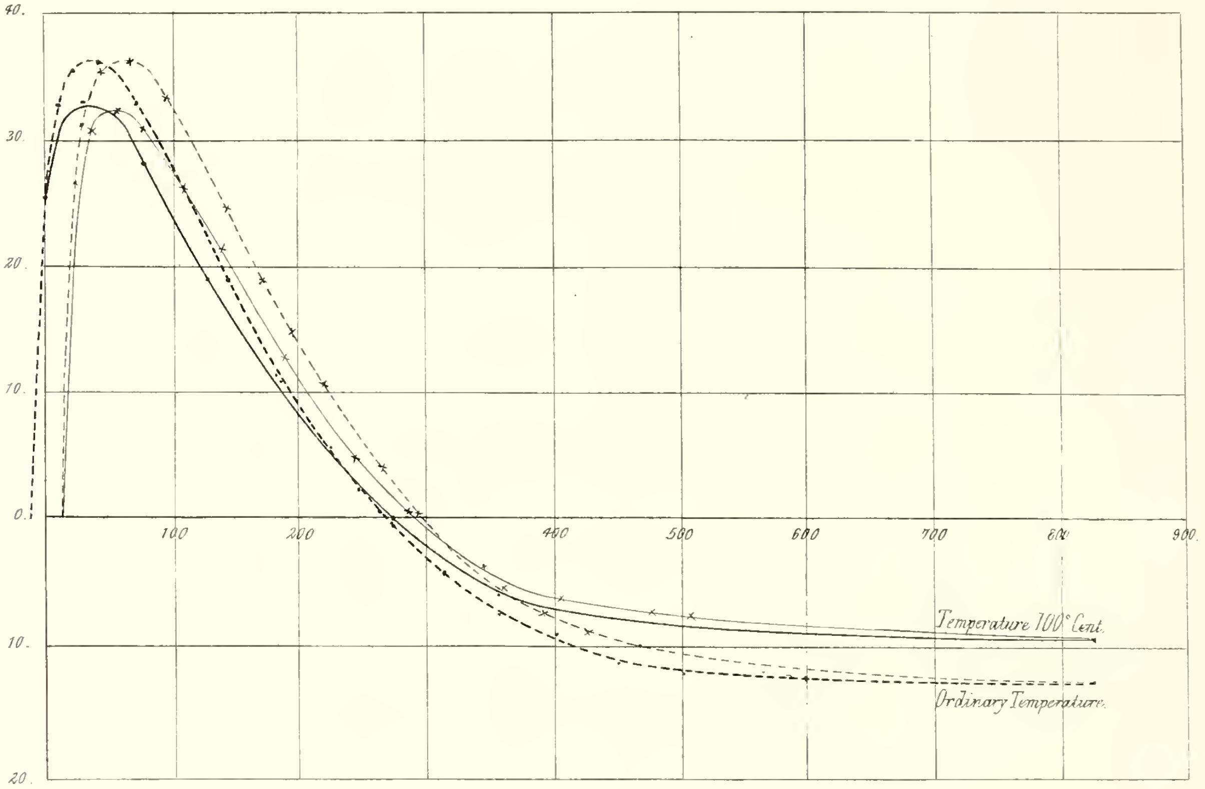
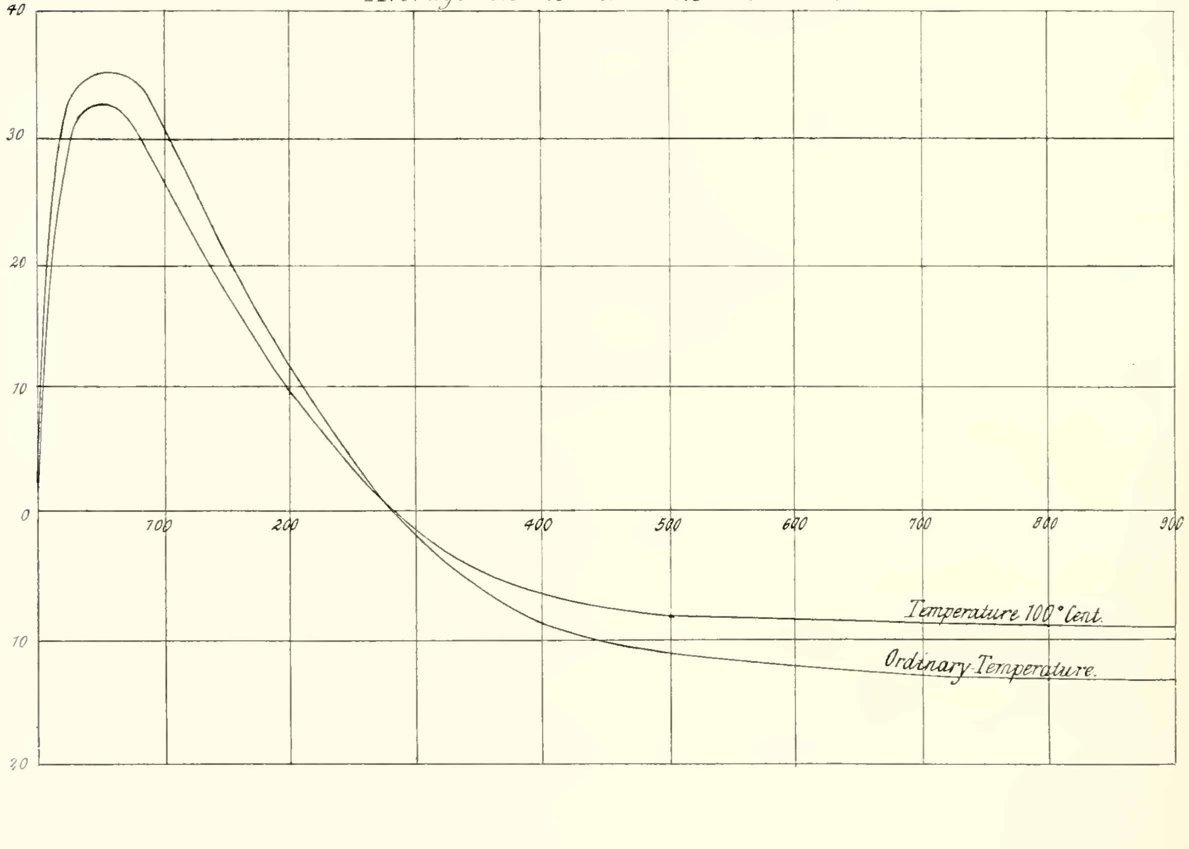
II.



III.

Average curves for 14 lbs. "on and off."



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approximation to the value of the vertical component of the earth's magnetic force at Glasgow. The true value, as said above (Part VI., § 183), is nearly ·43. (Compare also § 243 below.)

208. Immediately after the results given in Table I. were obtained, the experiments were repeated with the water in the cistern B kept constantly at the temperature of 100° C. Thus the water flowing through the channel between the coils may be considered as having been nearly at the boiling point. The results are given in Table II.

TABLE II.—Temperature 100° C.

Results for +M.			Results for −M.		
Operations.	Deflections.	Current strengths.	Operations.	Deflections.	Current strengths.
14 on	+26	0	14 on	0	11
,, off	−26		,, off	0	
,, on	+33	27	,, on	−31	34
,, off	−33		,, off	+31	
,, on	+28·5	75	,, on	−32·5	56
,, off	−28·5		,, off	+32·5	
,, on	+19	127	,, on	−31	86
,, off	−19		,, off	+31	
,, on	+11	185	,, on	−26·5	109
,, off	−11		,, off	+26·5	
,, on	+ 6	220	,, on	−22	138
,, off	− 6		,, off	+22	
,, on	0	274	,, on	−13	186
,, off	0		,, off	+13	
,, on	− 6	357	,, on	− 5	242
,, off	+ 6		,, off	+ 5	
,, on	− 8	453	,, on	− 0·75	283
,, off	+ 8		,, off	+ 0·75	
,, on	+ 9	830	,, on	+ 3	340
,, off	− 9		,, off	− 3	
			,, on	+ 6	406
			,, off	− 6	
			,, on	+ 7	478
			,, off	− 7	
			,, on	+ 7·5	508
			,, off	− 7·5	
			,, on	+ 9	830
			,, off	− 9	

209. For convenience of comparison, the results given in the above tables are represented graphically by the accompanying curves (Diagrams II. and III.), in which the abscissas represent magnetizing forces, and the ordinates augmentations and diminutions of the magnetism of the wire. In Diagram II., a separate curve is given for +M and −M, at both temperatures; the curves for the higher temperature being drawn in full lines, and those for the lower temperature in dotted lines. In their general features they are similar to the curves given in the preliminary notice above referred to; but they include a more exact determination of the amount of the magnetizing force which

gives maximum effect in each case, and of the points in which the curves cut the line of abscissas, and hence of the relation between the magnetizing force of the earth and the VILLARI critical value. The magnetizing force for which 14 lbs. "on" or "off" gave maximum effect was now found for the same wire to be about four or five times, and the VILLARI critical value about twenty-three times, the vertical component of the earth's magnetic force at Glasgow. These results also disprove the negative maximum indicated by the curves of that notice, and show (as there stated in the Note at the end) that for higher magnetizing forces than the VILLARI critical value, the effect approaches a constant amount and the curves become asymptotic.

210. The curves in Diagram III. were drawn by taking for ordinates the mean for each temperature of the effect for $+M$, and the effect for $-M$, of 14 lbs. "on" or "off," and for abscissas the corresponding current strengths, and therefore show approximately the effect which would have been produced by "on" or "off," had the wire not been affected by the magnetizing force of the earth.

By comparing the curve for cold water with the curve for hot water, we see that when the wire is at the temperature of 100° C., the average maximum effect of "on" or "off" is less than at the ordinary temperature of cold water by about 8 per cent. of the effect in the latter case, and that also, when the VILLARI critical value has been exceeded, the constant value to which the effect of "on" or "off" approaches is less for the higher temperature than for the lower, but in this case by about 30 per cent. of the amount of the effect for the lower temperature. The two curves also cross one another at a point above the line of abscissas, thus showing a greater critical value of the magnetizing force for the higher temperature than for the lower.

211. The curves of Diagrams IV. and V. give at both temperatures the average for each strength of magnetizing current of the effects on $+M$ and $-M$ of applying and removing stresses of 7 lbs. and 21 lbs. respectively.

A comparison of these curves with the average curves for 14 lbs. "on" and "off" (II. above) shows—

(1.) That the effect at both temperatures of the application and removal of the stress is greater with 14 lbs. than with 7 lbs., and much greater with 21 lbs. than with 14 lbs.; the maximums at the ordinary temperature in these three cases being respectively 31, 35, and 54.

(2.) That the VILLARI critical value is much greater for 7 lbs. "on" and "off" than for 14 lbs.: and, though by a smaller difference, greater for 21 lbs. than for 14 lbs.

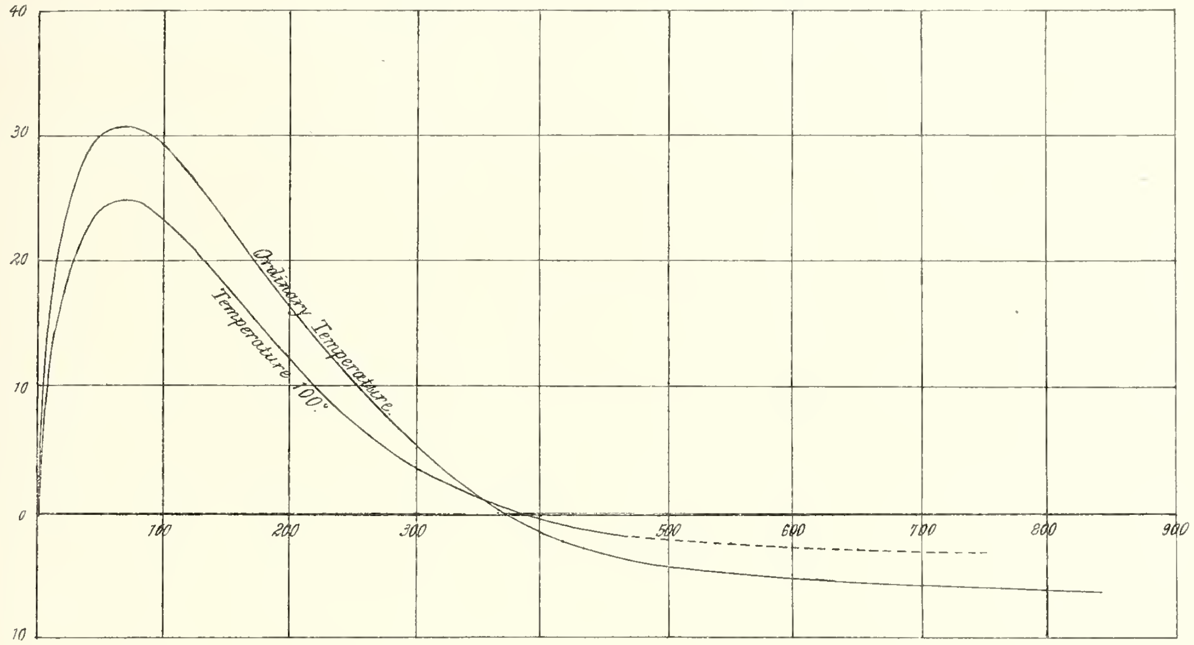
(3.) The difference between the maximum effects of "on" or "off" for the high and low temperatures is greater for 7 lbs. than for either 14 lbs. or 21 lbs., and seems to be greater for 21 lbs. than for 14 lbs.

212. A series of observations of the effects of alternately making and breaking the circuit of the magnetizing coil and battery were made at both temperatures and for both $+M$ and $-M$. The method of procedure was as follows—

With no current flowing, a weight of 14 lbs. was placed in the scale-pan, and

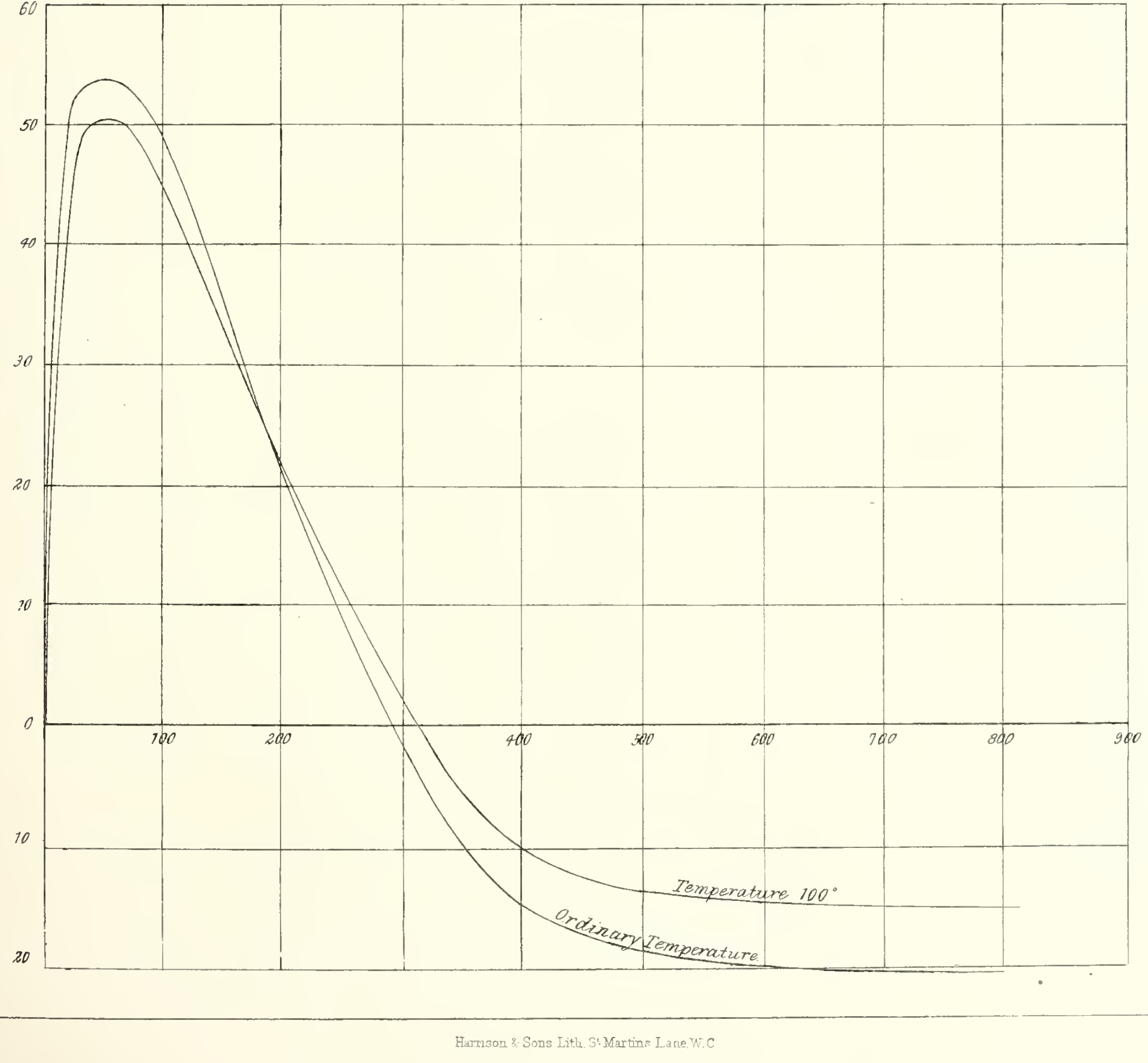
IV.

Average Curves for 7 lbs. "on and off.



V.

Average Curves for 21 lbs. "on and off.



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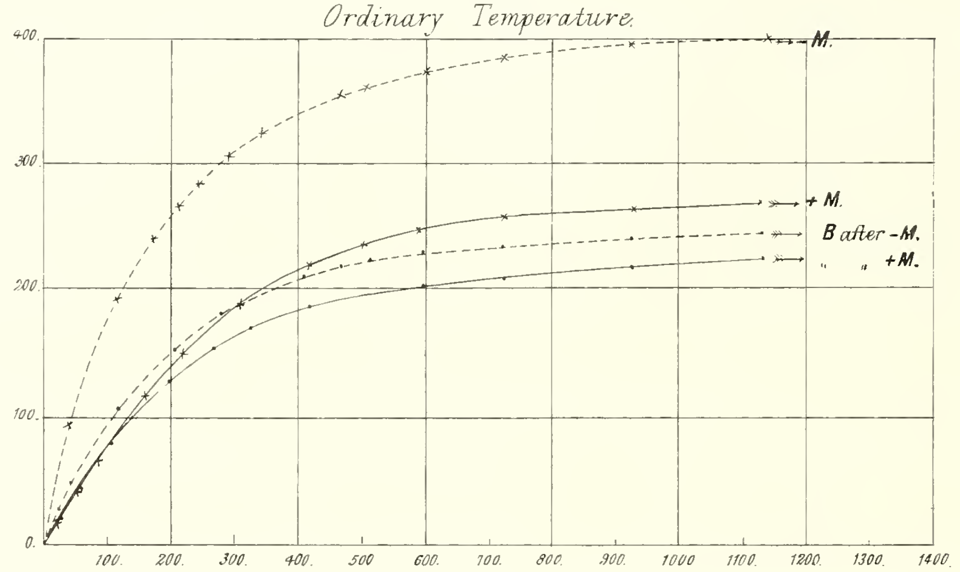
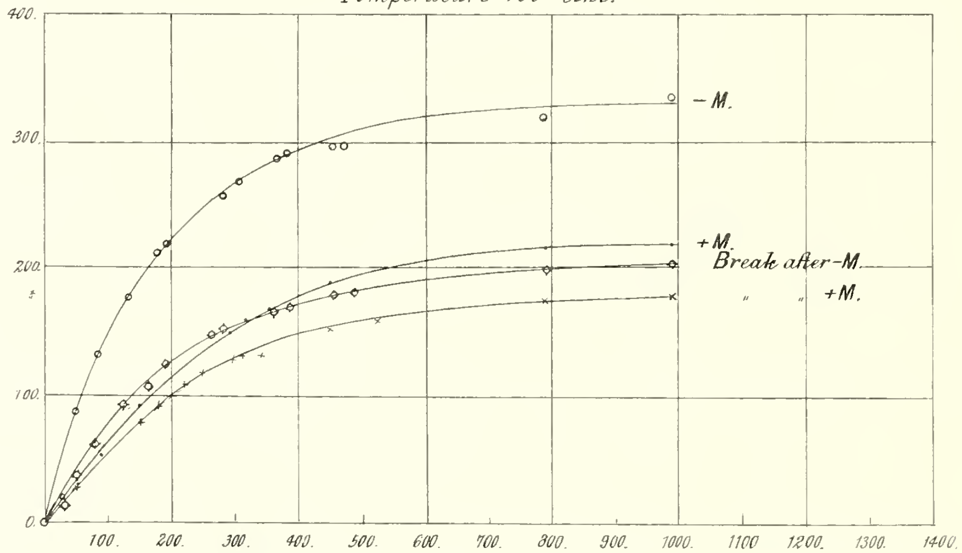
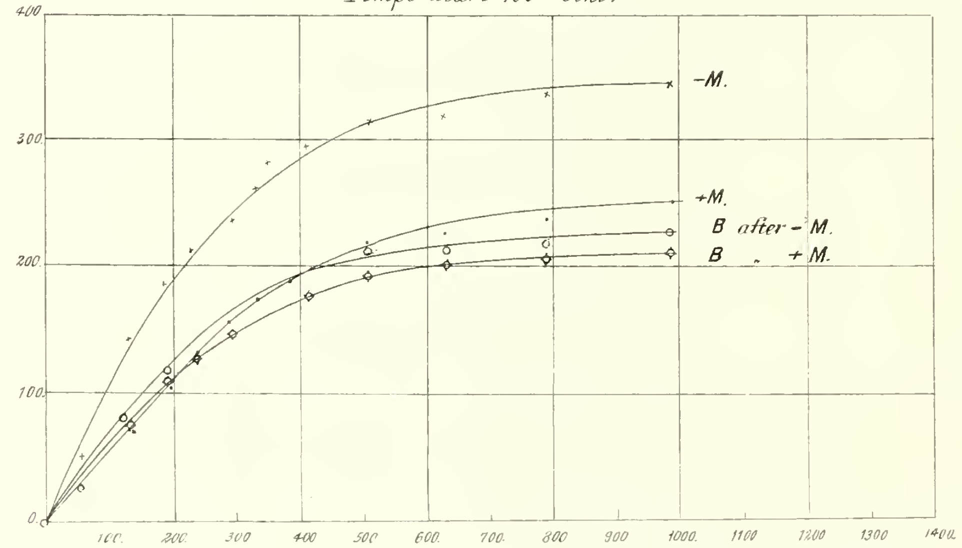
VI(1). 14 lbs.
Ordinary Temperature.



VI(1). 14 lbs.
Temperature 100° Cent.



VI(2). 21 lbs.
Temperature 100° Cent.



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removed ten times in rapid succession, and the wire finally left with only the permanent weight of 1 lb. hanging on it. Then beginning with +M, a magnetizing current of small amount was applied, and the effect measured by the "throw" of the ballistic galvanometer. The weight of 14 lbs. was then applied and removed ten times in succession, the circuit broken with nothing but the permanent weight of 1 lb. hanging on the wire, and the deflection of the ballistic galvanometer again noted. The same cycle of operations was then repeated for higher and higher strengths of current until ten cells were placed in circuit with the magnetizing coil.

The same process was then followed with the −M.

These experiments were repeated also at both temperatures with 21 lbs. as the weight applied and removed ten times before each operation. Curves (1), of Diagrams VI., exhibit the results for 14 lbs., and curves (2) those for 21 lbs.

A very striking feature of these results is the great excess of the deflection produced by −M over the deflection produced by +M. It cannot but be due to the terrestrially-induced magnetism existing in the wire each time before the current is made in either direction.

Comparing the results for the ordinary temperature with those for 100° C., we see that the effect at the higher temperature is always considerably less than at the lower temperature. Thus, taking the 21 lb. curves VI. (2), the deflection after −M with the greatest magnetizing force is 320 for the lower temperature, and about 250, or 22 per cent. less, for the higher, and for the same magnetizing force the other deflections are less at the higher temperature than at the lower, in nearly the same proportion.

213. Immediately after the results for the temperature 100° C., shown in curves (2), had been obtained, an experiment was made to determine the amount by which (as stated in the Preliminary Notice of June 10, 1875, § 7) the effects of making and breaking the circuit of the magnetizing coil and battery when the wire is pulled exceed the effects of the same operations when the wire is free from pull. The process was the same as that described in § 212, except that after each ten "ons" and "offs" the weight of 21 lbs. was put on and left in the scale-pan and the circuit made or broken before it was again removed. The experiment was made at only one temperature, 100° C. The results are given in the curves VI. (3).

A full examination of the results shown in these Diagrams VI. (1), (2), (3), must be reserved for a later communication; and further experiments will be necessary to elucidate them. Meantime it is interesting to see by comparing the curves of VI. (3) with the curves of VI. (2) for the same temperature (100°), that the effect of the −M is greater with the pulled than with the unpulled wire for every degree of magnetizing force; while the effect of the +M is greater in the pulled wire for magnetizing forces less than 250, and greater in the unpulled wire for magnetizing forces exceeding 250. This was to be expected from the previously proved (§§ 209, 210) greater magnetic susceptibility in the pulled than in the unpulled wire, when the magnetizing force is less than a critical value of 280 or 290 ; and greater susceptibility in the unpulled than in the pulled wire when the magnetizing force exceeds the critical value; and from the fact that the difference in one direction of the susceptibilities in the pulled and the unpulled wire when the magnetizing force is the Glasgow vertical force, is about three times as much as the difference in the other direction when the magnetizing force is 80 times the Glasgow vertical force. The effect of the $-M$ includes a reversal of the natural vertical force. That of the $+M$ is merely an addition to it.

§§ 214–222. *Preliminary investigation by the direct Magnetometric Method of the effects of transverse stress on the Magnetization of an Iron Tube.*

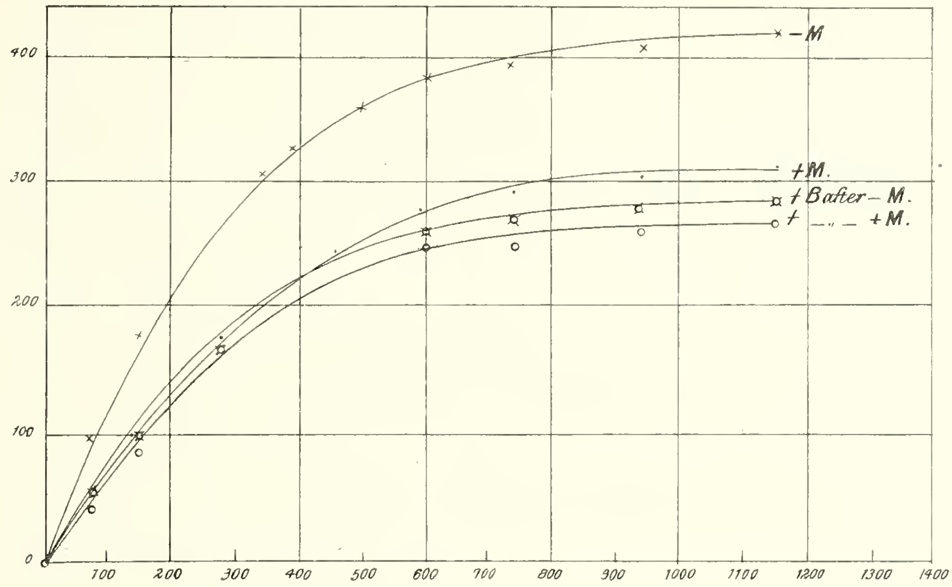
214. In order to test qualitatively, in the first instance, the effects of transverse stress on the magnetization of iron, experiments were made on a smooth gun-barrel, said to be made of tolerably soft iron. The barrel was fitted at its muzzle with a piston working watertightly in a Bramah stuffing-box, and served round with a magnetizing coil of silk-covered copper wire, separated from the barrel by a copper tube, and containing within it an induction coil, in order that the ballistic method might be used if this was deemed advisable. The barrel was then fixed rigidly to a stone pier in the laboratory, with its breech end resting on a large block of stone which formed the base of the pier. On a shelf, also attached to the pier, and at a convenient distance due magnetic west of the barrel's axis, a small reflecting magnetometer was placed, with its needle on a level with the top of the barrel. A dead-beat galvanometer, the resistance of which was only a small fraction of an ohm, was used to measure the magnetizing current. The barrel having been filled with water was subjected to hydrostatic pressure, applied by means of a lever carrying a weight, the lever itself being counterpoised by a weight attached to a cord passing over a pulley above. The pressure (the friction of the piston in its collar being neglected) was measured by the amount of the applied weight and the multiplication of the lever. The effects of the application and removal of the pressure were measured by the deflections of the galvanometer needle, read on a scale of half millimetres, placed at a distance of one metre from the mirror.

215. The first experiments after the apparatus had been got into working order brought out the remarkable result that the effects of transverse stress on the magnetization of iron are, as to quality, the opposite of those of longitudinal stress ; that is to say, when the magnetizing force is less than a certain critical value, the effect of applying transverse stress is to diminish, and of removing it to increase, the induced magnetization ; and when this critical value has been exceeded, the effect of the application of transverse stress is to increase, and of its removal to diminish the induced magnetization.

The curves (1) of Diagram VII. show (after the manner of Diagram II. above) the effect for both $+M$ and $-M$ on the magnetometer needle, when placed on a level

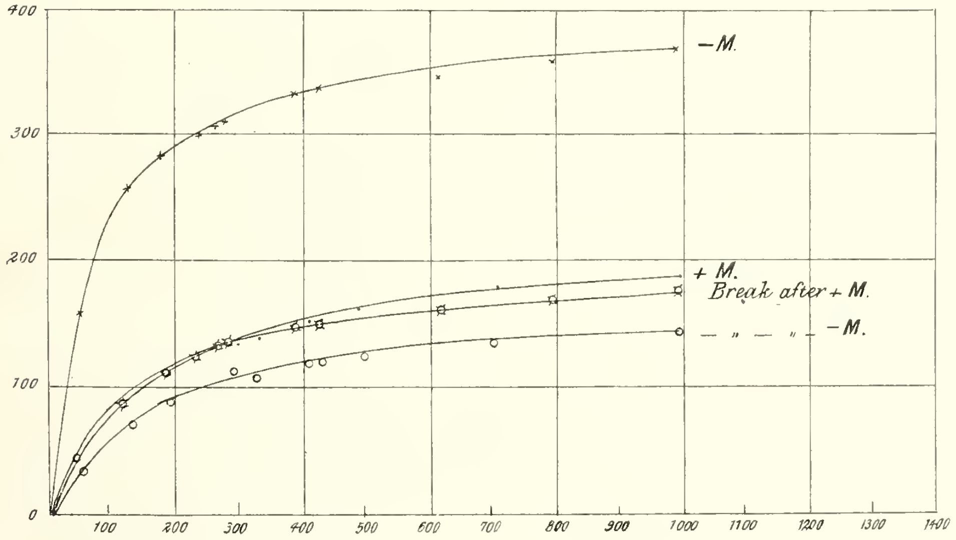
VI (2). 21 lbs.

Ordinary Temperature.



(3)

Temperature 100° Cent. M & B with 21 lbs on.



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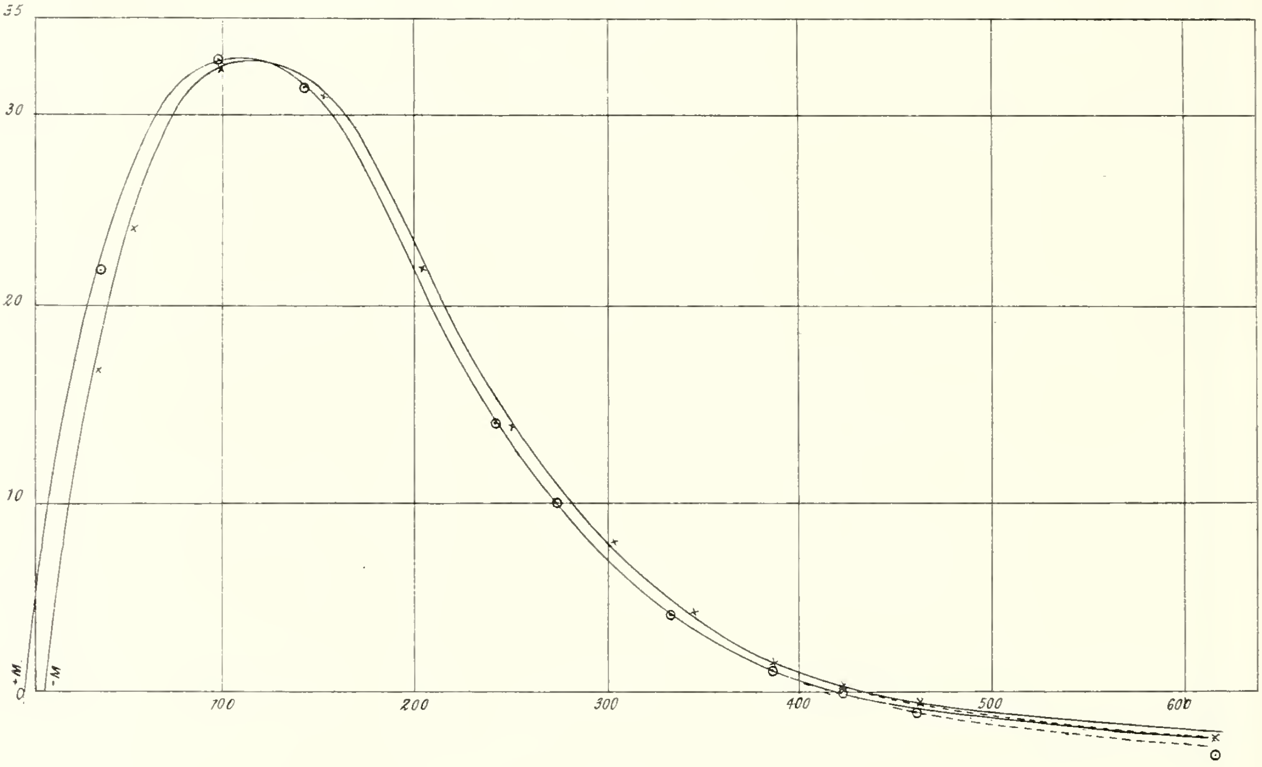
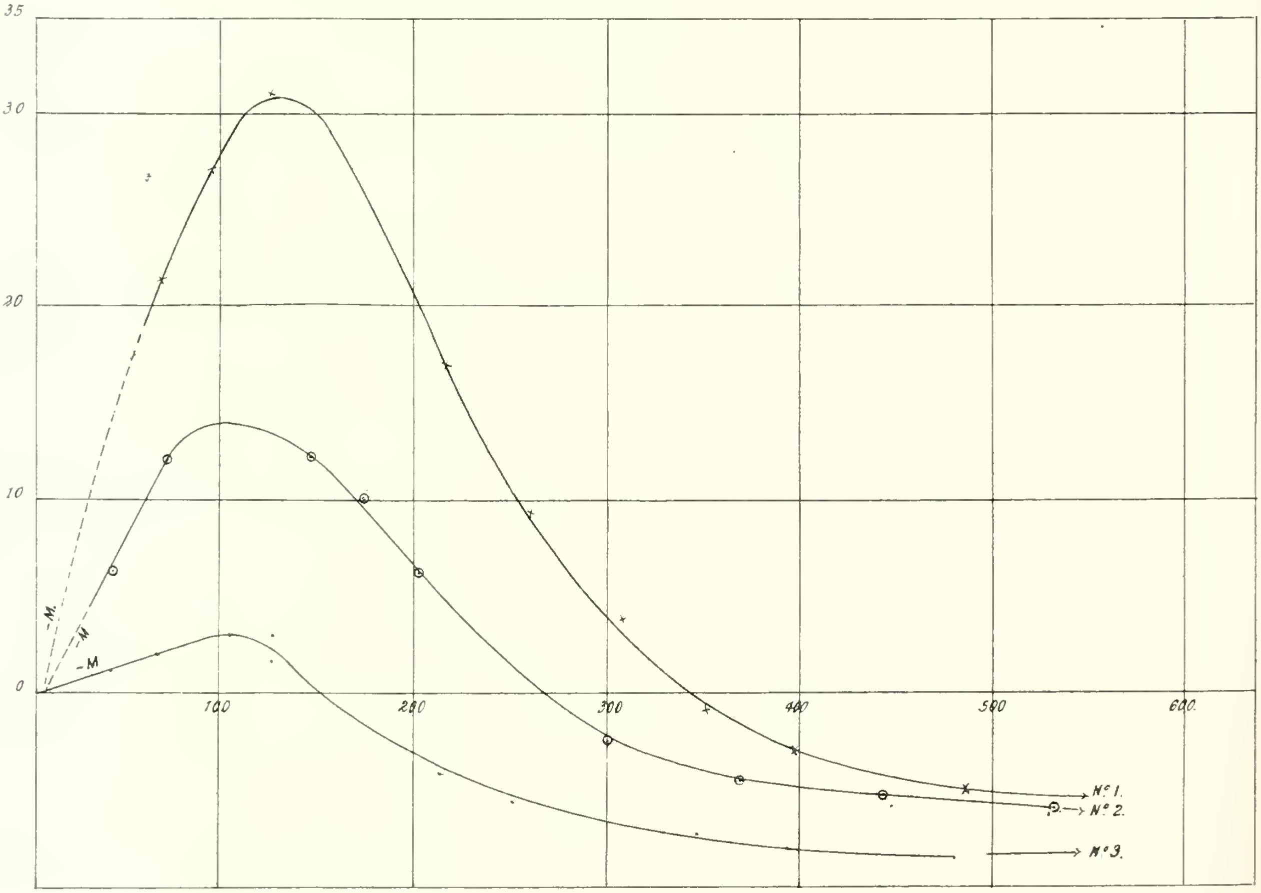
VII.

(1)



(2)



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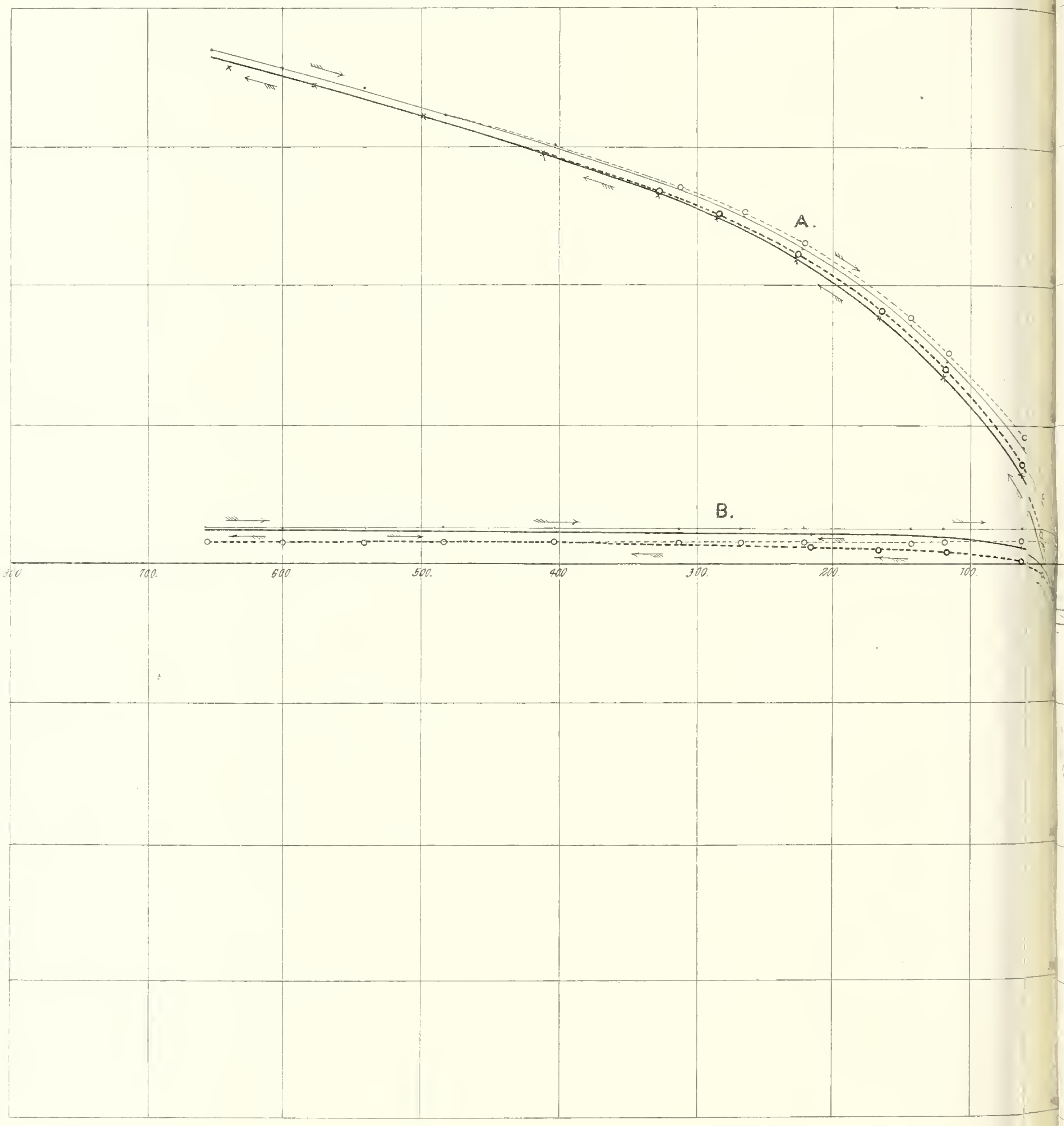
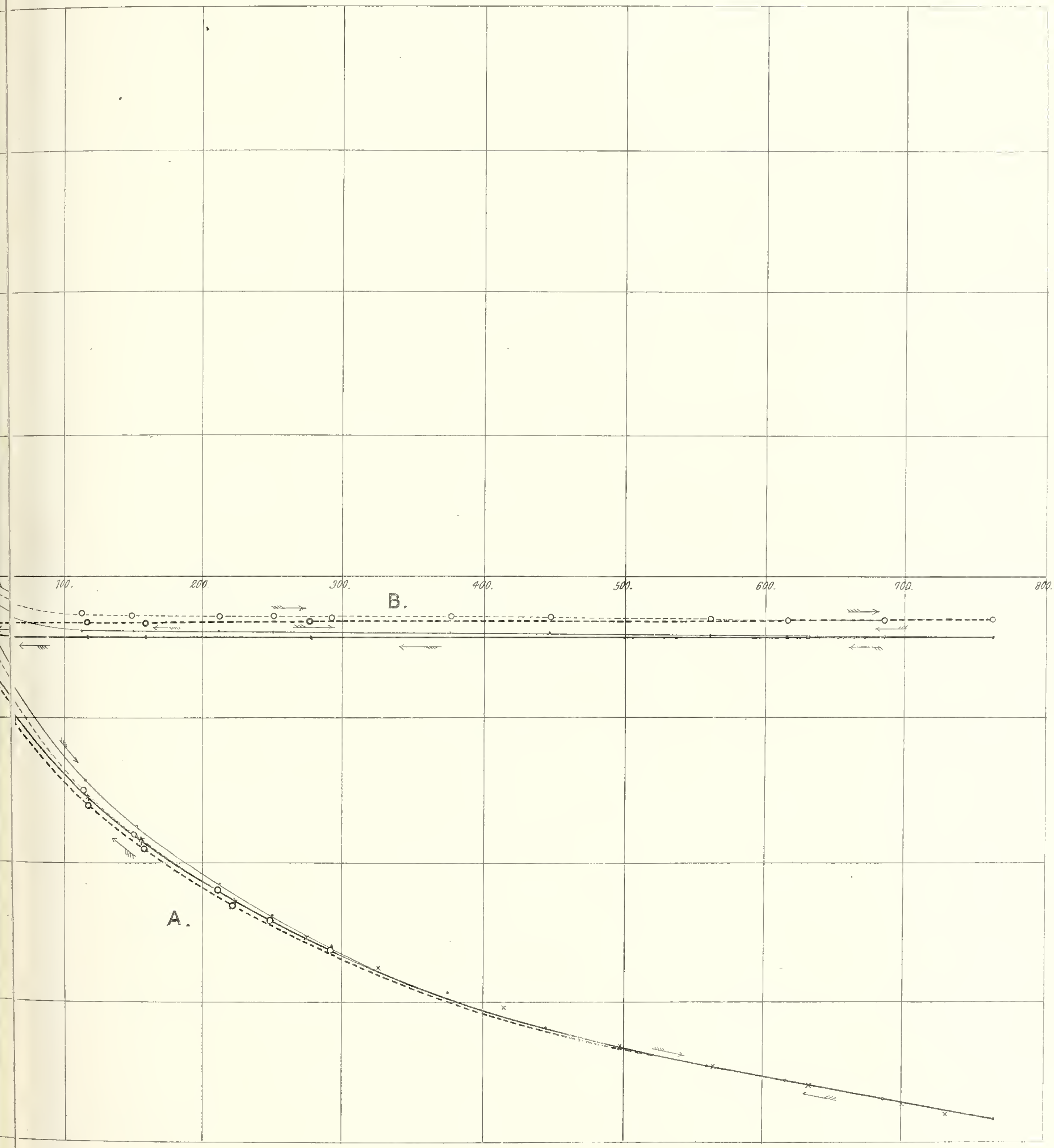
Sir W. Thomson




100.
200.
300.
400.
500.
600.
700.
800.
B.
A.

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with the top of the barrel, produced by applying a transverse outward pressure of (approximately) 1000 lbs. per square inch to the iron of the barrel. The ordinates are given in divisions of the magnetometer scale, and the abscissas in divisions of the scale of the battery galvanometer. It is interesting to note that the critical value of the magnetism is for this position of the magnetometer nearly 80 times the vertical component of the earth's magnetic force at Glasgow.

216. By placing the magnetometer at lower levels relatively to the barrel, it was found that the values of the current required for maximum and for zero effect, by "on" and "off," were less the nearer the level of the magnetometer needle was to the middle of the length of the bar. The following table shows the principal effects for —M at four different levels of the magnetometer needle :—

Distance* of top of barrel above level of magnetometer needle.	Maximum deflection before critical value was reached.	Magnetizing current for maximum deflection.	Critical value of magnetizing current.
Centims.			
0	32·8	110	430
10·5	31	130	345
21·0	14	100	265
31·5	3	100	147

The whole of the results for the —M in the last three cases are given in curves (2) of Diagram VII.

217. The magnetometer needle was placed on a level with the top of the barrel, and at a distance of two metres from its axis. Experiments were then made to find the total magnetization produced by different strengths of magnetizing current, and the effects on it of ten successive applications and removals, while the current was still flowing, of a hydrostatic pressure of 1000 lbs. per square inch; also to find the total residual magnetism after the removal of the magnetizing current, when the tube was left solely under the influence of the earth's vertical magnetic force, and the effect on it of ten successive applications and removals of the same hydrostatic pressure, while the bar was still under this magnetizing influence. The general character of the results of these experiments is difficult to describe in words, but can be seen by inspecting Diagram VIII., in which the curves marked A show the total magnetization, those marked B the residual magnetization. The abscissas of these curves are proportional to the magnetizing forces and the ordinates to the observed magnetization: but they have inadvertently been so drawn that their negative ordinates show magnetization of the same polarity to that produced by the inductive influence of the earth's vertical force, and their positive ordinates magnetization of the opposite polarity. The full lines in each set of curves show the magnetization before, the dotted lines after, ten "ons" a "offs" of and pressure of 1000 lbs. per square inch.

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* The length of the barrel and of the magnetizing coil was about 90 centims.

The results in these experiments were obtained by beginning with a *negative* (*i.e.*, opposed to magnetizing force of earth) magnetizing current of about 700 divisions of the battery galvanometer scale, which was gradually diminished to zero and then increased until a *positive* current of 700 (20 cells) divisions was reached. This process was then exactly reversed. The results may be examined in the order in which they were obtained, by beginning at the left hand ends of the curves A and B, and passing to the right along the thinner lines, returning to the left along the thicker lines.

218. The magnetometer was found in this position to be at too great a distance from the barrel to show the residual magnetism with accuracy, and accordingly its distance from the barrel was reduced to one metre, and the effects of residual magnetism alone observed. These are shown by curves marked B′ of Diagram VIII′. The method and order of experimenting were here the same as described in § 217; and the explanation of Diagram VIII., curves B, applies also to Diagram VIII′., except that in the latter the directions of the ordinates are reversed from the former; thus in VIII′. positive ordinates indicate positive magnetization, negative ordinates negative magnetization (see § 205).

219. Beginning at the extreme left of curves marked B′ on Diagram VIII′., and following the arrows, it will be seen that the residual magnetism remains nearly constant in amount until the magnetizing current has been diminished to about 300 divisions, when it begins to take a greater negative value, and continues to do so until the current is brought down to 50 divisions, when it begins slowly to diminish. After the reversal of the current the full residual magnetism diminishes with great rapidity, passing through zero at about 15 divisions of positive current. It then becomes positive, preserving nearly the same rapidity of change for some distance beyond zero. After a positive current of 150 divisions is reached, the full residual magnetism increases very slowly, and the curve becomes asymptotic towards a value of about 440. The curve showing the residual magnetism after ten "ons" and "offs" has similar characteristics to those for the full residual magnetism; the increase on the left side of zero at about 300 divisions of negative current is, however, more decided.

The general character of the return curves is similar to that just described. It is to be remarked, however, that the zero of current in each of them is much further passed before the zero of magnetization is reached. This difference between the going and returning curves would be done away with, and the curves from left to right and right to left in the diagram would be perfectly symmetrical about the zero of magnetizing current, if the influence of the earth's magnetic force were eliminated. I intend to return to this subject with a modification of the experimental arrangements, to allow the residual magnetism to be observed unaffected by any influence due to magnetizing force in the direction of the length of the tube; instead of as here with the tube always under the magnetizing action of the vertical component of the earth's magnetic force, when the electro-magnetic current is not flowing.

The curves, except in the positions corresponding to zero of current, show that the

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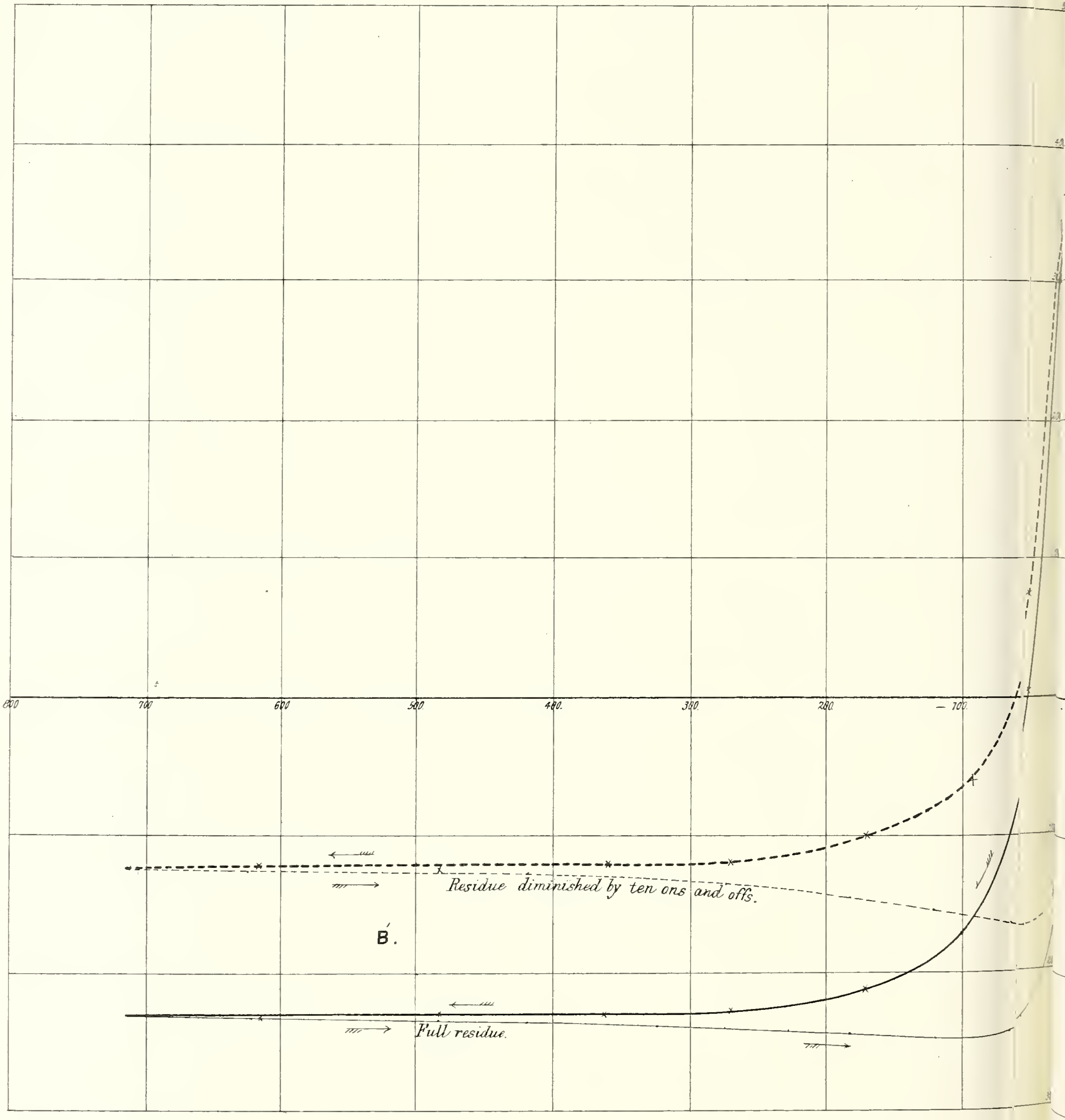
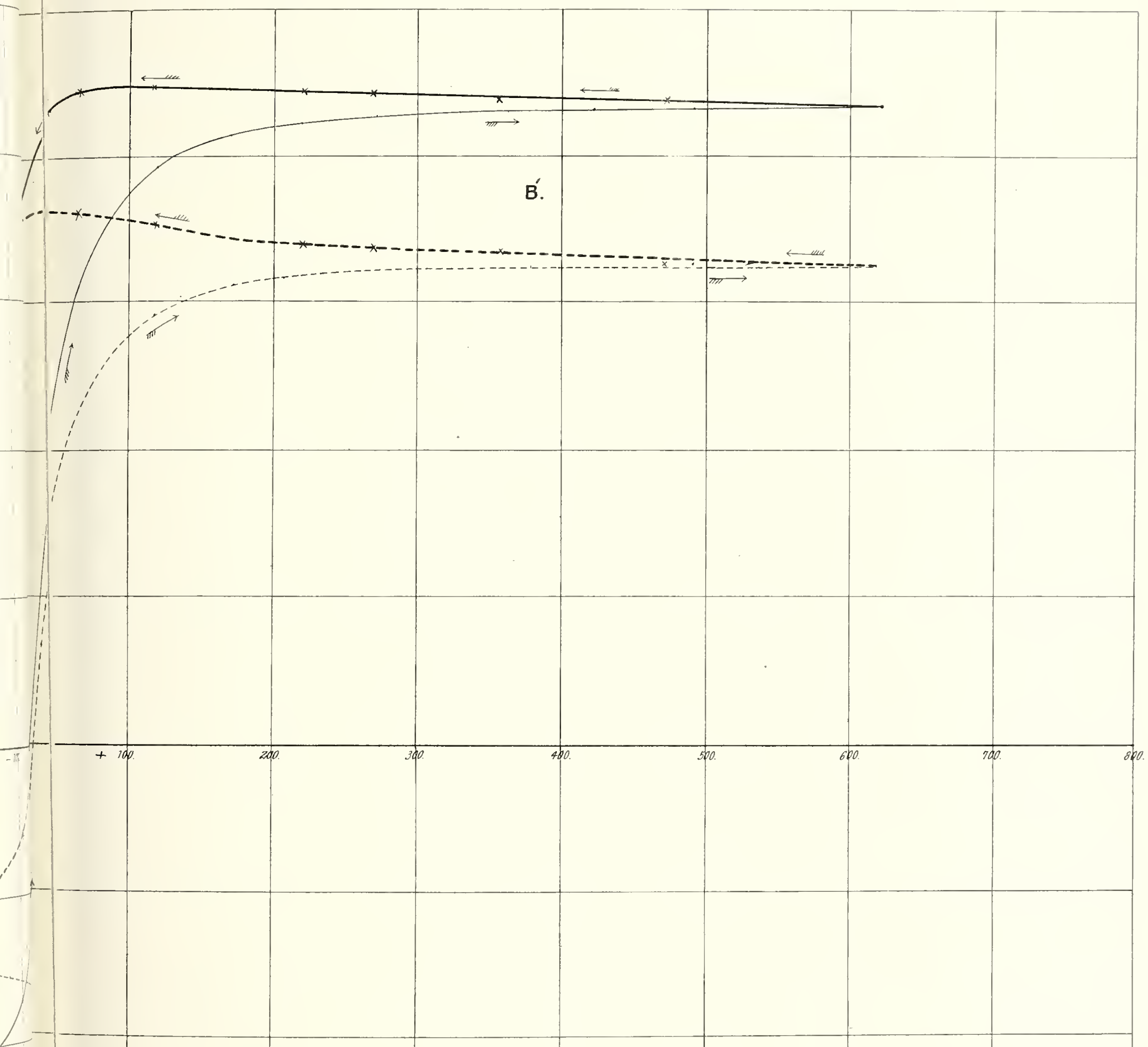

800
700
600
500.
400.
300.
200.
— 100.
Residue diminished by ten ons and offs.
B'.
Full residue.

Harrison Sc.



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ten "ons" and "offs" were not sufficient to shake out the effect of the powerful magnetizing force of the current, and allow the barrel to take the magnetization due to the vertical component of the terrestrial magnetic force. Indeed, they only diminished the residual magnetism in the curves both from left to right and from right to left by about 100 divisions, out of total residuals from 230 to 440. It is interesting to see by contrasting the right hand ends of the curves with the left hand ends, how much stronger the residual magnetism is when helped, than when opposed by the vertical magnetic force of the earth.

220. Comparison between the results of the effects of longitudinal and transverse pull shows that an aeolotropic property of different magnetic inductive susceptibility in different directions is temporarily developed in soft iron by aeolotropic stress (that is to say by stress not consisting of positive or negative pressure equal in all directions). The results show that with low magnetizing forces, negative pressure perpendicular to one set of parallel planes of soft iron produces an augmentation of magnetic susceptibility in the direction of the pressure and diminution of the susceptibility in all directions at right angles to it. The effects of positive pressure have not yet been tested experimentally, but it is certain that they will be opposite to the effects of negative pressure. Independently of experiment, we may also infer that the effects of infinitely small positive pressure perpendicular to one set of parallel planes, and infinitely small negative pressure of equal amount perpendicular to a set of parallel planes at right angles to them, must be equal and opposite in the directions of these pressures, and therefore must leave the magnetic susceptibility unaltered in the directions inclined 45° to them. This is exactly the stress which is experienced in a twisted wire of circular section; the amount of the stress being zero in the axis of the wire, and being elsewhere in simple proportion to distance from the axis. The directions of the positive and negative pressures at any point of the substance are two lines in the tangent plane of the cylindric surface through it, co-axal with the boundary of the wire, and inclined at 45° to the normal plane section. Hence, when the torsion is infinitely small, the magnetic susceptibility of the wire in the direction of its length must be unaltered, and if finite amounts of torsion produce any change in the magnetic susceptibility, the amount of this change must ultimately (for very small torsions) vary inversely as the square of the amount of torsion, as we see by remarking that whatever effect is produced must be independent of the direction of the torsion, there being nothing of helicoidal quality in longitudinal magnetization.

221. In WIEDEMANN'S 'Galvanismus' (vol. II., §§ 476–498) an abstract is given of researches in this subject by MATTEUCCI, WERTHEIM and EDMUND BECQUEREL.* One main result of all these investigations is that torsion in either direction diminishes the temporary inductive longitudinal magnetization of soft iron.

222. Nearly two years ago I instituted a series of experiments on the subject

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* MATTEUCCI, 'Comptes Rendus,' t. xxiv., 1847; WERTHEIM 'Comptes Rendus,' t. xxxv., p. 702, 1852.

chiefly for the purpose of finding the influence of torsion upon the longitudinal magnetization of soft iron wire subjected to different amounts of pulling force. These experiments, in which the magnetizing influence was simply the vertical component of the earth's magnetic force, were carried out by Mr. DONALD MACFARLANE. The mode of experimenting and the results obtained are described in the following report.

§§ 223–229. *Experiments on the effect of torsion and stretching in altering the induced Magnetism of a very soft Iron Wire, subjected to various amounts of constant pull.*

223.—*Description of Apparatus.*—In the Diagram IX. (facing § 201 above) AB is a soft iron wire, gauge No. 22, 81 centims. long, to the ends of which were soldered pieces of No. 16 copper wire; the upper piece, AD, about 5 metres in length, was attached to the ceiling of the room with an arrangement for raising or lowering it through a small space, the lower piece, BC, about 50 centims., had attached to its lower end a scale-pan for holding the stretching weight.

E is a small mirror magnetometer, the mirror being 1 centim. diameter, carrying a magnet 8 millims. in length and suspended at N by a single silk fibre 10 centims. long; and I a lens close to the mirror; F is a paraffin lamp; and GH a scale (bent into a circular arc of which E is the centre), on which is formed the image of a fine wire placed in front of the lamp flame, at E.

L and M are two edges at right angles to one another, fixed to the stand carrying the magnetometer, and just in contact with the wire: their use is to make sure that the wire is maintained in the same position relatively to the centre of the magnet.

K is an arm soldered to the copper wire, BC, for applying torsion to the wire, and immediately below it is a circle divided at intervals of 45° 20′, with small holes at each division for inserting pegs to keep the arm twisted at any angle while readings are being taken.

A similar arm was soldered to the copper wire AD at O, the two ends of which were in contact with two vertical guides, thus confining the twist to the portion of wire between K and O.

The distance of the wire from the centre of the magnetometer needle was 8·2 centims.; distance of scale from mirror, 157 centims.; one division of scale 0·5 millim.

The experiments represented in the Diagrams No. X. to No. XXIX. were made in this way:—

Having removed the wire to a distance from the magnetometer, the zero reading of the latter was taken; the wire was placed in position, a stretching weight put in the scale-pan, and the reading on the torsion-circle noted when the wire was free from torsive stress; torsion was then applied by turning the lower end of the wire 20° at a time up to 320°, and the reading on the magnetometer scale at each step taken.

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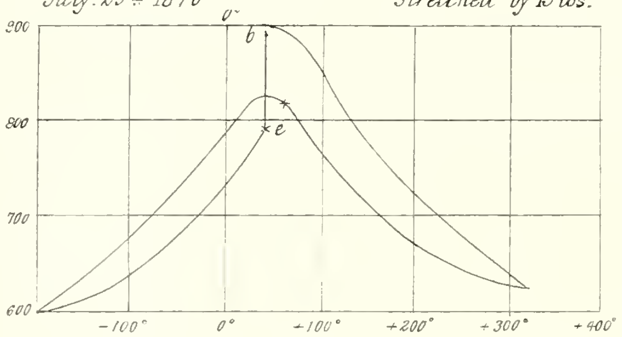
X.

July. 23rd 1876

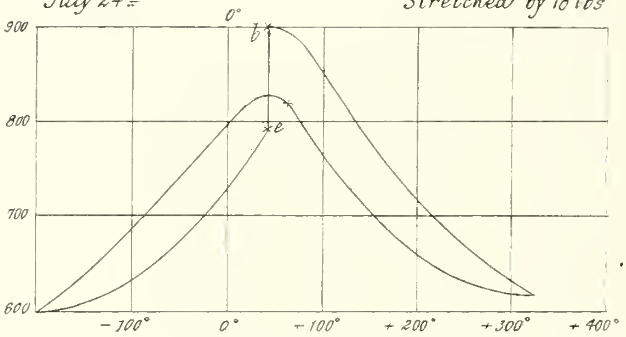
Stretched by 13 lbs.



XI.

July 24th

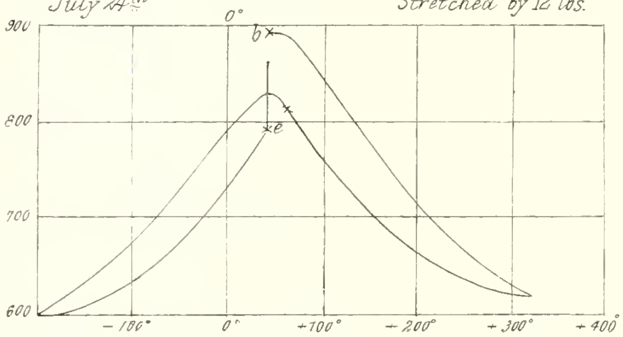
Stretched by 18 lbs



XII.

July 24th

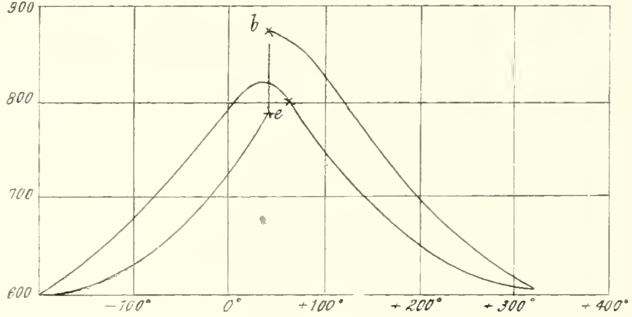
Stretched by 12 lbs.



XIII.

July. 24th

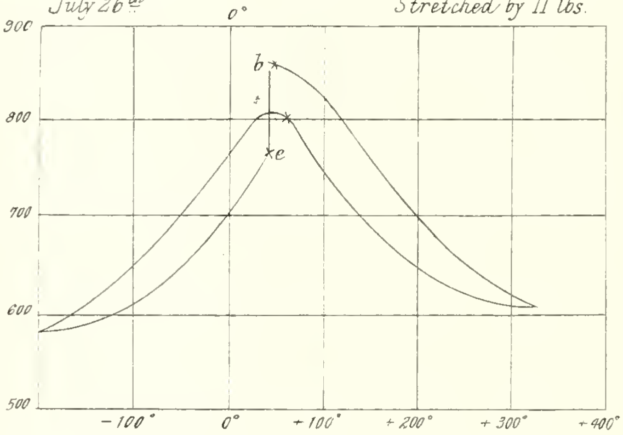
Stretched by. 12 lbs.



XIV.

July 26th

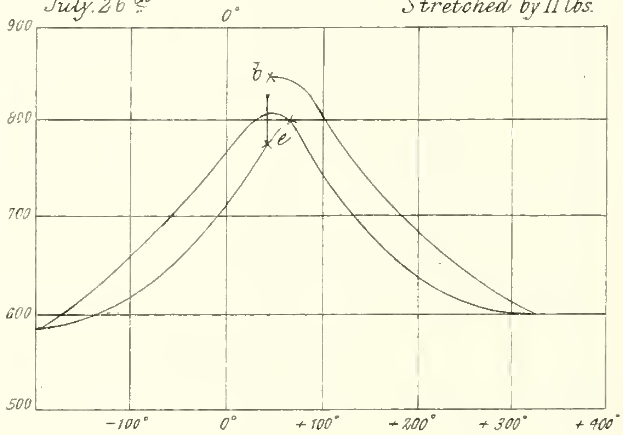
Stretched by 11 lbs.



XV.

July. 26th

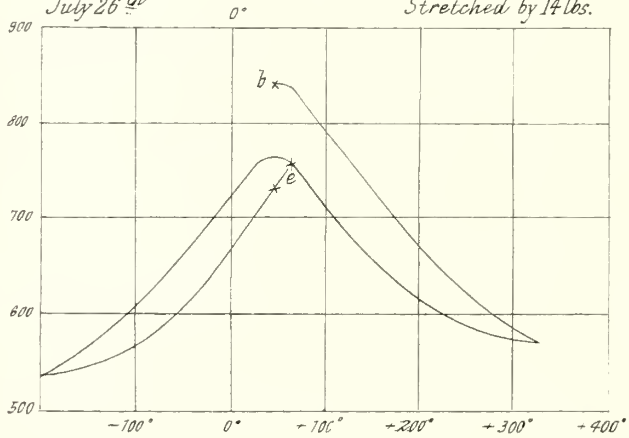
Stretched by 11 lbs.



XVI

July 26th

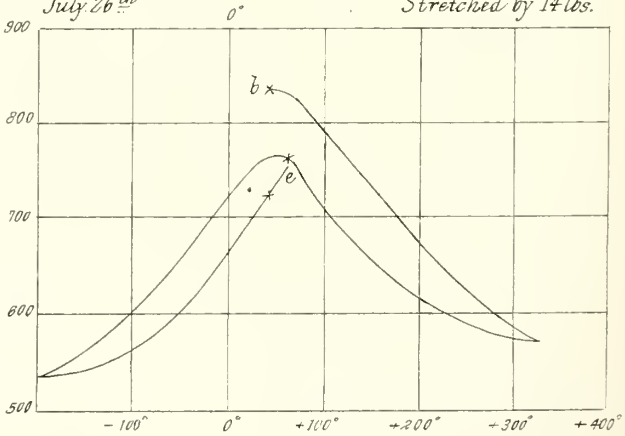
Stretched by 14 lbs.



XVII.

July 26th

Stretched by 14 lbs.



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Similarly, readings of the deflection were taken as the torsion was taken out, and then continued in the opposite direction as far as −200° and back to the original starting point. At the end of each of the first seven series the weight was taken off and put on again.

The results are represented in the accompanying diagrams of curves Nos. X. to XXIX. In each diagram the numbers at the side indicate the readings on the magnetometer scale, and those at the bottom the readings on the torsion circle, a cross (×) marks the positions where the couple of torsion was zero, *b* the beginning, and *e* the end of each experiment.

224. Explanatory remarks.

Diagram X.

Time, July 23, $4^h\ 12^m$ to $5^h\ 25^m$.
Zero of magnetometer, 90 divisions.
Stretching weight, 13 lbs.
Weight off and put on, final reading rose from 790 to 895.

Diagram XI.

July 24, $11^h\ 0^m$ to $11^h\ 55^m$.
Stretching weight, 13 lbs. left on from last.
Zero of magnetometer at end, 85.
Weight off and on, final reading rose from 815 to 896.

Diagram XII.

Time, July 24, $12^h\ 20^m$ to $1^h\ 5^m$.
Stretching weight, 12 lbs.
Zero of magnetometer, beginning 85, end 85.
Weight off and on, final reading rose from 820 to 874.

Diagram XIII.

Time, July 24, $1^h\ 15^m$ to $2^h\ 0^m$.
Stretching weight, 12 lbs.
Zero of magnetometer, beginning 85, end 90.
Weight off and on, final reading rose from 813 to 876.

Diagram XIV.

July 26, $12^h\ 50^m$ to $1^h\ 35^m$.
Stretching weight, 11 lbs. on for 46 hours.
Zero of magnetometer, at end 95.
Weight off and on, final reading rose from 800 to 850.

Diagram XV.

July 26, $1^h\ 45^m$ to $2^h\ 25^m$.
Stretching weight, 11 lbs.
Zero of magnetometer, at beginning 95, end 100.
Weight off and on, final reading rose from 803 to 835.

Diagram XVI.
July 26, 4^{h} 20^{m} to 5^{h} 10^{m}.
Stretching weight, 14 lbs.
Zero of magnetometer, at beginning 100, at end 100.
Weight off and on, final reading rose from 750 to 830.

Diagram XVII.
July 26, 5^{h} 20^{m} to 6^{h} 10^{m}.
Stretching weight, 14 lbs.
Zero of magnetometer, at beginning 100, at end 100.

Diagram XVIII.
July 28, 1^{h} 25^{m} to 2^{h} 15^{m}.
Stretching weight, 9 lbs.
Zero of magnetometer, at beginning 100.

Diagram XIX.
July 28, 4^{h} 45^{m} to 5^{h} 35^{m}.
Stretching weight, 9 lbs. left on from end of No. IX.
Zero of magnetometer, at end 105.

Diagram XX.
July 28, 5^{h} 20^{m} to 6^{h} 0^{m}.
Stretching weight, 10 lbs.
Zero of magnetometer, at beginning 105.

Diagram XXI.
July 30, 12^{h} 40^{m} to 1^{h} 20^{m}.
Stretching weight, 10 lbs. left on 42 hours from preceding No. XI.
Zero of magnetometer, at end 105.

Diagram XXII.
July 30, 4^{h} 20^{m} to 5^{h} 10^{m}.
Stretching weight, 16 lbs.
Zero of magnetometer, at beginning 105, at end 105.

Diagram XXIII.
July 30, 5^{h} 20^{m} to 6^{h} 10^{m}.
Stretching weight, 16 lbs.
Zero of magnetometer, at beginning 105, at end 105.

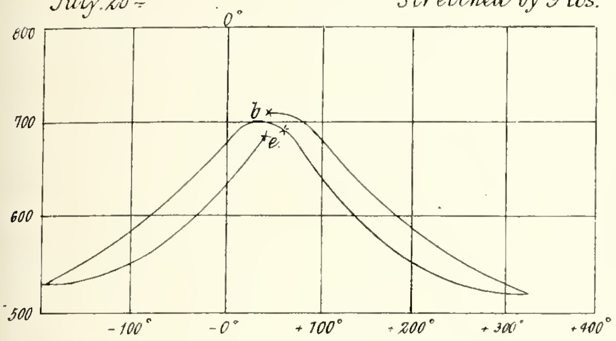
Note.—This experiment (XXIII.) was intended to be a continuation of the preceding. The magnetometer zero at the beginning was found, without taking off the weight, by drawing the wire aside; but the disturbance thus occasioned when the wire was replaced raised the final reading of No. XXII. from 690 to 712, the initial reading of No. XXIII.

The experiments represented in the six diagrams which follow were each repeated without stopping, and the repeat is represented by the dotted lines.

XVIII.

July 28th

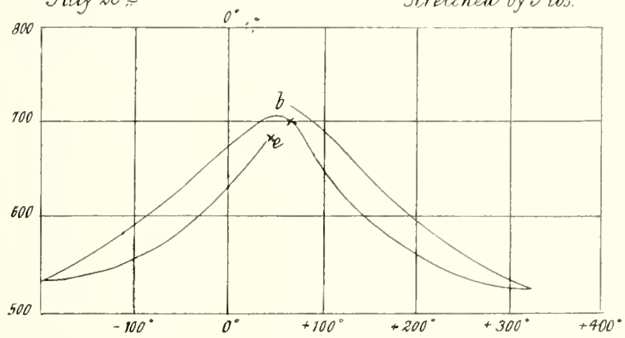
Stretched by 9 lbs.



XIX.

July 28th

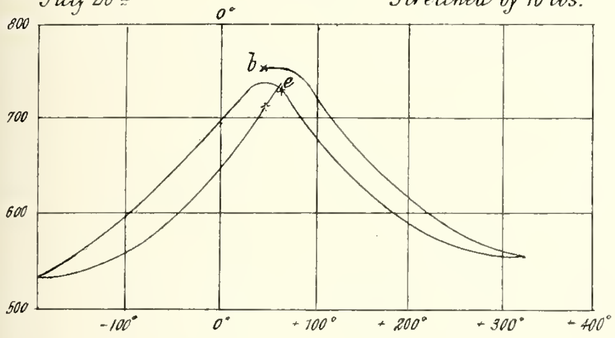
Stretched by 9 lbs.



XX.

July 28th

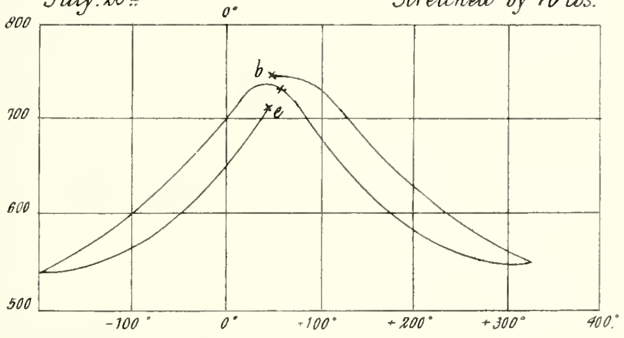
Stretched by 10 lbs.



XXI

July 28th

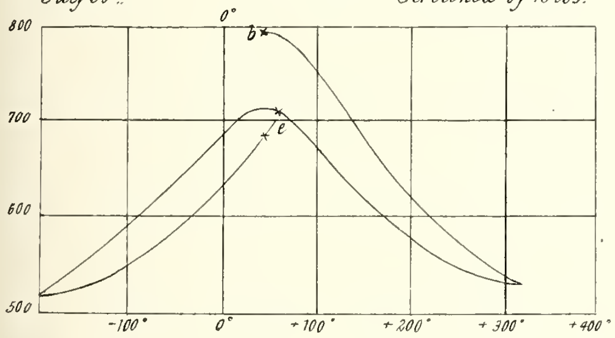
Stretched by 10 lbs.



XXII

July 30th

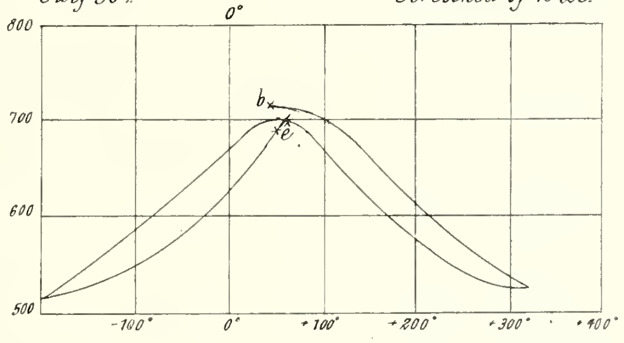
Stretched by 16 lbs.



XXIII

July 30th

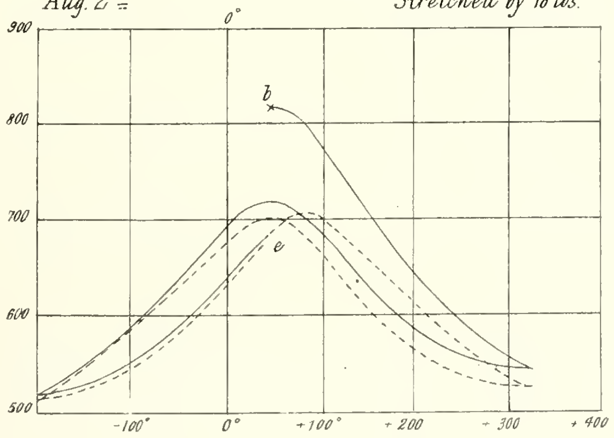
Stretched by 16 lbs.



XXIV.

Aug. 2nd

Stretched by 18 lbs.



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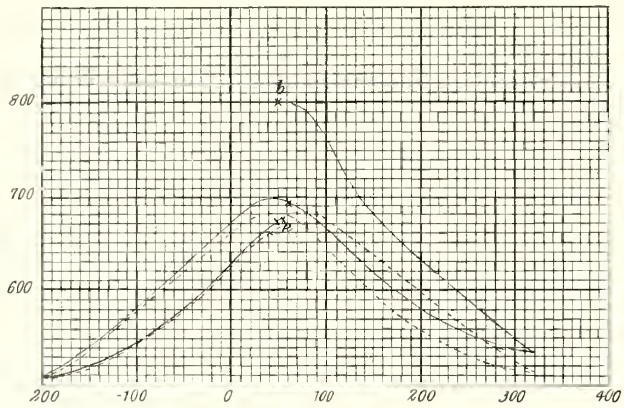
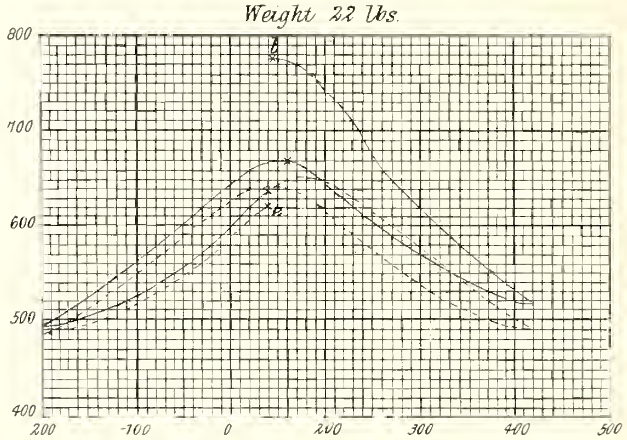
XXV.

Weight 20 lbs.



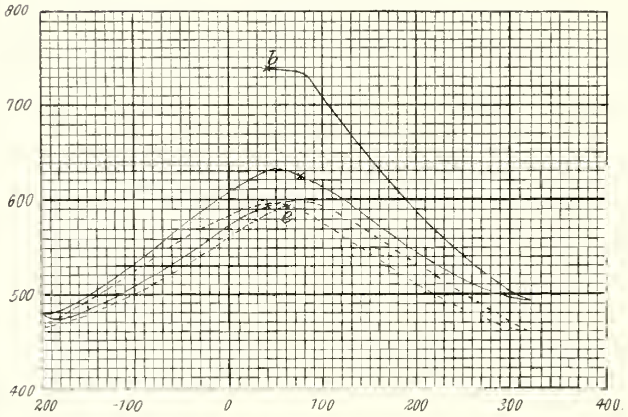
XXVI.

Weight 22 lbs.



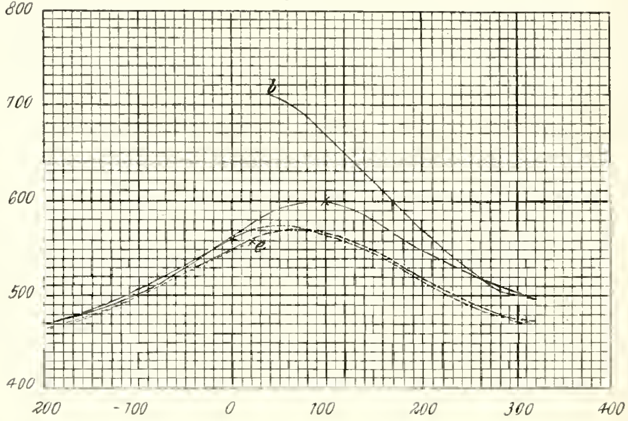
XXVII.

Weight 24 lbs.



XXVIII.

Weight 26 lbs.



XXIX.

Weight 28 lbs.

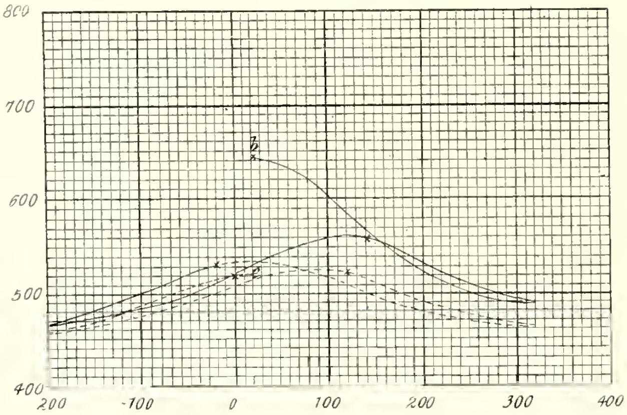


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Diagram XXIV.
August 2, $11^h\ 5^m$ to $12^h\ 40^m$.
Stretching weight, 18 lbs.
Zero of magnetometer, at end 95.

Diagram XXV.
August 2, $4^h\ 0^m$ to $5^h\ 20^m$.
Stretching weight, 20 lbs.
Zero of magnetometer, at beginning 95, at end 85.

Diagram XXVI.
August 3, $10^h\ 25^m$ to $11^h\ 45^m$.
Stretching weight, 22 lbs.
Zero of magnetometer, at end 90.

Diagram XXVII.
August 3, $12^h\ 0^m$ to $1^h\ 50^m$.
Stretching weight, 24 lbs.
Zero of magnetometer, at beginning 90, at end 100.

Diagram XXVIII.*
August 3, $4^h\ 0^m$ to $5^h\ 30^m$.
Stretching weight, 26 lbs.
Zero of magnetometer, at end 90.

Diagram XXIX.
August 4, $12^h\ 20^m$ to $1^h\ 50^m$.
Stretching weight, 28 lbs.
Zero of magnetometer, at beginning 90, at end 100.

225. From the curves it will be seen that the amount of the effect of torsion in diminishing the magnetization is not greatly influenced by the differences of pull from 10 lbs. to 20 lbs., but that it is greatly diminished by increase of the pull above 20 lbs. For simplicity and uniformity in the comparison, take the amount of the diminution of magnetization by the first application of torsion in the direction called negative, from the whole range of from $+50°$ to $-200°$ on the scale of torsion (abscissas of the curves).

We find with pulls of from 10 lbs. to 20 lbs. various amounts of from 180 to 230 magnetometer scale divisions. The seemingly irregular differences between these amounts showed no regular dependence on the amount of pull, but seemed rather to depend upon previous conditions of the wire. But when the weight exceeded 20 lbs. there seemed a somewhat regular diminution in the effect of torsion with increase of pull, as shown in the following table ; thus:—

With pull of	20 lbs.	the effect was	193
,,	22	,,	177
,,	24	,,	150
,,	26	,,	130
,,	28	,,	65

---

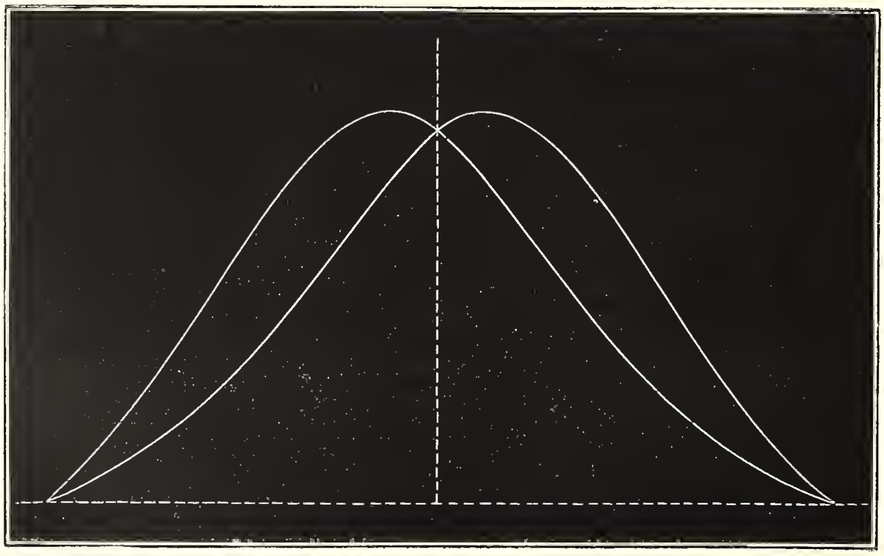
* In this diagram the two dotted lines between +40 and −200 coincide.

226. One very interesting feature common to all the diagrams, and presented even to some degree by the exceptional ones 9, 10, 12, and 14, shows that the effect of twisting the wire first in one direction and then in the other, and leaving it free from torsive force, was in every case to leave it with less magnetization than it had at the beginning.

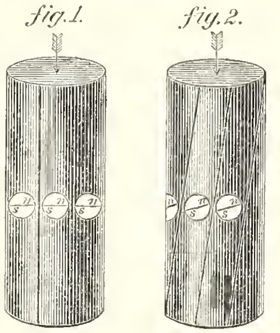
227. These exceptional diagrams and later continuation of the operations through a second positive torsion and a second negative torsion, represented in the latter halves of the curves of Diagrams XXIV. to XXIX., show what would be the general character of the effect of continued periodic applications of positive and negative torsion, through equal angles on the two sides of zero.

In every case there is a lagging of quality, showing a residue of effect from previously acting causes. Thus, beginning with a wire which has been reduced to a normal condition by having had the weight off and on, and having been left to itself for twenty-four hours, we found on the 24th of August that the magnetization fell from a normal value of 685 down to 550 as the result of twisting it to +260,° then to −260, and then to zero of torsion. The second application of positive and negative torsion reduced the magnetization further to 534. It is curious to find that not merely does torsion diminish the magnetization temporarily, but that it leaves so large a permanent diminution. Whether this permanence is absolute in respect to time or not is an interesting question to be solved by leaving a wire which has been thus dealt with absolutely quiescent from day to day, month to month, year to year, century to century. It seems, however, that but slight mechanical disturbance suffices to shake out the diminution of magnetization left at the end of each of these experiments.

228. The general lagging of effect is shown by the fact that in every ascending branch the curve is lower than in the immediately previous descending branch; and the dotted latter halves of the long curves of the Diagrams XXIV. to XXIX. show, by the intersections of their convex portions near the zero, that if the experiment was continued long enough, the history of the variation of magnetization would in every case be represented by a curve like that in the annexed sketch.



229. A very interesting discovery of Wiedemann's ('Galvanismus,' §§491 and 498), at first sight, seemed to find its explanation in the aeolotropic difference of magnetic susceptibility which I have found to be induced by aeolotropic stress in soft iron. The phenomenon consists in the development of longitudinal magnetization by twisting a wire through which a magnetic galvanic current is maintained longitudinally. The annexed double diagram, copied from Wiedemann's book,* describes the change of ideal magnetic molecules which would represent the actually observed effect, which is that the end of the wire by which the current enters becomes a true north pole when the twist given to it is right-handed (or that of an ordinary screw). If this effect were due to greater susceptibility in one direction than in another, the direction of greatest susceptibility would be the direction sloping at 45°, upwards to the right in the front of the right-hand diagram; that is to say, it would be the direction of positive pull in the stressed material. But the exceedingly intense magnetization by influence of circular lines of force round the cylinder, produced by currents of such strength as Wiedemann may be supposed to have used, must in all probability have been above the critical degree of magnetization at which the effect of pull becomes reversed, and therefore in all probability the direction of least susceptibility in the actual circumstances must have been that of positive pull. Hence, it seems almost impossible to admit the explanation of Wiedemann's result by aeolotropic magnetic susceptibility in the circumstances.† The true explanation is not easily conjectured: for another cause, also adverse to Wiedemann's result, is operative. The electric conductivity of the iron is probably least in the direction of the positive pull and greatest in the direction of the negative pull in the stressed material.‡ This aeolotropic quality in



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* I have altered the letters n and s of Wiedemann's book to Gilbert's old wholesome rule of putting n to represent true northern polarity, or the polarity of the same name as that of the earth's northern regions, and similarly s to represent true southern polarity.

† This experiment has been repeated for me since the communication of this instalment to the Royal Society, by Mr. Macfarlane, with currents, not hitherto measured or estimated in absolute measure, but strong enough to greatly heat the iron wire. The result was always the same as Wiedemann's, and was greatest with the strongest current used.—(W. T., May 22, 1878.)

‡ "Electrodynamic Qualities of Metals," §§ 145—153 (W. Thomson, Transactions of the Royal Society, 1856). "On the Increase in Resistance to the Passage of an Electric Current produced in Certain Wires by Stretching" (Tomlinson, Proc. Roy. Soc., No. 183, 1877).

respect to electric conductivity would cause the electric current, instead of flowing rectilineally along the wire, to flow in left-handedly helical lines in the case represented in fig. 2, and thus the central parts of the iron cylinder would become really magnetized by, as it were, an ordinary helix, but with very steep thread. The effect of such a helix is the same as that of a true solenoid superimposed upon a rectilineal current through the wire, and the direction of the current in the supposed circumstances is such that it would give a true south pole at the upper end of the iron rod in fig. 2.

(Received and read May 23, 1878.)

§§ 230–240. *On the effects of longitudinal stress on the Magnetization of Nickel and Cobalt.*

§ 230. Through the kindness of Mr. JOSEPH WHARTON, of Philadelphia, U.S., I was enabled to continue my experiments with malleable and cast bars of nickel, and of cast cobalt.

[Note added July 8, 1879.—A qualitative analysis of one of Mr. WHARTON's nickel bars, performed in the Chemical Laboratory of Glasgow University, by Mr. DONALD MACKENZIE, showed that the bar was not of absolutely pure nickel, but contained some carbonaceous matter and also a trace of iron. The amount of the latter, however, was very small, and probably could not vitiate to any sensible degree the results of the experiments described below.]

The apparatus used in the preliminary experiments and its arrangement are shown in the annexed diagram (XXX.). Each end of the rod experimented on was inserted into a ferule-shaped clamp, C (shown also detached in plan and elevation), the outer surface of which was conical and screw-threaded.

The clamp, which had in it three longitudinal slits, was then, by means of a conical nut working round it, made to grasp the rod tightly enough to admit of the application of great amounts of longitudinal pull, without much danger of pulling the clamps away from the rod. One of the clamps was then hung from a pin in a strong cross-beam of a frame, so that the bar hung vertically downwards.

A rope, R, made of copper wire, connected the other clamp with a point near the end of a long heavy lever, turning on a fulcrum at that end formed by a knife edge pressing upwards against a brass plate, which formed a bridge between two strong and rigid uprights attached to the floor of the room. A heavy weight of lead was hung on the lever, and could be slided along it to give different amounts of stress. The lever was graduated, and the effect of its own weight was measured, so that the stress applied at any time could be at once read off. When the bar was in position but not under stress, the lever rested on a support high enough to allow the wire rope R to be slack, and was gently removed from this support when the pulling stress was applied.

XXX.

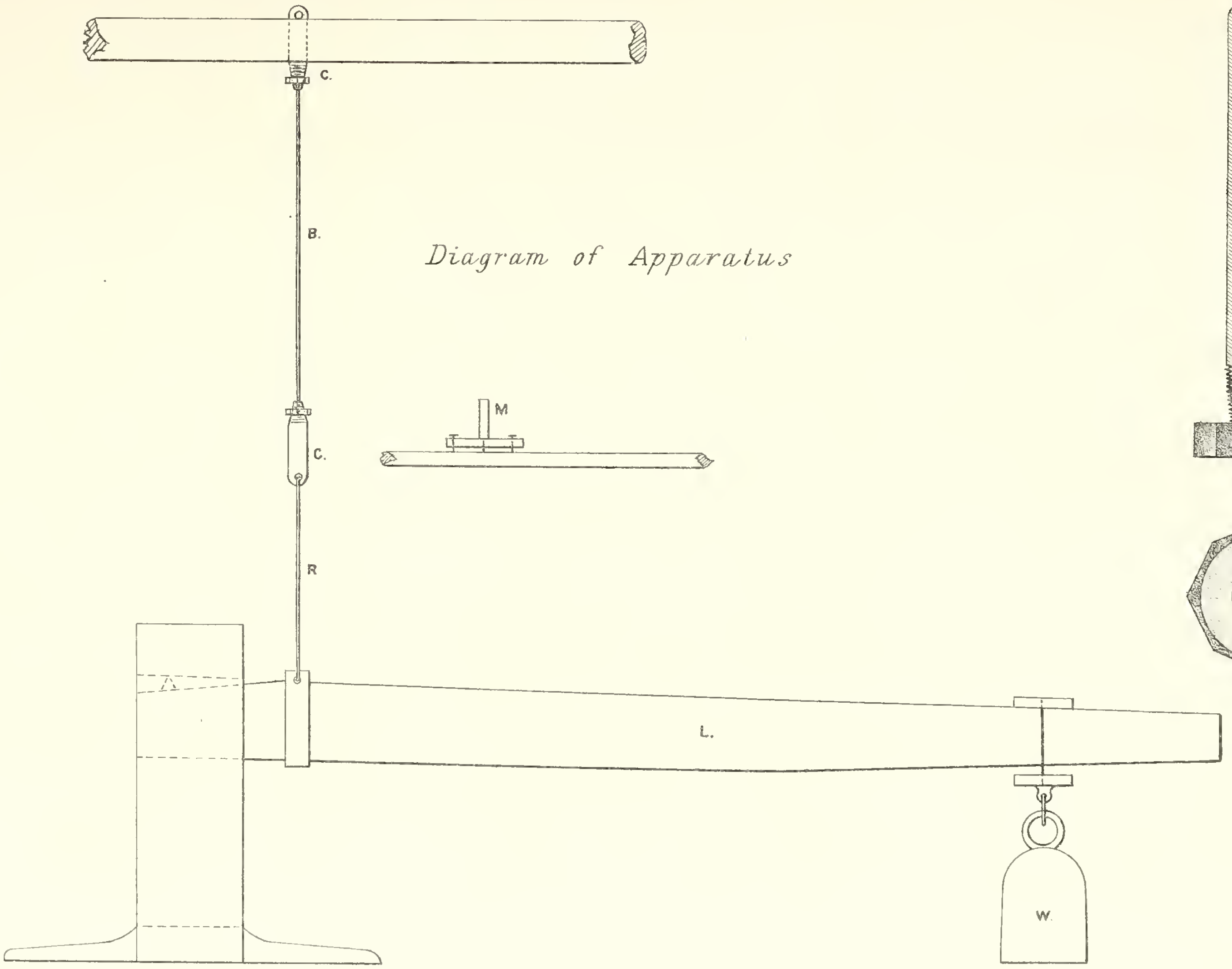
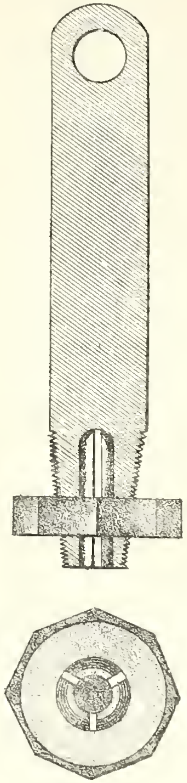


Diagram of Apparatus

Clamp.
Half Size.



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The effects of the various operations were measured by the deflections of the image on the scale of a reflecting magnetometer, M, the needle of which was on a level with the lower end of the bar, and at a distance due magnetic west from its axis of 12 centims. The scale of the magnetometer was at a distance of 135 centims. from the needle.

231. In these preliminary experiments the bars were under the influence of no magnetizing force except that of the vertical component of the earth's magnetic force at Glasgow. In order that the results obtained with the bars of nickel and cobalt might be readily compared with those obtained from iron in the same circumstances, the experiment was first performed on a bar of tolerably soft iron, of nearly the same dimensions as those of the bar of nickel or cobalt experimented on. The actual results of a set of these experiments are given in the following tables. Table I. contains the results of an experiment on an iron bar 60 centims. long and ·8 of a centim. in diameter, and Tables II. and III. the results respectively of two similar experiments performed immediately afterwards, one on a bar of wrought nickel and the other on a bar of cast cobalt. The total magnetization is reckoned positive when its polarity is the same as that produced by the inductive effect of the earth's magnetic force, and negative when its polarity is of the opposite kind.

Table I.—Iron Bar. Lower end of bar a true South Pole.

Operations.	Magnetometer readings.	Differences.	Total magnetization.
	Zero 463		
Bar placed in position . . .	Image off scale in negative direction		
Controlling magnet introduced	769		+1369
174 lbs. on	675	−94	+1463
,, off	687	+12	+1451
,, on	660	−27	+1478
,, off	675	+15	+1463
10 "ons" and "offs" .	667	− 8	+1471
174 lbs. on	649	−18	+1489
,, off	663	+14	+1475
259 lbs. on	571	−92	+1567
,, off	597	+26	+1541
10 "ons" and "offs" .	570	−27	+1568
259 lbs. on	540	−30	+1598
,, off	569	+29	+1569
325 lbs. on	470	−99	+1668
,, off	510	+40	+1628
10 "ons" and "offs" .	474	−36	+1664
325 lbs. on	421	−53	+1717
,, off	457	+36	+1671

TABLE II.—Wrought Nickel Bar. Lower end a true South Pole.

Operations.	Magnetometer readings.	Differences.	Total magnetization.
	Zero 460		
Bar placed in position . . .	450	−10	+ 10
146 lbs. on	421	−29	+ 39
,, off	411	−10	+ 49
,, on	416	+ 5	+ 44
,, off	410	− 6	+ 50
Clamp here slipped.			
Bar reclamped and replaced in position	Image at 415	−45	+ 95
146 lbs. on	412	− 3	+ 98
,, off	393	−19	+117
,, on	410	+17	+100
,, off	389	−21	+121
10 "ons" and "offs"	387	− 2	+123
146 lbs. on	410	+23	+100
,, off	388	−22	+122
174 lbs. on	408	+20	+102
,, off	384	−24	+126
10 "ons" and "offs"	383	− 1	+127
174 lbs. on	405	+22	+105
,, off	380	−25	+130
211 lbs. on	405	+25	+105
,, off	376	−29	+134
10 "ons" and "offs"	375	− 1	+135
211 lbs. on	403	+28	+107
,, off	378	−25	+132
249 lbs. on	403	+25	+107
,, off	368	−35	+142
10 "ons" and "offs"	375	+ 7	+135
249 lbs. on	395	+20	+110
,, off	365	−30	+140
,, on	398	+33	+107
,, off	364	−34	+141
287 lbs. on	397	+33	+108
,, off	357	−40	+148
10 "ons" and "offs"	353	− 4	+152
287 lbs. on	393	+40	+112
,, off	350	−43	+155

TABLE III.—Cast Cobalt Bar. Lower end at first a true *North* Pole.

Operations.	Magnetometer readings.	Differences.	Total magnetization.
	Zero 460		
Cobalt bar placed in position	952	+492	−492
146 lbs. on	895	− 57	−435
„ off	885	− 10	−425
10 " ons " and " offs "	867	− 18	−407
146 lbs. on	882	+ 15	−422
„ off	867	− 15	−407
Bar struck a few blows with a mallet. Magnetism reversed	370	−497	+ 90
146 lbs. on	395	+ 25	+ 65
„ off	370	− 25	+ 90
„ on	395	+ 25	+ 65
„ off	373	− 22	+ 87
174 lbs. on	403	+ 30	+ 57
„ off	375	− 28	+ 85
10 " ons " and " offs "	375	0	+ 85
249 lbs. on	Bar here broke, but being held in its place gave a reading of 360		

232. In Table I. it is stated that when the iron bar was placed in position the image on the scale of the magnetometer was driven off the scale in the negative direction. A controlling magnet, 15 centims. long, placed at right angles to the magnetic meridian in a horizontal line passing through the centre of the needle, was used to bring the image to the division 769 on the scale. When the nearer end of this magnet was at a distance of 23 centims. from the needle, the image rested at the division 323 on the scale, and when the magnet was brought 6 centims. nearer to the needle, at the division 773 on the scale. Hence, taking the tangent of the angle of deflection as equal to the angle itself, we may reckon the deflection caused by bringing the iron bar of Table I. into position as 1369 scale divisions, which may be taken as measuring the magnetism of the bar.* This increased during twelve successive

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* The method of calculation is as follows:—Let A denote the deflection produced by placing the bar in position; D and D′ the readings with the controlling magnet introduced, with its centre at the distances r and r' from the magnet; a the half length of the magnet; and B a constant depending on the controlling magnet; then we have,

$$\mathrm{D}=\mathrm{A}+\frac{\mathrm{B}}{(r-a)^2}-\frac{\mathrm{B}}{(r+a)^2}=\mathrm{A}+\frac{4\mathrm{B}ar}{(r^2-a^2)^2}$$

and similarly

$$\mathrm{D}'=\mathrm{A}+\frac{4\mathrm{B}ar'}{(r'^2-a^2)^2}$$

Eliminating B between these two equations, and solving for A we get

$$\mathrm{A}=\frac{\mathrm{D}+\mathrm{D}'}{2}-\frac{\mathrm{D}-\mathrm{D}'}{2}\,\frac{r(r'^2-a^2)^2+r'(r^2-a^2)^2}{r(r'^2-a^2)^2-r'(r^2-a^2)^2}$$

By taking for a the virtual half length of the magnet or the distance of either pole from the centre of applications and removals of a pulling stress of 174 lbs. to about 1470 scale divisions, when the bar was found to have been brought to a nearly permanent condition; and the average effect of applying this stress was to increase the magnetism by 16 divisions, and of removing the stress to diminish the magnetism by the same amount. The magnetism of the bar further increased during 11 successive applications and removals of a pull of 259 lbs. to 1568 scale divisions, and the average effect of applying and removing the pull was to increase and diminish the magnetism by 28 scale divisions. After 11 applications and removals of a stress of 325 lbs. the magnetism was found to have increased to 1664 divisions; and the average effect of "on" and "off" was found to be an increase and diminution of 45 divisions.

233. Passing now to the nickel bar of Table II., we see that when the bar was placed in position it showed 10 divisions of positive magnetism (or magnetism of the same polarity as that induced by the earth). The application of 146 lbs. of pull gave it 29 divisions, and the removal of this pull 10 divisions additional magnetism, that is, both "on" and "off" increased the magnetism. A second application of the pull gave a diminution of 5, and removal an increase of 6 divisions. The remainder of the procedure was similar to that followed in the case of the soft iron, and with the exception of the first result after the bar was placed in the clamps, the effect of "on" was always to *diminish* the magnetism of the bar, and of "off" to *increase* it.

After the bar was placed in the clamps the effect of the successive operations was on the whole to gradually augment the total magnetization of the bar from the value 10 to the value 155, at which it stood when the experiment was concluded.

234. From Table III. we see that the bar of cast cobalt, when placed in position, had its true north pole down, and gave a deflection of 392 divisions. Ten "ons" and "offs" with 136 lbs. diminished this deflection by 85 divisions. The effect of "on" was then to *increase* the magnetism by 15 divisions, and of "off" to *diminish* it by the same amount; that is to say, "on" increased the magnetism of the bar, "off" diminished it. A few blows of a mallet reversed this magnetism, and caused the bar to give a deflection of 90 divisions in the opposite direction. The effect of "on" was, as with the nickel bar, to *diminish* the magnetism, and of "off" to *increase* it, the effect of 146 lbs. being 25 divisions of the magnetometer scale, of 174 lbs. 29 divisions. The bar broke before the effect of the application of 249 lbs. could be observed. After the reversal of the magnetism by tapping the bar while under the influence of the vertical component of the earth's magnetic force, there was very little gradual change in the magnetization of the bar.

The seemingly anomalous effect obtained with the cobalt bar, when placed with its true north pole down, according to which the effect of the application of stress was to increase the magnetism of the bar, and of the removal of stress to diminish it,

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the magnet's length, instead of the actual half length, a somewhat nearer approximation to the value of A might have been obtained, but, as the approximation was at best a rough one on account of the size of the angle, it was not thought necessary to make this refinement.

was no doubt due to the magnetizing influence of the earth tending to reverse the retained magnetism of the bar. It will be further investigated in a continuation of experiments on cobalt.

235. The effect of longitudinal stress on the magnetization of nickel when magnetized by a current flowing in a coil surrounding the bar formed the subject of the next series of investigations. The magnetizing coil was 54 centims. long, and consisted of six layers of silk-covered copper wire of No. 22 B.W.G., each layer forming a solenoid containing 10·7 turns per centim. The resistance of the coil when cool was 7·2 ohms, and the resistance of the electrodes ·3 of an ohm. In this experiment the magnetometer was placed at a distance of 40 centims. from the axis of the bar, and on a level with its lower end; and in order that the deflection due to the total magnetization of the bar might be conveniently measured, the directive force on the magnetometer needle was increased by placing behind it, in the magnetic meridian, a bar magnet, with its true north pole turned towards the north. As in the previous experiments with iron and steel, −M indicates that the electromagnetic field was opposite in polarity to that of the earth, and +M that its polarity was the same as that of the earth. The dead-beat galvanometer used in the experiments on the effects of transverse stress on the magnetization of an iron tube was again employed to measure the strength of the magnetizing current.

236. The results of a series of these experiments are given in Table IV. It will be seen from that table that the effects of the application and removal of stress were respectively to diminish and to increase the induced magnetization, and that, as in the case of soft iron, this effect reached a maximum with a certain strength of magnetizing current, after which it slowly diminished. In this experiment the critical value of the magnetizing force corresponding to what has been called above, in the account of experiments in soft iron, the VILLARI *critical value*, was not reached.

TABLE IV.—Bar of Wrought Nickel.

Operations.	Readings. Zero 554.	Differences.	Total magnetization.	Strength of magnetizing current.
Bar put in position .	352	−202	+202	
−M . . .	583	+231	− 29	62
10 "ons" and "offs"				(2 double cells)
with 285 lbs. . .	637	+ 54	− 83	
285 lbs. on	623	− 14	− 69	
,, off	637	+ 14	83	
B	568	− 69	− 14	
285 lbs. on	554	− 14	0	
,, off	556	+ 2	− 2	
,, on	553	− 3	+ 1	
,, off	555	+ 2	− 1	

TABLE IV.—Bar of Wrought Nickel.—continued.

Operations.	Readings. Zero 554.	Differences.	Total magnetization.	Strength of magnetizing current.
+M	340	−215	+214	62
285 lbs. on	367	+ 27	+187	
,, off	332	− 35	+222	
,, on	363	+ 31	+191	
,, off	329	− 34	+225	
B	388	+ 59	+166	
285 lbs. on	426	+ 38	+128	
,, off	403	− 23	+151	
,, on	427	+ 24	+127	
,, off	404	− 23	+150	
−M	863	+459	−309	147
285 lbs. on	832	− 31	−278	
,, off	864	+ 32	−310	
,, on	833	− 31	−279	
,, off	864	+ 31	−310	
B	745	−119	−191	
285 lbs. on	690	− 55	−136	
,, off	716	+ 26	−162	
,, on	689	− 27	−135	
,, off	716	+ 27	−162	
+M	226	−490	+328	147
285 lbs. on	260	+ 34	+294	
,, off	227	− 33	+327	
,, on	260	+ 33	+294	
,, off	226	− 34	+328	
B	343	+117	+211	
285 lbs. on	400	+ 57	+154	
,, off	370	− 30	+184	
,, on	400	+ 30	+154	
,, off . . .	371	− 29	+183	
−M	953	+582	−399	282
285 lbs. on	927	− 26	−373	
,, off	950	+ 23	−396	
,, on	925	− 25	−371	
,, off	950	+ 25	−396	
B	777	−173	−223	
285 lbs. on	715	− 62	−161	
,, off	746	+ 31	−192	
,, on	715	− 31	−161	
,, off	744	+ 29	−190	
+M	155	−589	+399	282
285 lbs. on	181	+ 26	+373	
,, off	156	− 25	+398	
,, on	182	+ 26	+372	
,, off	156	− 26	+398	
B	327	+171	+227	
285 lbs. on	388	+ 61	+166	
,, off	357	− 31	+197	
,, on	388	+ 31	+166	
,, off	358	− 30	+196	
Zero changed to . .	500	Difference from zero		
−M	940	440	−440	432
285 lbs. on	925	− 15	−425	
,, off	940	+ 15	−440	

TABLE IV.—Bar of Wrought Nickel.—continued.

Operations.	Readings. Zero 500.	Differences.	Total magnetization.	Strength of magnetizing current.
285 lbs. on	923	− 17	−423	
,, off	939	+ 16	−439	
B . . .	726	−213	−226	
285 lbs. on	665	− 61	−165	
,, off	697	+ 32	−197	
,, on	664	− 33	−164	
,, off	696	+ 32	−196	
+M	48	−648	+452	432
285 lbs. on	63	+ 15	+437	
,, off	48	− 15	+452	
,, on	63	+ 15	+437	
,, off	47	− 16	+453	
B	268	+221	+232	
285 lbs. on	329	+ 61	+171	
,, off	297	− 32	+203	
,, on	330	+ 33	+170	
,, off	298	− 32	+202	
+M	13	−285	+487	580
285 lbs. on	25	+ 12	+475	
,, off	16	− 9	+484	
,, on	28	+ 12	+472	
,, off	18	− 10	+482	
B	268	+250	+232	
285 lbs. on	334	+ 66	+166	
,, off	299	− 35	+201	
,, on	334	+ 35	+166	
,, off	299	− 35	+201	
−M	963	+664	−463	580
285 lbs. on	954	− 9	−454	
,, off	963	+ 9	−463	
,, on	952	− 11	−452	
,, off	961	+ 9	−461	
B	724	−237	−224	
285 lbs. on	664	− 60	−164	
,, off	697	+ 33	−197	
,, on	663	− 34	−163	
,, off	696	+ 33	−196	
Reading brought to .	617			
−M	900	+283	−479	708
285 lbs. on	892	− 8	−471	
,, off	897	+ 5	−476	
,, on	889	− 8	−468	
,, off	897	+ 8	−476	
B	645	−252	−224	
285 lbs. on	582	− 63	−161	
,, off	616	+ 34	−195	
,, on	582	− 34	−161	
,, off	615	+ 33	−194	
Reading brought to .	765			
+M	74	−691	+497	708
285 lbs. on	81	+ 7	+490	
,, off	76	− 5	+495	
,, on	83	+ 7	+488	
,, off	76	− 7	+495	

TABLE IV.—Bar of Wrought Nickel—concluded.

Operations.	Readings.	Differences.	Total magnetization.	Strength of magnetizing current.
361 lbs. on	92	+ 16	+479	708
,, off	80	− 12	+491	
,, on	93	+ 13	+478	
,, off	80	− 13	+491	
285 lbs. on	91	+ 11	+480	
,, off	85	− 6	+486	
,, on	94	+ 9	+477	
,, off	85	− 9	+486	
211 lbs. on and off 5 times	90	+ 5	+481	
211 lbs. on	96	+ 6	+475	
,, off	90	− 6	+481	
,, on	96	+ 6	+475	
,, off	91	− 5	+480	
136 lbs. on	96	+ 5	+475	
,, off	93	− 3	+478	
,, on	97	+ 4	+474	
,, off	93	− 4	+478	
B	350	+257	+221	
136 lbs. on	380	+ 30	+191	
,, off	363	− 17	+208	
,, on	384	+ 21	+187	
,, off	365	− 19	+206	
,, on	384	+ 19	+187	
,, off	365	− 19	+206	
361 lbs. on	418	+ 53	+153	
,, off	377	− 41	+194	
,, on	420	+ 43	+151	
,, off	379	− 41	+192	
285 lbs. on	412	+ 33	+169	
,, off	379	− 33	+202	
,, on	411	+ 32	+170	
,, off	378	− 33	+203	

237. The preceding table shows that the effect of the application of pull was to diminish, and of the removal of pull to increase the magnetism of the bar, whether induced or residual; and that a series of these operations *increased* on the whole the magnetism induced by +M or −M, but diminished on the whole the residual magnetism after B. Further, as stated above, the effect of "on" or "off" on the induced magnetism of the bar increases up to a certain point, and then diminishes as the magnetizing force is increased from zero upwards; while, on the other hand, the effect of "on" or "off" on the residual magnetism goes on increasing as the residual magnetism is increased, and, as does also the residual magnetism, approaches more and more to a certain constant value.

238. On account of the thickness of the bar, a large amount of wire was required to make a coil which would give a sufficiently powerful magnetizing force to reach or pass the critical value to which the magnetizing force seemed in the preceding experiments

to approach. It was found more convenient to continue the experiment with a smaller bar of nickel, kindly lent for the purpose by Professor TAIT. This (which was a square bar 45·7 centims. long and ·30 centim. thick) was placed within a coil wound on a thin copper tube of just sufficient internal diameter to admit the bar. The coil contained six layers of silk-covered copper wire, of No. 22 B.W.G., each layer forming a solenoid 42·7 centims. long, containing 10·7 turns per centim. The total resistance of the coil when cool was 4·33 ohms. The resistance of the electrodes was, as before, ·3 of an ohm. The magnetometer needle was in this case placed on a level with the upper end of the bar, and at a distance of 25 centims. from the axis of the bar, and of 108 centims. from the scale of half millims. on which the readings were observed. The stress was not applied by means of the lever, but a weight of 14 lbs. was placed on a pan attached to the bar. This pan, which weighed 1 lb., was left hanging on the bar during the whole experiment.

239. In their general character the results are precisely similar to those shown in Table IV. A maximum effect of 20 divisions was obtained when the magnetizing force was 194 divisions of the battery galvanometer scale, or that due to about 4 cells. As the magnetizing force was increased beyond this value the effects obtained gradually diminished, and seemed to reach zero when the magnetizing force was about 1000 divisions, or that due to about 40 cells. The results at this point could not, on account of variations of the magnetizing force due to heating of the coil, be relied on as being accurate.

[Note added June 4, 1879.—This result has not been confirmed by experiments lately made with improved apparatus in which the effects of the heating of the magnetizing coil, formerly a great source of trouble, were to a great extent prevented. No sign of a neutral point was found, although battery powers of from 5 to 79 tray cells were employed for the purpose of magnetizing the nickel bar, which was the actual bar formerly experimented on.]

240. An experiment was then made to find whether this critical value could, as in the case of the soft iron tube, be obtained with a smaller degree of magnetizing force when the magnetometer was placed on a level with a point between the middle and end of the bar. Accordingly, the magnetometer was lowered 10 centims., and brought to a distance of 10 centims. from the axis of the bar. The scale was left in its former position, and thus the distance of the mirror from it was increased to 123 centims. The directive force on the needle was also increased to 6·08 times that due to the horizontal component of the earth's magnetic force. The maximum effect of the application and removal of stress was obtained with a magnetizing force of 50 scale-divisions, or that due to 1 cell. And the critical value was found without difficulty to be 428 divisions of the battery galvanometer scale, or the current due to 10 cells. Beyond that point the effect of "on" and "off" were respectively to increase and to diminish the induced magnetization of the bar, the effect with 707 divisions, or 20 cells, being about $3\frac{1}{2}$ divisions of the scale.

§§ 241–244. *Experiment by the direct Magnetometric Method on the effects of longitudinal stress on the Magnetization of Iron Wire.*

241. In this experiment the magnetizing coil described above (§ 201) was employed. The total length of the wire (which was a piece of Messrs. JOHNSON and Nephew's very soft iron wire, and cut from the same hank as that used in the former experiments) was 97 centims. As only the soldered fastenings of the wire projected beyond the ends of the coil, the magnetometer needle was placed on a level with the top of the coil, and at a distance from its axis of 25 centims., and the distance of the scale of the magnetometer from the mirror was 108 centims.

242. This experiment confirmed in all essential points the results obtained by the ballistic method in the experiments described above (§§ 200–212). The effects of applying and removing a weight of 14 lbs. were respectively to *increase* and to *diminish* by 31 divisions the magnetism induced in the wire by the vertical component of the earth's magnetic force. The amount of this induced magnetism, when only the pan was hanging on the wire, was also 31 divisions. Hence the application and removal of the 14 lbs. alternately doubled, and reduced to its previous amount, the magnetization induced by the earth's force.

When the magnetizing current was 4·25 divisions of the battery-galvanometer scale, the application and removal of pull produced no effect, thus showing that the influence of the earth's magnetizing force was exactly counterbalanced by that due to the current. The maximum effect of applying and removing stress was obtained when the magnetizing force was about 50 divisions (one cell gave 70 divisions). The VILLARI critical value was obtained with 215 divisions of magnetizing current, or 50 times the magnetizing force which balanced the influence of the earth's magnetic force. This is a much greater number than that obtained by the ballistic method; that it is so is no doubt due to the fact that the induction-coil then used was much shorter than the magnetizing coil, and was placed with the centre of its length coincident with that of the magnetizing coil. With the highest strength of current (567 divisions) the effect of "on" and "off" was 8·5 divisions, and the effects were, as in the former experiments, increasing very slowly.

243. The value in absolute units of the vertical component of the earth's magnetic force was calculated from the value of the magnetizing current which balanced it, and the total resistance of coil electrodes and battery, which were all measured for the purpose with great exactness. As before, the electromotive force of one tray cell was taken as one volt, or 10^8 on the C. G. S. system of units. The total resistance with one cell in circuit was 3·503 ohms, the number of turns per centimetre in the coil 19·628, and the fraction of the cell which gave a magnetizing force which just balanced the earth ·0607 of that due to one cell. Hence we have vertical component $=\frac{4\pi \times \cdot 0607 \times 10^8 \times 19\cdot 628}{3\cdot 503 \times 10^9}=\cdot 429$, which must be very nearly its true value at Glasgow. (Compare with § 207 above.)

244. It is stated above (§ 211) that the Villari critical value was higher for the smaller weights. This result was also verified by the magnetometric method.

The following are the numbers obtained with various amounts of pull :—

Soft Iron Wire.

Weights " on " and " off."	Villari critical value.
6 lbs.	248
10 ,,	227
14 ,,	215
18 ,,	190
26 ,,	185

IV. *On Repulsion resulting from Radiation.*—Part VI

By William Crookes, *F.R.S., &c.*

Received June 27,—Read November 21, 1878.

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DIRECTION OF THE LINES OF PRESSURE INSIDE THE RADIOMETER.

387. Although the general character of the reactions which cause repulsion under the influence of radiation is now understood, much light may be thrown on the subject by an experimental examination of the direction and strength of the lines of pressure inside the case of a radiometer on which light is allowed to fall. Radiation will pass almost unimpeded through a very thin, colourless and transparent substance such as mica, but molecular pressure or stress is arrested by such a body (232). By introducing fixed or movable screens in various parts of the case of a radiometer, the direction of pressure can be determined at will, and its force can be modified in many ways, whilst all the other conditions of the experiment remain unchanged.

In the present Part I propose to give the results of a long series of experiments on the action of thin mica screens in modifying the movements of the fly of a radiometer; I shall examine the action of the residual gas, the action of the sides of the glass case, and the applicability of the information so afforded to the construction of instruments of greatly increased sensitiveness for the purposes of research and illustration; and I shall also describe other experiments which have been tried from time to time during the last few years—experiments which at the time were isolated in their bearings, but which now fit into their places.

388. My first experiments were directed to ascertain whether the lines of pressure could be deflected from their path by a thin plate of mica fixed on the fly of an ordinary radiometer. A two-disk radiometer was made, having flat roasted mica disks, lampblacked on one side (fig. 1, *a*). The lampblack surface is represented in the figure by a row of dots close to the surface. In front of the black surface, and separated from it by a space of 1 millim., is fixed a thin disk of clear mica, *b*, about 1 millim. larger in diameter than the blacked disk. When properly exhausted and exposed to the radiation from a candle, the normal direction of movement would be opposite to that shown by the arrows, supposing no screen were present. The screen has, however, changed the direction; the pressure, instead of reacting between the black surfaces and the sides of the glass case opposed to them, is deflected back, as shown at *c*, and the result is rapid rotation in the direction shown by the arrows, the black side now approaching the light.

Fig. 1.

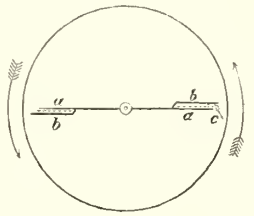
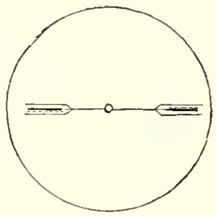


Fig. 2.



389. To test this action more completely, another thin mica disk was fixed on the plain side of the blacked disk, the latter being now guarded on each side, as in fig. 2. The effect of this is to cause an almost total loss of sensitiveness, the fly now only moving very slowly in full sunshine.

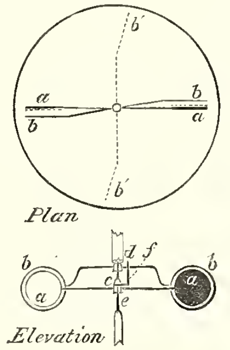
390. Two explanations of the action of the screen in fig. 1 suggest themselves. According to one, the radiation passes through the clear mica screen, and generates molecular disturbance on the black surface. The direction of greatest stress being prevented by the screen from exerting itself between the black surface and the glass bulb, is reflected back in the direction of the arrow, *c*, and produces negative rotation. Another view of the action is this. The radiation, falling on the compound fly, warms it in proportion to the amount absorbed. Molecular pressure is exerted on that side which is most easily warmed, and which most readily parts with its heat to the adjacent gaseous molecules. The roasted mica, *a*, coated on the inner side with lampblack, answers these conditions better than the clear mica; it warms up as a whole, and gives up its heat to the gaseous molecules on the outside, because there they are free to carry off the heat. This side, therefore, becomes the driving surface.

This view is rendered less probable on referring to the behaviour to a standard light of silver-flake mica blacked on different sides. I quote the following from par. 237 of Part V. of this series :—

Material of which the experimental disk in the torsion apparatus is composed.	Amount of repulsion under the influence of a standard light.
Lampblack (standard disk, pith)	100·
Mica, silver-flake, lampblacked on one side, black side exposed ..	151·
Mica, silver-flake	12·5
Mica, silver-flake, blacked on one side, black turned away from light ..	— 9·8

391. It would throw much light on these two hypotheses, and probably enable the true law governing the movements to be disentangled from the individual results, if the screen were easily movable. An instrument was accordingly constructed as shown in fig. 3. *a a* are disks of silver-flake mica, lampblacked on one side. They are mounted on aluminium arms, in the centre of which is a glass cap, *c*, balanced on the needle point, *e*, of the radiometer. The cap, *c*, has a small needle point fused into the top, and this serves as the support of a second glass cap, *d*, to which arms are attached, carrying the clear mica disks, *b b*. The upper cap, with the clear disks, is therefore pivoted on *c*, and will revolve freely round until arrested in one direction by striking against the blacked disks, and in the other direction by meeting the stop, *f*, which prevents the two pairs of arms from separating beyond a right angle, as shown by the dotted line, *b′ b′*. The two pairs of disks will rotate together on the cap, *c*, and will not separate if no repulsive force is exerted between them; but if pressure is generated at the black surface, the disks will be pushed asunder.

Fig. 3.



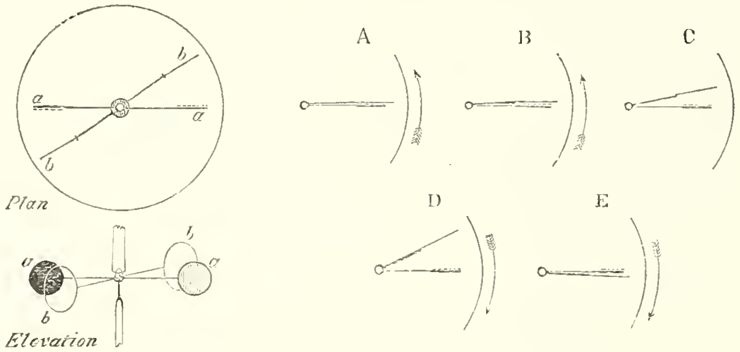
Experiment shows that, in whatever way the light is allowed to shine on the disks, there is an actual repulsive force produced between them, which causes them to separate to the furthest possible extent. By gentle tapping in a faint light, the clear disks can be brought within 2 millims. of the lampblacked surfaces; a candle is now brought near, shaded by a screen; on removing the screen, so that the rays pass through one of the clear disks and fall on the black, the black disk immediately retreats, the clear disk remaining stationary for a moment, and then approaching the light. As soon as the disks have swung apart their full distance they keep there, and revolve like the fly of an ordinary four-vaned radiometer. The reason why the clear disk does not immediately move towards the light in proportion to the movement of the black disk away from the light, appears to be that the slight friction between the upper needle point and cup opposes, at first, such movement.

Another explanation has frequently occurred to me, which, however, I give with diffidence. This is, that when radiation first falls on a black surface the repulsion is produced, not by the chain of molecular impacts between the black surface and the solid body in face of it, but by the reaction between the black surface and the layer of normally-moving gaseous molecules near it, these forming a resisting cushion for the extra active molecules to react against and press back the black surface. This cushion of inactive molecules is, as it were, rapidly eaten into by the active molecules, until in a very short space of time the pressure spans across from one surface to the other.

If the candle is allowed to shine on the plain side of the black disk no movement takes place at first. Very soon the disks both move in the same direction away from the candle, the speed of the clear disk gradually increasing over that of the blacked disk.

392. In the instrument just described, the two pairs of disks were freely pivoted on separate points. Another radiometer was now made, in which the plain and blacked disks could be fixed beforehand in any relative positions. Fig. 4 shows the instrument; the arms carrying the black disks, *a a*, are supported on a cup revolving on a needle point; the arms carrying the clear disks, *b b*, have a circular hole in the centre, which fits over the glass cap. The relative positions of the arms can easily be adjusted by inverting the radiometer and tapping it; when replaced in position they are held together by friction, and the fly moves as a whole.

Fig. 4.



Experiments with this radiometer fully confirm the results obtained with fixed screens. In position A, with the two plates touching, the black surface being inside, the movement under the influence of radiation is similar to that observed when a mica screen was fixed on each side of the blacked disk, the rotation being slow, or altogether stopping if the candle is five or six inches off.

When, by tapping, the disks are removed to about 1 millim. apart, as in position B, the sensitiveness is about at the maximum, and the rotation is in the negative direction, the same as in the radiometer shown in fig. 1.

The speed of rotation in the positions between B and C gradually diminishes as the plates approach the latter position; and in position C, the plates being about 7 millims. apart, the sensitiveness has vanished, no rotation taking place even in a strong light.

Between 7 millims. and 12 millims. apart there is an almost total absence of sensitiveness. Beyond 12 millims. apart the repulsion begins to act most on the black surface, causing positive rotation, as shown in position D. When the two arms are at right angles to each other, positive rotation is strong, and it increases in speed as the clear disk approaches the plain side of the blacked disk; the speed being at its maximum in position E.

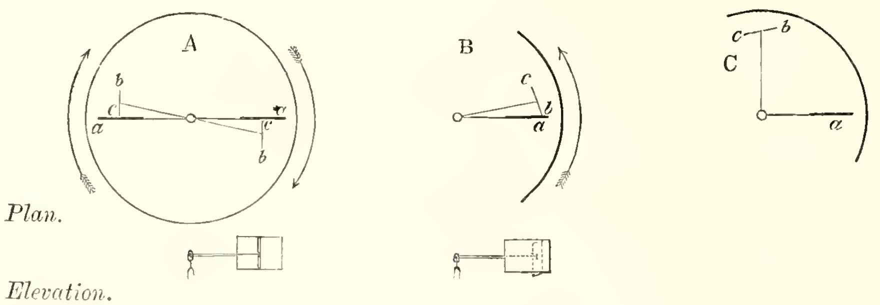
393. I am now able to decide between the two hypotheses advanced in par. 390. The result obtained when the screen is close to the black surface (fig. 4, A), in which the speed was considerably less than when they were 1 millim. apart, shows that the negative rotation obtained in these positions is not due to a molecular disturbance generated on the plain side of the blacked disk, and reacting directly between that and the glass case (the second hypothesis), but that it is caused by the warming up of the black surface by radiation falling direct on it through the clear screen, and the deflection backwards of the lines of molecular pressure thereby generated. At 1 millim. apart, sufficient of this pressure is deflected to produce good negative rotation; at less distances much of the force is taken up in molecular beating to and fro between the disks; while at greater distances the negative rotation due to the reflected force is more or less counterbalanced by the positive rotation due to the molecular impacts exerted directly between the black surface and the glass bulb.

The cause of the maximum positive rotation taking place when the clear screen is close to the plain side of the blacked disk is, that in this position the difference of temperature between the outer sides of the compound disk is greatest.

394. The action of these radiometers is somewhat complicated, owing to the surfaces of the fixed disks being different in absorptive power. Another instrument was therefore made in which the vanes were of polished aluminium, perfectly flat and symmetrical to the bulb. The screens were of clear mica, and instead of being parallel to the vanes, as in the former instruments, they were at right angles to it, as in fig. 5. The aluminium plates are shown at *a a*; the mica screens are at *b c*, they are fixed to their arms at a slight angle, so that in position A the screens touch the centre of the vanes at *c*, and project from them at right angles, whilst in position B the screens touch the vanes at *b*, near the outer edge, meeting them at an acute angle. The arms carrying the screens move easily on the centre cap, as in the previously-

described instruments (392), whilst they retain any position into which they are put by tapping.

Fig. 5.



The candle in these experiments was 3 inches from the bulb. In position A the fly revolved in the direction of the arrows at a speed of 40 revolutions a minute. When the edge *c* of the screen was 3 millims. from the vane *a*, the revolutions were 30 a minute. When the distance was increased to 7 millims. the speed was reduced to 13 revolutions a minute, and at 15 millims. distance the speed was only 4 revolutions a minute. A little beyond this distance the rotation stopped.

The screens were now turned round till they touched the other sides of the vanes, as shown in position B. The speed was thereby greatly increased, being more than 60 revolutions a minute. When the edge *b* was separated 3 millims. from the vanes, the speed was 44 revolutions a minute; at 7 millims. apart the speed was 21 revolutions a minute; and at 15 millims. there were 10 revolutions a minute. At greater distances the speed rapidly diminished, and when the arms were at right angles to each other, as in position C, no rotation could be obtained.

395. The explanation of these movements is not difficult. At position C, when the vanes are not screened, the molecular pressure reacts equally on each side between the bright aluminium and the glass bulb, and no movement takes place. When the mica screen is put in position A, it offers obstruction to the lines of pressure between the inner half of one side of the aluminium plate and the side of the bulb, whilst the corresponding pressure is free to act on the other face. In position B the same occurs, but more of the aluminium plate is obscured by the mica screen. Hence, in each position the balance of pressure is exerted on the unscreened face of the plate, which retreats from the light, the available force being greater in position B than position A.

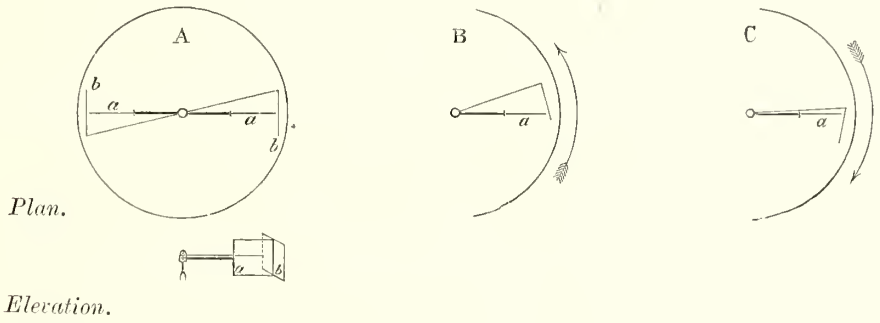
396. The screens were put into position A, and the whole bulb was heated with a spirit lamp. Strong negative rotation took place (screened side retreating); this kept on for 2·5 minutes; the fly now changed its direction and rotated positively, keeping on slowly for 12 minutes.

When the screens were in position B, the same phenomena occurred on heating the bulb and allowing it to cool, the first negative rotation being very rapid, and lasting 3 minutes, and the succeeding positive rotation being slower, and lasting 13 minutes. The negative rotation lasts whilst the glass bulb is giving heat to the fly; when the bulb and fly are at the same temperature the movement stops; and as soon as the bulb and fly are cooling together, the fly being necessarily a little warmer than the bulb (306), the rotation is positive.

A hot iron ring applied equatorially to the bulb, the screens being in position B, gave strong negative rotation, changing to positive on cooling. A hot ring applied to the top of the bulb gave very slight negative rotation, also changing to positive on cooling (298 to 301).

397. The screens in the above-described radiometer were found to produce the greatest sensitiveness to radiation when in position B (fig. 5). Another instrument was therefore made, in which the indications thus given were followed out, the screens being placed in a different position, so as better to obstruct the molecular reaction between the vanes and the glass bulb. Fig. 6, A, shows the arrangement. The bright aluminium vanes, *a a*, are connected by thin arms, and are pivoted on a glass cap and needle point in the usual way. The mica screens, *b b*, are supported on independent arms, so that by tapping they can be put in any relative position in respect to the vanes *a a*. The screens are so fixed to the arms that they will pass between the vanes and the glass bulb, and be at right angles to them, as shown in the elevation. In position A, when exposed to light, or when heated with a lamp, no movement was produced. In position B strong rotation was given in the direction of the arrow, and when the screen was adjusted to shade off the other side, as in position C, equally strong rotation was produced in the opposite direction.

Fig. 6.



On heating the bulb with the screens in position B or C, strong negative rotation was caused, reversing to positive on cooling.

398. In a preliminary notice[*] sent to the Royal Society, November 16, 1876, when discussing the action of light upon the cup-shaped vanes of a radiometer, I advanced the hypothesis that some of the phenomena might be explained on the assumption that the molecular pressure acted chiefly in a direction normal to the surface of the vanes; and concluded that "it would not be difficult to test this view experimentally, by placing a small mica screen in the focus of a concave cup, where the molecular force should be concentrated."

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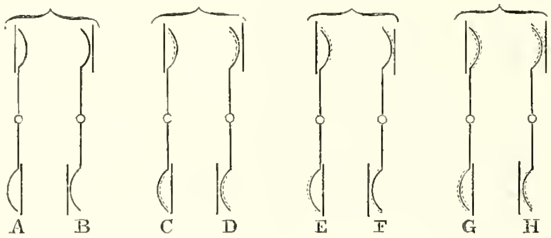
[*] Proc. Roy. Soc., No. 175, 1876, vol. xxv. p. 304.

Following out this idea, a series of eight radiometers were made, having flies as shown in plan in fig. 7, A, B, C, D, E, F, G, H. The cups were of thin aluminium, and clear mica disks were attached in the positions shown in the drawings. In the pairs A and B the aluminium cups were bright on each side, and the mica screens were, in A, facing the concave side, and in B facing the convex side. In the pairs C and D the aluminium cups were blacked on the concave side; in E and F they were blacked on the convex side; and in G and H on both sides. The screens in each pair followed the arrangement in A and B. The whole series compares with the similar series without screens given in Part V., par. 322.

Fig. 7.



Experiments were tried with each of these in succession; the radiometers being exposed—1, to the direct rays of a candle, 3 inches from the bulb; 2, to the same candle, after having cut off the light from the concave side by an opaque screen; and 3, after screening the light from the convex side.

I will not describe each experiment in detail, but will briefly record that in each case, when the eight radiometers were exposed to the candle shining on both sides of the fly, the movement was one of rotation, being very strong with G; less so with B, C, F, and H; moderate with E; and only slight with D.

When the opaque screen was placed so as to shade the candle light from the concave side of the fly, there was strong rotation in the case of E, F, G, and H; moderate rotation with A, B, and C; but no rotation with D.

When the light was cut off from the convex side of the fly, there was strong rotation with B, C, G, and H; moderate with D and F; and no rotation with E.

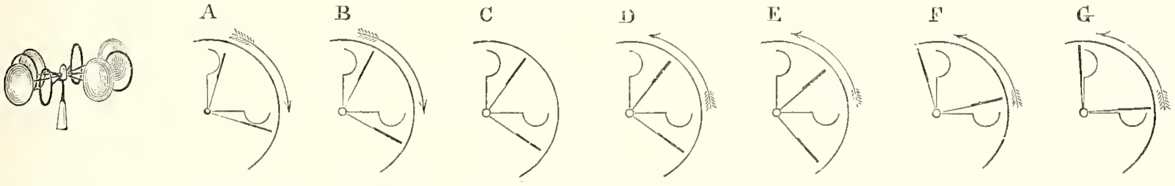
In every case where rotation was produced the direction of movement was the same, the mica screen approaching the light and the aluminium cups being repelled, irrespective of the side turned to the light or of the position of the black coating.

399. On reference to the previously-quoted experiments on similar radiometers without mica screens in front (322), it will be seen that in two instances only was there no rotation, viz., 1, when the cups were blacked on the concave side, and the light was only allowed to shine on the convex side; and 2, when the cups were blacked on the convex side, and the light was allowed to shine only on the concave side. It will be seen in the present experiments that rotation is produced in all but the two analogous forms, viz., D and E, the former when the concave side is screened

from the light, and the latter when the convex side is screened from the light. The screen in these cases, being interposed between the cups and the light, could not assist rotation. In other cases the mica screen appears to intensify the action of repulsion.

400. It was now necessary to ascertain what effect would be produced by varying the distance of the screens from the metallic cup-shaped vanes. The simplest form of cups, with no lampblack on them, was chosen, and to avoid the difficulty which was sometimes found in starting the rotation and keeping it uniform, four vanes were used instead of two. The instrument had the form shown in elevation in fig. 8, the aluminium cups are affixed to rigid arms pivoted on a glass cap and needle point, and the four clear mica disks, acting as screens, are also pivoted on a quadruple arm capable of adjustment in any relative position in respect to the arms carrying the cups.

Fig. 8.



Elevation. 6 *per min.* 3 *per min.* 0. 6·5 *per min.* 12 *per min.* 30 *per min.* 50 *per min.*

The candle was kept at a distance of 4 inches from the bulb. The first experiment was tried with the disks as close as possible to the convex side of the cup, but not touching, as in position A. Rotation took place in the direction of the arrow, at the rate of 6 revolutions a minute.

In position B the screens were adjusted 2 millims. from the convex surface. The direction of rotation was as before, but the speed was reduced to 3 revolutions a minute.

The space between the convex side and the screens was now increased to 3 millims., as in position C. No rotation took place.

The vanes were still further separated from the cups, being now 6 millims. from the convex sides, as in position D. The direction of movement changed, rotation being produced, in the direction of the arrows, at the rate of 6·5 revolutions a minute.

When the screens were midway between the cups, *i.e.*, 10 millims. from the convex side and 10 millims. from the plane of the concave side, as in position E, the revolutions in the direction of the arrow were 12 a minute.

In position F, with the screens 5 millims. from the convex sides of the vanes, the speed in the direction of the arrows was 30 a minute.

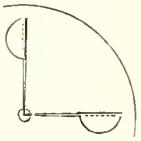
When the disks were brought as close as possible to the concave sides of the cups, as in position G, the speed was 50 a minute.

401. These experiments, taken in conjunction with those immediately preceding, prove that, under the influence of radiation, a pressure is exerted between each side of the metal cup and the side of the glass case facing it, the pressure from the convex side

being greater than that from the concave side. The proportion is roughly 50 to 6, for when the molecular pressure is as completely as possible cut off between the convex side of the cup and the glass bulb, as in position A, fig. 8, the rotation due to the repulsion generated by the concave side is at the rate of 6 revolutions a minute; but when the screen is in position G, fig. 8, cutting off the pressure between the concave side and the bulb, and allowing that from the convex side fully to act, the revolutions become 50 a minute. In position C, the screen is at such a point that the active pressure is nearly equal from each side of the cups, and the result is neutrality. In the other positions a balance of pressure, represented by the stated revolutions per minute, is allowed by the screens to escape and react against the glass bulb.

402. I now endeavoured to ascertain if there was a difference between clear mica and blacked mica screens in front of metal cups. A radiometer was made similar to the one last experimented with (fig. 8), but the mica screens were lampblacked on the sides facing the concave surfaces of the cups. This altered the action considerably. At an exhaustion of 47 M, and with the candle 3 inches off, there was no movement when the screen was close to the concave surface, as at fig. 9. When the distance was increased to 2 millims., there was very slow positive rotation at the rate of 1 in $1\frac{1}{2}$ minute. With the screen midway between the disks, the speed was scarcely increased, and when it was 2 millims. from the convex surface the rotation was only at the rate of 1 per minute.

Fig. 9.

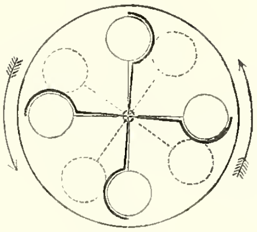


The exhaustion of 47 M was chosen, as that is near the point of maximum sensitiveness for ordinary radiometers (334, 382). I next tried the effect of exhausting the instrument to a higher point, at which an ordinary radiometer would begin to lose sensitiveness. At an exhaustion of 26 M, the speed was found to be 5 revolutions per minute in the negative direction, when the screens were in the position of fig. 9. With the screens 2 millims. from the concave side there was no motion, and beyond this point the more the screens approached the convex sides of the cups the more decided was the positive rotation.

403. The mica screens were now arranged in a horizontal plane, the metal cups remaining vertical (fig. 10). The exhaustion was 8 M, the candle remaining as before. In the position shown by the black lines the speed in the positive direction was 15 revolutions per minute. When the screens were moved to the positions shown by the dotted lines, the speed was reduced to 6 a minute; and at any intermediate distance between these two positions, the velocity of rotation likewise varied in proportion.

I was unable to produce negative rotation, or a position of rest, with this form of instrument.

Fig. 10.



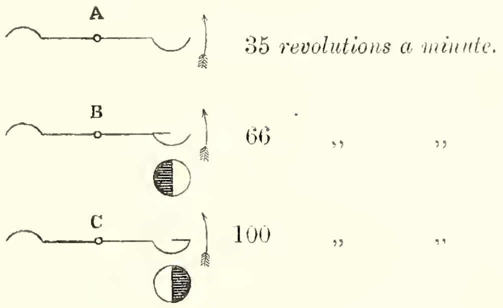
Plan.

404. My next endeavour was to ascertain what difference in action was caused by an alteration in the size and shape of the screens. A radiometer was constructed like the one shown in fig. 8 (400), except that the mica disks which acted as screens were 2 millims. smaller in diameter than the metal cups. The screens were adjustable. This instrument was found to be less sensitive than the one described in par. 400, where the screens and cups were of the same diameter, and under the influence of the candle it behaved differently. When the screens were close to the concave side of the cup, as shown at G, fig. 8, the revolutions were 20 per minute. When they were midway between the cups, as at E, fig. 8, the revolutions were 10 per minute. With the screens close to the convex sides, as at A, fig. 8, there was no movement at all. When rotation occurred it was in each case positive. The rotation when the screens were 2 millims. from the convex sides of the cups, as at B, fig. 8, was positive at the rate of 3 revolutions a minute. With the larger screens, as described and figured in par. 400, the rotation in this position was *negative*, at the rate of 3 per minute.

The exhaustion in these experiments was 12 M.

405. Other radiometers were made of the forms shown in fig. 11. In A, the mica screen was absent; in B it was in the form of a half disk covering the inner half of the cup, and in C a similar screen covered the outer half of the cup. The form A was the least sensitive, and the form C most so; the speed in forms A, B, and C being 35, 66, and 100 revolutions a minute.

Fig. 11.



These experiments show that the inner half of the concavity of the cups sends a considerable amount of molecular disturbance to react on the glass case, although it is not so effective in this respect as the outer half of the concavity.

406. The metal cups used in these experiments were now lampblacked on the inside, and their speed compared with what it had been when they were bright, as at fig. 11, A. It was increased from 35 to 75 revolutions per minute. A mica disk, lampblacked on the inside, was now fixed close to the concave side of the cup, when the speed rose from 75 to 150 revolutions per minute. Had the blacked mica screens been some little distance from the cups, this doubling of the speed would have been easily understood, as the screens would have acted as additional driving vanes to the radiometer; but being close to the cup they can only act by absorbing and radiating back the heat given out from the bright concave surface of the cups, and thereby increasing the difference between the temperature of the black convex and bright mica surfaces.

407. The experiments hitherto described have been with mica screens. Mica being a bad conductor and radiator of heat, I next tried replacing it by aluminium, to see if the employment of a good conductor as a screen made any radical change in the phenomena. A radiometer was constructed like the one shown at fig. 8, with the exception that the screens as well as the cups were of aluminium, and the outsides of the cups were coated with lampblack, as at fig. 7, E. The diameters of the cups and the disks were the same.

The experiments were tried with the screens in the same positions as are shown in fig. 8, so I will refer to that figure in illustrating the results. The screens could not be got closer than 4 millims. to the black convex surfaces (intermediate in position between C and D, fig. 8), so I could not ascertain if there was a point of neutrality; in this position the speed was at the rate of 6 per minute in the positive direction. In position G, the screens being as close as possible to the concave surfaces of the cups, the positive rotation was at the rate of 120 revolutions a minute; and at any intermediate point between these two extreme positions the speed, in the positive direction, was proportionally intermediate between 6 and 120.

Making allowance for the increased sensitiveness communicated to the cups by blacking their convex surfaces, these results agree fairly well with those illustrated in fig. 8, and prove that the action of the screens is almost entirely one of mechanical obstruction, and does not depend on their special power of conducting, absorbing, or radiating heat.

408. In most of the instruments hitherto experimented with, the screens have been in such a position in respect to the glass bulb that the secondary action which the candle may set up by reason of its shining on the bulb has tended somewhat to complicate the results. The cup-shape of the vanes is also not the simplest form with which to investigate these reactions. A careful examination of the results obtained in the foregoing experiments with screens, throws much light on the physics of the

attenuated gas in the radiometer, and on consideration it seemed likely that by a slight modification in the shape of the vanes and screens, much that was obscure and contradictory might be rendered clear and harmonious.

Instead of cups I employed hemi-cylinders for the vanes, and the screens were square plates of mica held at right angles to the supporting arm, so that they should always be in a plane parallel to a tangent to the curvature of the bulb which they faced. Thus any reaction which might take place between the bulb and the surface of the screen could have no tendency to cause rotation in either direction.

Fig. 12.

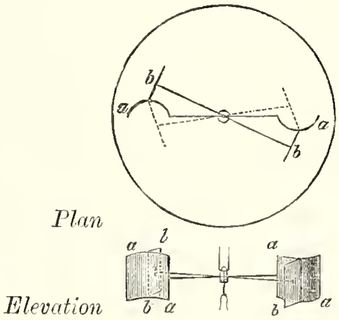
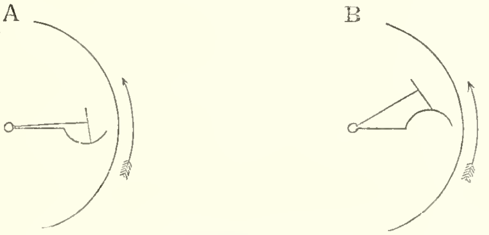


Fig. 12 shows the radiometer in plan and elevation. *a a* are half cylinders of bright aluminium; *b b* are the mica screens on an adjustible arm, so that by tapping they can be shifted into any relative position in respect to the hemi-cylinders, from touching the convex surfaces, as shown by the black lines, to touching the concave surfaces, as shown by the dotted lines. The radiometer was kept attached to the pump during the experiments, so that the degree of exhaustion could be varied, whilst the distance between the screens and vanes was at the same time altered.

409. The vanes and mica screens were set as shown in fig. 13 *a*, A, the screens touching the concave surfaces. A candle was placed 3 inches from the bulb, and the pump was set to work. At a pressure of 540 M, positive rotation commenced in the direction of the arrow. A tendency to positive rotation was observed at a lower exhaustion, but it is best to take the first point of permanent rotation. When the screen was moved away from the vane, rotation stopped.

Fig. 13 *a*.



Exhaustion 117 M.

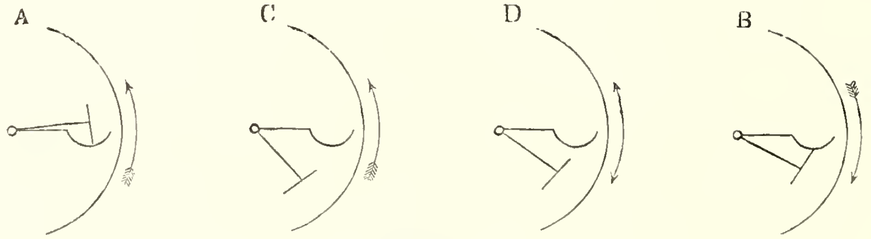
The vanes were now set touching the convex side as at fig. 13 *a*, B. The exhaustion continued, and when it had reached 117 M, permanent negative rotation commenced in the direction of the arrow. As the position of the screen was gradually shifted from B to A, the exhaustion remaining 117 M, the rotation got quicker and quicker, showing that for this exhaustion, position A is most sensitive, and position B least sensitive.

410. The screen was put into position 13 *a*, B, touching the convex side of the vane, and the pumping was continued. The positive rotation got a little quicker, until at 82 M the speed was 5 revolutions a minute. The screen was moved over to position A, the pressure remaining the same, when the speed in the positive direction was 13 a minute. As the screen was moved round from touching the convex side to touching the concave side, the speed of positive rotation of the fly increased.

411. The screens were now replaced in position B, touching the convex side of the vanes, and pumping was continued. As the exhaustion increased the rotation got slower and slower, until at a pressure of 87 M, the vanes stopped. Before the movement ceased the vanes acted as if, in revolving, they came successively under the influence of a strong attractive force. Each vane went quickly up to the spot of warm glass opposite the candle, then hesitated, and got past with difficulty. This hesitation became more and more decided, till at last a vane refused to pass the dead centre, when it swung back and the fly set equidistant from the candle.

On continuing the exhaustion a tendency to negative movement was soon observed, and at an exhaustion of 12 M, permanent negative rotation commenced, the screens all this time remaining in position B. At this pressure of 12 M, the speed in position A was 46 times a minute, positive. As the screen was moved from the concave side the speed got less. When it was 6 millims. from the convex surface (position C, fig. 13 *b*), the speed was 3 revolutions a minute, positive; when they were 3 millims. apart (position D), the rotation stopped; and when the screens and convex surfaces touched (position B) there was negative rotation, at the rate of about 1 turn a minute.

Fig. 13 *b*.

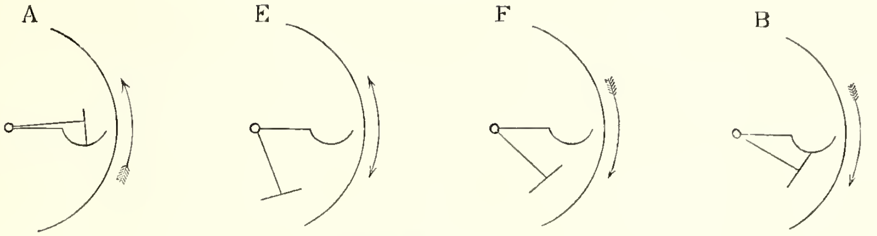


Exhaustion 12 M.

412. When in position 13 *b*, A, no change in direction of rotation takes place, whatever the exhaustion. There is an increase of speed up to a rarefaction of about 15 M, after which further exhaustion has little or no effect on the speed.

413. The exhaustion was now carried to ·18 M, and the following observations were taken :—

Fig. 13 c.



Exhaustion ·18 M.

In position A (Fig. 13 *c*), the speed was 53 revolutions a minute positive.

In position E, the screen being 13 millims. from the convex side of the vanes, there was no rotation.

In position F, the screen and vanes being 5 millims. apart, there was negative rotation of 4 revolutions a minute.

In position B, the vanes and screens touching on the convex side, there was negative rotation at the rate 12·5 turns a minute.

414. From these observations, which are in conformity with several hundreds of experiments tried at different distances and pressures, the following laws can be traced.

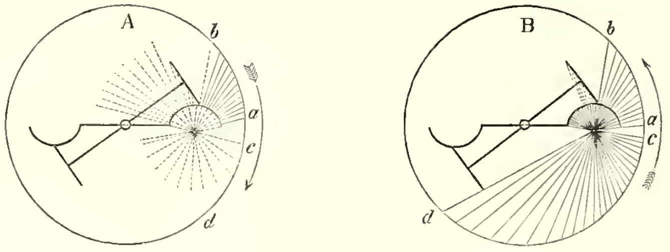
a. When the screen touches the concave surface of the vanes, the rotation is always positive. It commences at a low exhaustion, increases in speed till the rarefaction is so high that an ordinary radiometer would begin to lose sensitiveness, and afterwards remains at about the same speed up to the highest rarefaction yet obtained.

b. At any rarefaction after 87 M, there is a neutral position for the screen. When it is on the concave side of this neutral position the direction of rotation is positive, and when on the convex side of the neutral position the direction of rotation is negative. The speed of rotation is greater as the vanes are further removed from this neutral position on either side.

c. The position of this neutral point varies with the degree of exhaustion. Thus, at 12 M the screens must be 3 millims. from the convex side ; at ·18 M they must be 13 millims. from the convex side. The higher the exhaustion the greater the distance which must separate the convex side of the hemi-cylinders and the screens.

415. The various behaviours of these screened flies can be explained in the following manner. Fig. 14, A and B, represents the instrument in full sized plan, when the screen touches the convex surface of the hemi-cylinder. The radial black and dotted lines are supposed to represent the direction and extent of lines of pressure in a manner sufficient to enable the mind to follow the train of reasoning, but without implying that they indicate more than a very limited part of the whole action. It will be remembered that in this position, when the exhaustion is 117 M, the fly rotates positively (409), and that at an exhaustion of ·18 M, the fly rotates negatively (414).

Fig. 14.



Exhaustion 117 M. *Exhaustion* ·18 M.

I have already shown (312) that when thin aluminium vanes are exposed to candle light the metal becomes equally warm throughout, and a layer of molecular pressure is generated at its surface. The thickness of this layer of pressure, or the length of the lines of force of repulsion, varies with the degree of exhaustion, being longer as the exhaustion increases. The lines of force radiate from every part of the warm metal, and being strongest in a direction normal to the surface it will be convenient only to recognise these in a discussion of their action. The force of repulsion is also greater the closer the repelled body is to the generating or driving surface, and the force diminishes rapidly as the distance increases, according to a law which has not yet been accurately determined,* but which does not appear to be the law of "inverse squares." It will also simplify matters if the interfering action caused by the pressure generated by the warm glass bulb is neglected.

In fig. 14, A, the exhaustion is low (117 M), and the thickness of the layer of pressure I assume to extend to a distance of 10 millims. from the surface of the aluminium vane. The lines in the diagram are represented as extending for that length from the hemi-cylinder in directions normal to the surface. Where the lines of force extend from the fly to the glass case, as at *a* to *b* and *c* to *d*, pressure is exerted along the line, and repulsion ensues; these are represented by black radial lines. But where the lines of force do not reach to the glass, as shown by dotted lines, no pressure is exerted. In the position and under the conditions shown in fig. 14, A, the rotation must be in the direction of the arrow or positive, as the pressure in the positive direction exerted by the lines of force between *a* and *b* more than counterbalances that in the negative direction exerted by the lines between *c* and *d*. Inasmuch as some active lines of force are cut off by the screen, increase of the distance between the screen and the convex surface of the vane will allow more rays of force to become active, and will increase the speed in the positive direction.

416. In fig. 14, B, I have represented the action when the screens and vanes are in the same position as in 14, A, but at an exhaustion of ·18 M. Here the thickness of the layer of molecular pressure is supposed to extend to a distance limited by the

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* See the Author's "Experimental Contributions to the Theory of the Radiometer." Proc. Roy. Soc., November 16, 1876, No. 175, vol. xxv. p. 310.

size of the bulb. The positive pressure between *a* and *b* is now overweighted by the negative pressure between *c* and *d*, and rotation is therefore produced in the direction of the arrow, or negatively. If we imagine the arm carrying the screens to be rotated on the axis, it will be seen that as one screen leaves the convex side of the vane, more of the positive rays of force are rendered active, whilst negative rays are cut off by the lower screen. Hence, as the screens are moved round, the fly will gradually rotate less strongly in the negative direction ; it will become neutral when the screens are in such a position that the opposing forces balance ; and ultimately it will rotate positively. This is what experiments prove to be the case (413).

I have tested this explanation by comparing with it the whole of the phenomena obtained with movable screens in various positions and at different pressures as described in pars. 391 to 413, and in no case does it fail. There is therefore high probability that it is true.

417. An apparatus (fig. 15) was constructed not differing in principle from the last, but having, in addition to the aluminium hemi-cylinder and movable mica screen, a small rotating fly made of clear mica, mounted in such a way that it could be fixed, by means of an exterior magnet, in any desired position inside the bulb ; the screen was also capable of adjustment by means of another magnet. The aluminium hemi-cylinder was in this apparatus immovable. The adjustible indicator, being very small in diameter, in comparison to the other parts of the apparatus, and being easily placed in any part of the bulb, was expected to afford information as to the intensity and direction of the lines of pressure when a candle was brought near the bulb.

It will be impossible within reasonable limits to give more than a few general results which I have obtained with this apparatus. Experiments have been tried :—

a. With the screen in different positions in respect to the hemi-cylinder ;

b. With the indicator in different parts of the bulb ;

c. With the candle at different distances from the hemi-cylinder, on one side or the other ; and

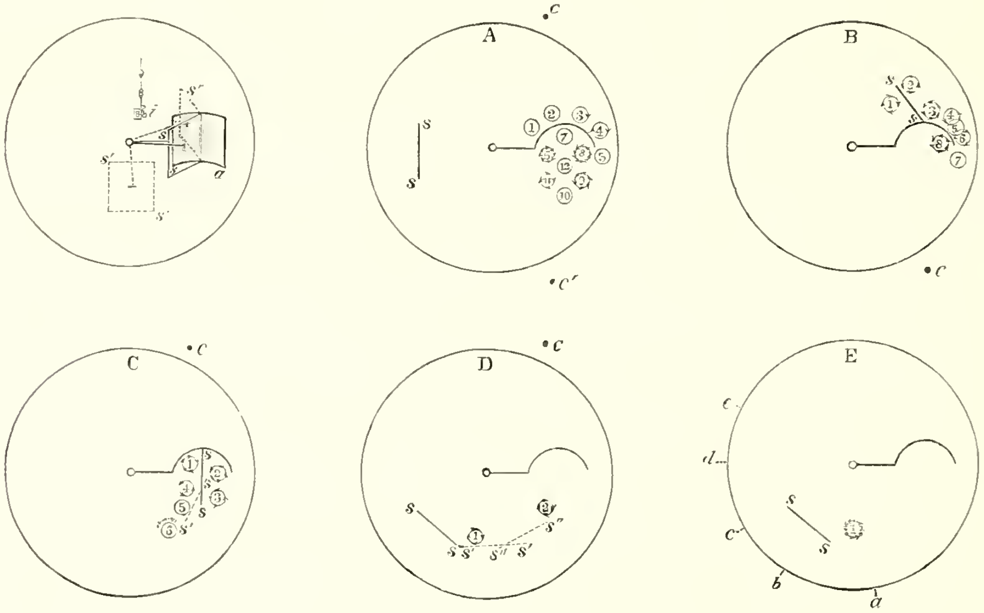
d. With the degree of exhaustion varying between wide limits.

These four conditions have been varied in numerous ways, and each time experiments have been tried both with and without an alteration in some or all of the other conditions. Although much time has been spent in this mode of experiment, it will be understood that, with so vast a number of possible combinations, only a few of the most obvious can be thoroughly investigated.

418. I will neglect the results at inferior exhaustions, and will confine myself to results obtained with a rarefaction of 12 M.

Fig. 15 gives a perspective view of the working parts of the apparatus ; *a* is the aluminium hemi-cylinder rigidly fastened to an arm fixed to the bulb ; *s s* is the mica screen, drawn up close to the concave side of the hemi-cylinder ; *s′ s′* shows the same screen when moved some distance away. The screen can be moved right round the bulb, till it comes into position *s″*, where it is shown touching the convex surface of the aluminium. The indicator *i*, which can be put into any part of the bulb, is a very small and light fly of a radiometer, the vanes being of clear mica not blacked on either side, and each being about 2 millims. square. They rotate under the influence of a very slight force, and the direction of rotation gives the direction of the force, and also shows on which side it is increasing or diminishing.

Fig. 15.



419. Fig. 15, A, is a plan of the hemi-cylinder *a*, with the mica screen *s s* put as far from it as possible, so as to exert no appreciable influence on the movement of the indicator. The small circles, 1, 2, 3, &c., show the different positions in which the indicator was placed, the candle being at *c* when the indicator was (apparently) below the hemi-cylinder, and at *c′* when it was above—the object being to prevent the light from the candle, or the force from the heated glass near it, from falling direct on the indicator; my wish being that the indicator should be moved as much as possible by the force generated by the hot aluminium.

The direction of the arrow heads on the circles shows the direction of rotation taken by the indicator, and the number of arrow heads show the relative velocity. In positions 1, 2, 5, 7, 12, 10, there was no rotation. A reference to the hypothetical lines of force given in fig. 14, B, shows that along the line of positions 7, 12, 10, there would be equality of force on each side. At 5 also is another point of neutrality, the indicator being midway between the positive rays of force *a b*, fig. 14, B, and the negative rays *c d*, in the same figure. Positions 8 and 6 are those of greatest movement; this also is intelligible, as the indicator is struck almost entirely on one side. Position 8 is slightly the most favourable to sensitiveness; this is probably owing to that end of the aluminium hemi-cylinder being a little warmer than the other, from the conduction of heat away along the arm. The direction of movement in 8 and 6 is also in accordance with theory. The slight action on the convex side of the curved metal plate is probably due to the lines of force acting here in both directions to and fro between the metal and the glass. This would cause the pressure here to be much greater than on the concave side, but at the same time would cause much less rotation. The slight movement in position 4, and its direction, is caused by the molecular disturbance generated on the surface of the aluminium, where it is nearest the glass, being struck back in somewhat greater quantity than at position 3, where the interval between the metal and glass is greater. In positions 1 and 2 the lines of force issuing from the convex surface strike each side of the indicator equally, the opposing face of the glass being too far off to cause a reaction.

420. Fig. 15, B, represents the position in which the mica screen, *s s*, was next placed, touching the convex side of the hemi-cylinder. The candle was at *c*, and the circles represent the successive positions of the indicator. The screen has now entirely altered the disposition of the lines of pressure. Position 3, where in the last experiment the indicator was almost stationary, is now the position of greatest speed. The indicator, in fact, has the rays which, in the absence of the screen, struck it on both sides, now acting on one side only, and the result is a rapid rotation in the direction of the arrows. No. 1 is the next best position, and then 2. The velocity and direction of movement in these two positions are clearly due to rays of force not acting so strongly on the side next the screen. As the indicator is moved in the other direction, away from the most sensitive position 3, the influence of the screen diminishes; at 4 the rotation is slight; and at 5 there is no movement. In position 6 the influence of the screen has almost disappeared, the motion now being opposite to that at No. 4. At 7 there is another position of neutrality, and at 8 the motion is again reversed. The movements in the last three positions correspond with those in similar positions when the screen was absent (fig. 15, A).

421. The screen was now moved round till it touched the hemi-cylinder on the concave side, as shown at *s s* in fig. 15, C. On comparing the direction taken by the indicator with that in fig. 15, A, it will be seen that no change is produced in positions 1, 2, and 3 (corresponding to 6, 8, 9, fig. 15, A), the speed only being reduced. In positions 4 and 5 the interference of the screen has caused the direction of rotation to change. The lines of pressure, being deflected by the mica, now radiate towards the centre of the bulb, and being stronger the nearer they are to the generating surface of aluminium, cause the indicator to rotate, as shown at 4 and 5. In position 6 there is no movement when the screen is close to the hemi-cylinder, but when it is brought into the position shown by the dotted line *s′ s′*, rays are deflected, and the indicator is rotated in the direction shown by the dotted arrow.

422. The screen thus renders evident the existence of active lines of force at great distances from the generating surfaces. A reference back to fig. 14, B, shows certain hypothetical lines of force stretching across the bulb from the generating hemi-cylinder

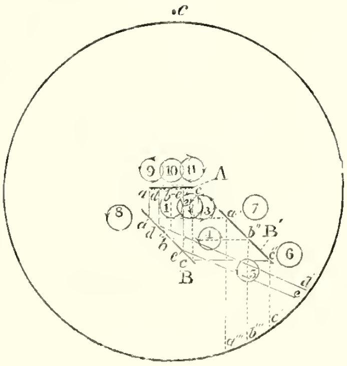
to *d*, and being obstructed by the screen. The present apparatus seemed capable of rendering those rays sensible. The indicator was accordingly placed in position 1, fig. 15, D, the screen being quite away from it and the candle being at *c*. The rays of pressure are here supposed to start from the hemi-cylinder, and strike both sides of the indicator equally; the result is that no movement is produced. Without moving the indicator, the screen was brought into position *s s*. The indicator immediately rotated rapidly in the direction of the arrow heads. The action of the screen here is to strike back the impinging molecules, and thus cause a neutralisation of the pressure acting on the left side of the indicator. The line of pressure acting on the right side of the indicator passes on unimpeded, and therefore causes rotation. This explanation was verified by putting the indicator in position 2, and gradually bringing the screen nearer to it. With the screen at *s*, there was no movement. When the screen got to position $s' s'$, movement of the indicator commenced; and when it reached position $s'' s''$, the rotation of the indicator was rapid. A glance at the lines of force in fig. 14, B, will show how closely theory and experiment agree.

423. The action of heat applied to the bulb was now tried, the indicator being in position 1, and the screen at *s s*, fig. 15, E. A spirit flame was applied for a few seconds to the bulb at *a b*. There was rapid rotation of the indicator in the direction of the arrow heads, with strong negative rotation on cooling. When the action had ceased, the part of the bulb between *c* and *d* was heated; the result was rapid rotation in the opposite direction to the arrow heads. These movements are perfectly explicable on the supposition that lines of pressure radiate from the hot internal surface of the glass, and, being partly cut off by the screen, strike the left or the right side of the indicator as the case may be. When the bulb is heated in such a part that the lines of force are entirely cut off, or are not at all obstructed by the screen, as at *d e*, or *b c*, no rotation of the fly is produced.

424. In the hope of deciding whether the force was capable of true reflection from a surface on which it impinged, or whether it was only deflected out of its course as a current of air would be, the apparatus represented in plan in fig. 16 was fitted up. A represents a plate of aluminium, lampblacked on each side, firmly fixed in the centre of a large glass globe capable of good exhaustion. B and B′ are clear mica screens, a little larger than the aluminium plate, and capable of being held in any desired position by exterior magnets. A little exploring fly, as in the last experiment, shows the direction and strength of the lines of pressure. The positions into which this indicator was brought are shown by small circles. The candle was at *c*. It was found that similar results were obtained wherever the candle was placed, allowance being made for the interfering actions of the screens, but at *c* there was less interference than at any other part outside the bulb. The force is supposed to issue from the plate A in a direction normal to its surface. The dotted lines $a\ a'\ a''\ a'''$, $b\ b'\ b''\ b'''$, $c\ c'\ c''\ c'''$, show the direction taken by the lines of pressure, supposing specular reflection to take place from the surfaces of B and B′.

In positions 10 and 1 no rotation was produced; rotation in one direction was produced at 9, and in the other direction at 11; these results agree with those given with apparatus 15. In position 2, rotation was obtained in the direction of the arrow heads, and it was still stronger in the same direction at 3. These also agree with theory. At 4 and 5 there was slow rotation in the direction of the arrows. Were the lines of force to follow the path shown by the dotted lines *b b′ b″ b‴*, &c., there should have been no rotation in these positions, but supposing the line *c′ c″* to have struck the indicator in position 5, there might be slow rotation in the opposite direction to what is here shown.

Fig. 16.



The black lines *d d′ d″*, *e e′ e″*, show the direction which the lines of pressure might take, supposing they were merely deflected out of their course by the screen B. On this supposition the movements of the exploring fly are quite reasonable. At 4 the force, striking one side only, would cause rotation in the direction as shown, and at 5 rotation in the same direction would also take place, but not with the same speed, owing to its being more surrounded by the stream. On moving away the screens, the fly remaining at position 5, the rotation stopped. The indicator was successively brought to a great many positions in respect to the screens, and the screens were also moved about to different parts of the bulb, and the results were in all cases easy of explanation on the *de*-flection theory, whilst they were sometimes contradictory on the *re*-flection theory. At 8 there was rotation due to the screen B cutting off some of the force from one side of the fly. At 6 and 7 there was no rotation, the screen B′ having obstructed all the force.

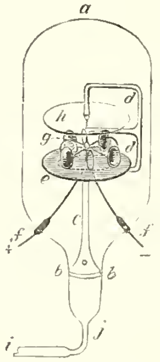
The molecular pressure at the exhaustion found best for these experiments diminishes greatly as the distance from the plate A increases; it is not easy, therefore, to say whether a diminished speed, as in positions 2, 4, and 5, is due simply to increased distance, or to some action of the screens; but from the indications given generally by the exploring fly, I have come to the conclusion that the force is not reflected in the sense that light is reflected, but that it is merely deflected from a surface against which it strikes, as would be the case with a molar wind.

Received October 16, 1878.

ACTION OF HEAT APPLIED INSIDE THE RADIOMETER.

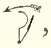
425. In a previous paper it has been shown that phenomena, feeble and contradictory when caused by radiation external to the bulb, became vigorous and uniform when the radiation was applied internally by the agency of an electrically heated wire (336 to 384). It was hoped that some of the more obscure phenomena shown by the deep cups with movable screens in front (400, *et seq.*) might be intensified if set in action by a hot wire.

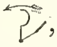
Fig. 17.



An apparatus was made similar to the one described and figured in a former paper (360, fig. 32, C D), but instead of the movable fly being of sloping mica vanes, it was composed of deep metallic cups with mica screens in front, capable of being brought by tapping any desired distance from the cups. This fly is similar to the one shown in fig. 8, paragraph 400, the complete instrument being shown in the annexed cut, fig. 17. It consists of a wide glass tube, *a*, *b b*, sealed off and blown round at *a*, and drawn off narrow at the end *b b*. Inside is a stem *c d d*, with branches. A disk of silver-flake mica, *e*, lampblacked on the upper side, rests on the platinum wire ring, the ends of which are joined to thicker platinum wires passing through the glass at *f f*. The disk *e* is pierced in the centre, through which passes the stem *c*, carrying a needle point on which the compound fly *g* rotates. Above the fly is a flat disk of clear mica, *h*, having a glass cap in its centre, and rotating on a needle point. The fly and the clear mica disk are supported independently of each other on separate needle points held in the glass rods *c*, *d d*. The tube *i* connects the apparatus with the mercury pump, and when the apparatus is exhausted to the requisite degree it is sealed off at *j*. The wire is heated by two GROVE's cells connected with *f f*, the current passing through a contact key, resistance coils, and galvanometer, to keep it constant, as explained in a former paper (359).

The plan of the adjustable fly with aluminium cups and mica screens is shown in a previous drawing (fig. 8, par. 400). The different positions of the screen are also shown on the same figure, so I shall refer to the positions there given to illustrate the results here obtained.

426. The mica screen was in the first experiment adjusted in the position shown at fig. 8 A, or as close as possible to the convex surfaces of the cups. The rarefaction was brought to 12 M. Electric contact being made, the wire ring became red hot, heating the blacked mica disk which rested on it. The fly commenced to revolve in the negative direction , at first slowly, then quicker, and afterwards slowly again, finally coming to rest after it had completed 34 revolutions. It now oscillated a little, and then remained quiet, although the battery contact was kept down, and the wire was glowing red. After remaining thus for a little time battery contact was broken. Instantly the fly acted as if released from a state of tension, and rotated in the positive direction , coming to rest only after a considerable time.

427. The screens were now adjusted midway between the cups, as shown at fig. 8, E, and the foregoing experiment was repeated. As soon as the wire was made hot the fly commenced to rotate negatively, and after making $20\frac{1}{4}$ revolutions it stopped. Now, however, instead of remaining quiet as in the last experiment, the fly commenced rotating in the opposite direction , and kept moving at an uniform speed of 50 revolutions a minute, as long as battery contact was maintained. On breaking contact the fly considerably increased its speed for some time, then gradually got slower, and finally came to rest.

428. In this experiment the screens were brought close to the concave faces of the cups, as in fig. 8, G. The fly made 15 revolutions in the negative direction , it then stopped and rotated positively at a speed of 86 revolutions a minute, keeping at this rate as long as the wire remained hot. On allowing the wire to cool, the speed got much faster at first, and then declined till the whole apparatus was cold, when all movement ceased.

429. Experiments were now tried with the screens in various positions intermediate between those described in the foregoing paragraphs. The results differed in no material respect from those given above, the number of preliminary negative revolutions and the speed of the continuous positive revolutions being the chief variations.

430. An alteration was made in the fly, the clear mica screens being lampblacked on the sides facing the concavity of the cups. On repeating the experiments described in pars. 426 to 428, it was found that the blackening had caused a change in the order of the phenomena. When the screens and cups were in position A, fig. 8, and no lampblack was on the screens, the action of the hot wire was to cause the fly to remain at rest, after having revolved a certain time negatively (426). Now, however, with the mica screen lampblacked, and in the same position, *i.e.*, close to the convex surface of the cups, the permanent rotation was at the rate of 30 a minute positively, after having made 22 preliminary negative revolutions. Again, when the blacked mica

screen was brought close to the concave surface of the cups, as at fig. 8, G, the preliminary negative revolutions were too rapid to count; after a time they slackened and ceased, and the fly then refused to move till the current was turned off, when it seemed to be suddenly released from constraint, and revolved rapidly in the positive direction. A reference to par. 428 will show that when no lampblack was on the mica screens there was permanent rotation in this position at a somewhat rapid rate, after the first preliminary turns had taken place.

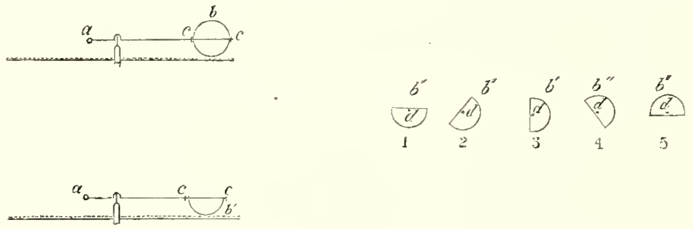
431. The fly and screens were now altered in the several ways described in pars. 403, 404, 405, 406, 408, and numerous experiments were tried with them, the relative positions of cups and screens being altered each time. To recapitulate the several results would be wearisome. I will therefore shortly say that they confirm what has already been said as to the effect of the hot wire. In all cases whilst the fly was receiving heat from the wire the direction of motion was negative. When the fly had ceased to increase in temperature the continuous rotation was positive, but occasionally a position of fly and screens was met with at which no movement took place (426, 430). On breaking battery contact, and allowing the whole instrument to cool, the first effect was to cause the fly to move at a greater speed positively, gradually slackening till it had cooled down to the ordinary temperature, when all movement ceased.

Besides these I tried two other forms of fly, which yielded results of sufficient interest to be worth giving more in detail.

432. The first fly consists of a single arm, supported in the centre by a glass cap working on the needle point, and having at one extremity a small metal counterpoise, and at the other a single aluminium cup. The cup is supported on the horizontal arm, which forms an axis traversing the centre of gravity of the cup through holes pierced in the sides.

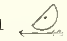
Fig. 18 shows this arrangement of fly; *a* is the counterpoise balancing the cup *b*;

Fig. 18.

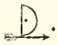


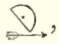
the axis passes through holes shown at *d*, on the side view of the cup *b″*. Stops *c c* on the axis keep the cup in position, and just sufficient friction is secured to retain the cup in any desired position, shown at *b″*, 1, 2, 3, 4, 5, into which it may be brought by tapping the apparatus. The remainder of the apparatus, the black mica disk, the platinum wire ring, &c., is the same as before.

When the cup was set horizontally with the concavity upwards, as shown at *b′*, and also at b'' 1, no movement took place after exhaustion, when the platinum wire was ignited. In the reverse position, the concavity being downwards, as at b'' 5, there was also no movement.

With the cup in position 2, there was no movement on igniting the platinum wire and keeping it hot. When the current was cut off, and the wire and inside of the apparatus allowed to cool, good rotation took place in the direction shown .

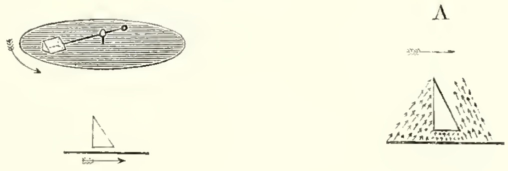
When the cup was turned over a little less, so as to get it about halfway between positions 1 and 2, no movement was produced either on heating or cooling.

With the cup in position 3, heating the wire caused the fly to make 18 revolutions in the direction . It then oscillated and came to rest, the wire remaining hot. On turning off the battery current no further movement took place.

In position 4 the hot wire gave a rotation in the direction , at the rate of 150 a minute. When still further turned over, till the cup was halfway between positions 4 and 5, the speed was 120 a minute in the same direction.

433. The other fly experimented with also consists of a single arm, with cap in the centre and counterpoise at one end. Instead of an adjustable cup, the other end supports rigidly a right-angled hollow mica prism, shown at fig. 19 in end view and perspective.

Fig. 19.



After exhaustion the current was turned on, and the fly immediately rotated in the direction shown by the arrows, keeping up continuously as long as the wire remained hot.

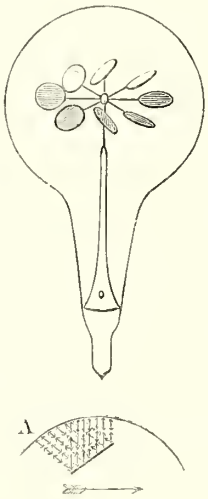
This experiment, I think, proves that the direction of the pressure is not wholly normal to the surface on which it is generated. Were it so, no movement would have taken place, as the base of the prism was in a parallel plane to the blacked mica disk. A tangential direction of the lines of force will, however, exert more pressure on the perpendicular face of the prism than on the hypotenuse, and will drive the fly round, as shown at fig. 19, A, in the direction given by experiment.

THE TURBINE RADIOMETER.

434. Experiments tried with the apparatus described in a former paper (336, 345, 354), where thin mica vanes, inclined at an angle of 45° to the horizontal plane, form the fly of the radiometer, show that this form of fly possesses advantages in cases where the incident radiation falls at an angle on the instrument; and the favourably-

presented radiometers, when tested with the hot metal rings applied equatorially and at the poles (298 to 305), showed that the action of radiation was in some cases entirely different, according as it fell on the vanes horizontally or nearly vertically. The experiments just described with the sloping adjustable cup (432) show still more forcibly the great difference in speed of rotation caused by a little alteration in the angle presented by the vanes to the direction of pressure. These experiments were, however, tried with somewhat complicated apparatus. A simpler form of instrument, in which full advantage is taken of the peculiarities of sloping vanes, was exhibited before the Royal Society on April 5, 1876, under the name of the turbine radiometer.

Fig. 20.



In the earlier radiometers of this kind the vanes were of mica, blacked on both sides, and inclined at an angle like the sails of a windmill, instead of being in a vertical plane. These are not sensitive to horizontal radiation, but move readily, in one or other direction, to a candle held above or below. If one side only of each vane is blacked, the fly becomes more sensitive to radiation falling on the black side, but is not sensitive when radiation strikes it on the other side. In the ordinary form of radiometer, the number of disks constituting the fly is limited to six or eight, a greater number causing interference one with the other, and obstruction to the incident light. In the turbine form of fly there is no such difficulty; the number of vanes may be increased to a considerable amount without overcrowding, and with corresponding advantage. The action of the vanes is evident. Fig. 20 shows the turbine vanes; light falling from above generates molecular pressure between the surface of the vanes and the top of the exhausted bulb, and this pressure, acting as at A, drives the fly round in the direction of the arrow. A vertical light gives the strongest action, but rotation takes place whatever be the incident angle, provided the light is caught by one surface more than by the other. If the finger, or any warm substance, touches the top of the bulb, so as to generate pressure from the inner surface of the exhausted bulb, the fly is strongly driven round in the direction shown at fig. 20, A (which I will call *positive*). The lower half of the bulb grasped in the hand causes pressure to be exerted on the vanes from below, and the resulting rotation is *negative.*

435. Ether dropped on the top of the bulb to chill it causes rapid negative rotation. If the radiometer is floated in a vessel of ice-cold water, and exposed to the air of a warm room above, it rotates rapidly in the positive direction, acting as a heat-engine, and continuing so to act until the rotating fly has equalised the temperature of the upper and lower portions of the bulb.

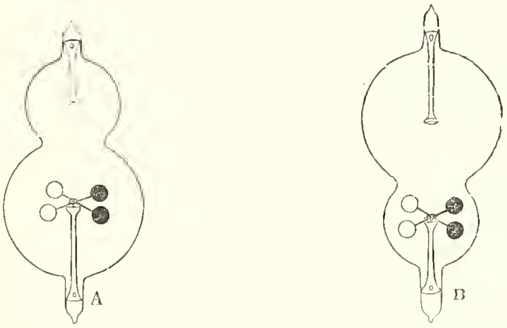
By reversing the cycle of operations—by floating the radiometer in hot water, and cooling the upper portion of the bulb—the fly instantly revolves in the negative direction.[*]

436. The turbine fly mounted vertically works equally as well horizontally. In this form it is supported on a double pointed needle, working horizontally in glass caps. A little care is required in balancing the vanes, otherwise one side tends to fall to the lowest point, and there is a difficulty in first starting the movement. This *windmill* form is very sensitive to radiation falling on it parallel with its axis of rotation, and it has the peculiarity of continuing its movement in any position. When once started it may be turned round, upside down, or on either side, without interfering with the rotation of the fly so long as the sloping vanes catch the light.

If mica vanes are used in the turbine radiometer they should be oval, the arms of the support passing through the minor axis, so that when viewed from the direction in which the light should fall on them their appearance will be circular.

437. The cause of the movement of the radiometer being pressure between the driving surface and the glass case of the instrument, it would follow that, other things being equal, the fly should revolve faster in a small bulb than in a large one.[†] This cannot well be tested with two different radiometers, as the weight of the fly and other essential points would not be the same in each, but I have constructed a double radiometer which shows this fact in a very satisfactory manner. It consists of two

Fig. 21.



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[*] Radiometers working in this manner were exhibited, with appropriate descriptions, at the soirée of the Royal Society, April, 5, 1876.

[†] Proc. Roy. Soc., November 16, 1876.

bulbs, one large and the other small, blown together so as to have a wide passage between them, as shown in fig. 21. In the centre of each bulb is a cup, held in its place by a glass rod, and in the bulb is a small four-armed fly with roasted mica vanes blacked on one side. The fly can be balanced on either cup, so as to admit of experiments being tried in position A or position B. In the larger bulb there is about half an inch between the vanes and the glass, whilst in the smaller bulb there is a space of a quarter of an inch. When exposed to the same source of light under identical circumstances, the mean of several experiments shows that in the small bulb (position B) the fly rotates about 50 per cent. faster than it does in the large bulb (position A).

438. [The following experiments show very clearly the influence of an alteration of density in the residual gas on the movement of a radiometer.

A small sensitive radiometer was fitted in a bulb $1\frac{1}{2}$ inch diameter, and a large bulb 3 inches diameter was connected to the lower limb of the radiometer by a narrow glass tube. The whole was then exhausted to about 35 **M**, and sealed off.

The large bulb was then placed in an air bath so arranged that it could be raised or lowered in temperature without altering the temperature of the radiometer, except by convection or conduction through the connecting glass tube. A candle was placed 3 inches from the radiometer, and observations of speed were taken whilst the temperature of the large bulb was raised or lowered. The mean of several observations showed that the speed of the radiometer was always reduced by heating the supplementary bulb, the expansion of the residual gas contained in it causing increased pressure in the radiometer bulb. On cooling the bulb, the speed of the radiometer increased. Taking the speed at the lowest temperatures at 10 a minute, the variations are as follows :—

Temperature.	Revolutions per Minute.
0°	10·
15°	10·
200°	6·9
300°	5·

W. C., November 21, 1878.]

439. Advantage has been taken of the fact, elicited in the experiment described in par. 437, in the construction of radiometers having flies of a spiral form.* By cutting a thin disk of aluminium into the form of a spiral, then drawing it out corkscrew fashion, and suspending it on a needle point in a tube in the usual way, the spiral rotates very quickly on exposure to light after proper exhaustion. The upper surface of the spiral is blacked, and it is kept of as large a diameter as will conveniently go in the tube. In this form, as in the turbine form of radiometer, the black surface is always exposed to the incident light instead of being alternately in light and in darkness. The driving surface is greater in this form than in the usual kind of fly,

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* Proc. Roy. Soc., November 16, 1876.

and the distance between the reacting surfaces may be very small with corresponding advantages of increased speed.

Spiral flies of mica are much more sensitive than those of aluminium, but are less easy to make. A very sensitive spiral radiometer may be made by threading flat mica disks on a thin aluminium wire. The blackened surfaces must in all these instruments make an angle with the inner surface of the glass tube, or there will be no tangential action of the molecular pressure.

440. The following experiments were tried with an aluminium spiral radiometer exhausted to the most sensitive point. It was completely immersed in water at 20° C., and when quite still it was suddenly plunged into water at 60° C. Rapid negative* movement took place, continuing for a few minutes; the fly then became still. When it was quiet in the hot water the radiometer was suddenly plunged into cold water (20° C.). Rapid positive movement took place, stopping when the temperature was uniform.

The radiometer was now heated with a lamp over the upper half of the tube; the spiral immediately rotated positively, and continued so to do till quite cold. Heat applied in a similar manner to the lower half of the radiometer case caused the spiral to rotate rapidly in the negative direction. Whilst so rotating negatively, a lighted candle was brought near; the fly stopped and then turned positively, resuming its negative rotation when the candle was removed, and continuing it till the tube and fly were cold. When the lower half only of the radiometer was immersed in hot water, the upper half being allowed to remain cool by radiation to the air, the negative rotation of the fly continued at an almost uniform speed, not coming to rest till the water was nearly cold.

These positive and negative movements of the spiral fly under the influence of partially applied heat are in strict accordance with the explanation already given, that when the glass case is heated its inner surface becomes the generating surface for the molecular pressure, which then acts as if it were a molar wind, repelling whatever happens to be within the sphere of its action (218, postscript).

441. In the foregoing experiments the results have in all cases testified to the truth of the theory that the glass case of the radiometer is essential to the movement. It was of interest to ascertain whether the substitution of another kind of surface for that of polished glass would have any effect on the motion of the fly. A radiometer was accordingly constructed with a four-vaned mica fly, the disks being blacked on one side. The case of the radiometer was made so that it could readily be opened and sealed again when a change of contents had to be made. Round the inner surface of the case an equatorial band of aluminium was fixed a little wider than the diameter of the disks on the fly, and coated with lampblack on the inner side. Fig. 22 shows the instrument complete. It was attached to the pump and

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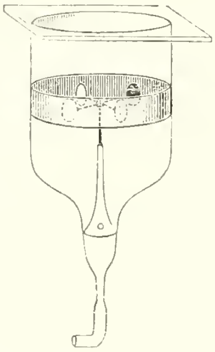
* In all cases the positive direction is that in which the fly moves when exposed to candle or day-light; the reverse direction being negative.

exhausted. A candle was placed 2 inches from the glass, and sufficiently above the level of the band to allow the rays to pass over it on to the fly. The exhaustion was continued until the maximum sensitiveness was obtained. Air was let in, and the speed at the pressure of greatest sensitiveness was taken several times; it was found to be 40 revolutions a minute, becoming less if the exhaustion exceeded or fell short of the most sensitive point.

Fig. 22.



The aluminium band was now removed and the experiment again repeated. At the pressure of maximum sensitiveness the revolutions of the fly were only 8·5 revolutions a minute, diminishing on each side of this pressure.

The black aluminium band was replaced, and a fly, with clear mica vanes, not blacked, and favourably presented (273), was suspended in the instrument. The experiments were now repeated as in the last case, and the result was as before—that the fly rotated more quickly with the aluminium band in, than when it was absent.

The increased speed in these experiments must not be entirely attributed to the fact that the blacked aluminium band presents a better reacting surface for the rebounding molecules than does the polished glass. In the first place, the introduction of the metal band slightly diminished the distance between the fly and the reacting surface; and in the second place, the rays from the candle not only shone on the fly, but also on the opposite hemi-cylinder of black aluminium, and warming it converted it into a driving surface. Each of these actions would cause an increase of speed in the fly; but added together, they could only account for a part of the jump from 8½ to 40 revolutions. The diameter was only to a slight extent diminished, and it has been shown that a much greater reduction of diameter (437) had only power to add 50 per cent. to the speed. The effect which the light of the candle shining on the blackened hemi-cylinder would have on the speed was ascertained by placing the candle in such a position that it would illuminate the inner half of the aluminium band, while its direct action on the fly was cut off by screens. The driving action of the pressure from the cylinder was shown in this way to be only equal to about 13½ revolutions a minute.

ROTATION OF THE CASE OF THE RADIOMETER.

442. Granting the existence of pressure acting between the fly and the case of the radiometer, it follows that one being held fast the other must move. Usually, the bulb is fixed and the fly rotates, but if the conditions are changed and the fly fixed while the globe was free to move, the latter must rotate. On March 30, 1876, I brought before the Royal Society an experiment in which this was shown to be the case. I quote the following opening sentence from the short note of this experiment which was published at the time*:—"During the discussion which followed the reading of Professor Reynolds's and Dr. Schuster's papers at the last meeting of the Royal Society, I mentioned an experiment bearing on the observations of Dr. Schuster. I have since tried this in a modified form, and as the results are very decided, and appear calculated to throw light on many disputed points in the theory of these obscure actions, I venture to bring a description of the experiment, and to show the apparatus at work before the Society."

The experiment was as follows: A radiometer was employed, the fly of which carried a magnetised needle. The radiometer was floated in water, and four candles were brought near. The fly immediately rotated, carrying round the magnetic needle. A powerful magnet was held over it, when the rotation of the fly was arrested, and the glass envelope rotated in the opposite direction to that in which the fly had been moving, the rotation keeping up as long as the candles were burning.

443. I have tried numerous forms of radiometer in the endeavour to ascertain the best kind for showing this rotation of the envelope. Turbines with pith or with metal flies, and arranged to rotate when floated on hot or cold water, did not give good results; neither did any variety of radiometers with metallic cup flies. By far the most sensitive form is one in which the fly carries four silver-flake mica disks, lampblacked on both sides. Over the glass cap carrying the fly, another fly stifly works, carrying four other disks made of clear mica. A reference to fig. 4, par. 392, will explain the arrangement, it being understood that the instrument now described is four-vaned instead of two-vaned, and the opaque mica disks are blacked on both sides instead of on one side. The arms carrying the clear mica screens also carry a magnetic needle. By tapping the instrument the screens can be brought close to the black vanes on either side. When exposed to light, rapid rotation is produced, the exposed side of the lampblacked vanes being repelled. By altering the position of the screens, so as to obscure either one or the other black surface, rotation can be produced in either direction. By increasing the distance between the screens and the blacked mica, the rotation is made slower; and by putting the screens midway between the vanes rotation ceases. Reference to par. 392 shows that the present results entirely confirm those there given. Shots are sealed in the lower part of the case to act as ballast and keep it upright when in water.

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* Proc. Roy. Soc., No. 168, March 30, 1876.

444. The radiometer was floated in water contained in a large beaker, and a cover was placed over it to prevent interference from air currents. The screens were brought close to one side of the vanes, and the whole was exposed to sunlight. The force of repulsion at once overcame the power of the magnet, and the compound fly rotated rapidly. A strong magnet was then brought near the bulb, the motion of the fly was arrested, and immediately the bulb rotated in the opposite direction, making about 6 revolutions a minute. So strong was the force of rotation that when the cover was removed, and a paper arrow 3 feet long was cemented to the top of the bulb, as long as the sunshine lasted this paper index was swung round the room 3 times a minute. On tapping the screens to the other side of the vanes, rotation took place equally well, but in the opposite direction.

445. The radiometer was brought into a dark room, and four candles placed near, the outside control magnet being in position. The bulb rotated about 3 times a minute. When its speed was uniform the control magnet was removed, liberating the fly, which then revolved under the influence of the internal pressure, the case and the vanes going opposite ways. After the bulb had made one-third of a revolution it stopped, and went back for about the same distance. Then it again stopped and made one-third of a revolution in the original direction. These oscillations kept on for a considerable time ; they seemed to get gradually less and less, but I did not observe that they stopped altogether. The motion of the bulb in the opposite direction to the fly was always more rapid than when it went with the fly. The movement of the fly kept up at a uniform speed as long as the experiment lasted.

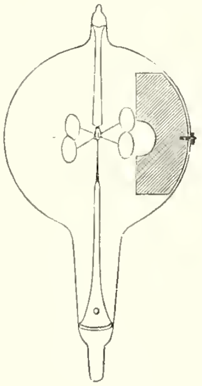
446. It would seem that this oscillatory movement was due in whole or in part to internal friction, either of the steel point on the glass socket or of the vanes against the residual gas, or to both these causes combined. To ascertain what power this friction possessed the candles were removed, and as soon as the whole instrument had come to rest, a bar magnet was moved alternately from one side of the radiometer to the other, so as to cause the fly to rotate as if it had been exposed to light. The fly rotated rapidly, and the internal friction carried the glass envelope round in the same direction at the rate of about 1 revolution in 3 minutes, in opposition to the friction of the water against its sides.

447. The rotation of the envelope, the fly being fixed by a magnet, was also effected in another manner. Oblique mica vanes were fixed round the inner horizontal circumference of the bulb, and the movable fly, which carried a small magnet, was furnished with metallic vanes favourably presented (273), so that molecular pressure acting between them and the envelope should cause the fly to rotate. Inside the fly, coiled round the supporting glass stem, but not in contact with the fly, was a platinum wire spiral, the extremities of which were connected with thicker wires passing through and sealed into the glass bulb. By connecting these terminals with a battery the spiral was ignited, and the heat warming the metallic fly caused pressure to be exerted between the fly and the outer envelope, and produced rapid rotation. The

whole instrument was floated in water, and one of the outer terminal wires was caused to dip into a mercury cup at the bottom of the water, whilst the other terminal dipped into a ring-shaped trough of mercury surrounding the upper part of the bulb. Battery wires were connected with the two mercury cups, when the fly rotated rapidly. On arresting the movement of the fly by means of a magnet, the bulb was seen to rotate in the opposite direction. The movement, however, was feeble, and with difficulty could be kept up for any length of time, owing to the friction of the platinum wires in the mercury cups. Having demonstrated the fact of rotation, further experiments seemed unnecessary.

448. The fixed vanes in the apparatus just described appearing to act in a very decided manner, further experiments were tried in this direction, as it was thought that the results might throw further light on the theory of the movement.

Fig. 23.

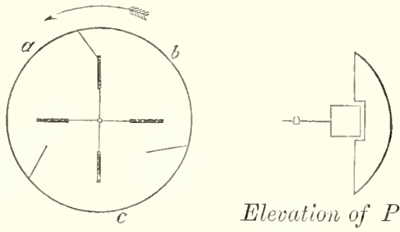


A radiometer was furnished with a fly, the four vanes of which were cut from thin transparent mica, and were mounted symmetrically with the axis of rotation, not being favourably presented. At the side of the bulb, in a vertical plane, a plate of mica was fastened in such a position that each clear vane in rotating should pass it, clearing it by about a millimetre. The mica screen was cut away in the middle to allow the vanes to pass, as shown in fig. 23. The screen is a double one, formed of two plates of mica about a millimetre apart, the outer surface of one being lampblacked, the other being clear. On bringing a candle near, and allowing the light to shine on the clear side of the mica screen, no effect is produced. If by means of a shade the light is allowed only to shine on the clear mica vanes, there is still no action; but if the light shines on the blacked side of the mica screen, the fly rotates rapidly as if it were blown round by a wind issuing from the black surface, and keeps on moving as long as the light is near.

A similar instrument contained a mica screen, blacked on both sides. The fly carried four bright aluminium cups (318). On shading the light of a candle from the blacked mica screen, and allowing it to shine only on the cups, the fly rotates positively, the convex sides retreating from the light. If the candle is allowed to shine only on the black mica screen, the cups being unexposed, the fly is driven rapidly round in one direction or the other, according to which black side is illuminated, the cups going with almost equal readiness whether the convex retreats, as in an ordinary cup radiometer, or the concave retreats, as in ROBINSON's anemometer. When the light shines on both screen and cups the rotation of the fly is always positive; but the speed is much greater when the molecular pressure from the black screen conspires with, than when it opposes the positive motion of the fly.

449. It was suggested to me by Professor STOKES that it might be possible to get rotation in a radiometer with a perfectly flat fly, alike on both sides, by throwing the obliquity from off the fly on to the case. Following out the suggestion, three vertical partitions of thin clear mica were fixed in the bulb of a radiometer, with their planes not passing through the axis of rotation, but inclined, as shown in fig. 24, and slightly cut away to let the fly pass closely. The fly had four aluminium vanes, polished alike on both sides. Candles at a, b, c make the fly revolve rapidly in the direction of the arrow. Indeed, one candle is sufficient to produce rotation. Breathing gently on the bulb causes negative rotation. A hot glass shade inverted over the instrument causes strong negative rotation, changing to positive on cooling. When the fly is furnished with clear mica, or with silver-flake mica vanes, the same results are obtained as when aluminium vanes are employed. The strongest action is produced by warming the bulb.

Fig. 24.



Elevation of Partition and Fly.

THE OTHEOSCOPE.

450. The experiments on the rotation of the envelope (442 to 447) show that the surface generating the molecular pressure need not be the one which rotates. The apparatus shown in fig. 23, as well as the one described in a former paper (360), prove that it is an advantage to have the driving surface stationary. In the radiometer the surface which produces the molecular disturbance is mounted on a fly, and is itself driven backwards by the excess of pressure between it and the sides of the containing vessel. Regarding the radiometer as a heat-engine, it is seen to be imperfect in many respects. The black or driving surface, corresponding to the *heater* of the engine, being also part of the moving fly, is restricted as to weight, material, and area of surface. It must be of the lightest possible construction, or friction will greatly interfere with its movement; it must not expose much surface, or it will be too heavy; and it must be a very bad conductor of heat, so as to retain the excess of pressure on one side. Again,

the part corresponding with the *cooler* of the engine (the side of the glass bulb) admits of but little modification. It must almost necessarily be of glass—one of the worst materials for the purpose; it is obliged to be of one particular shape; and it cannot be brought very near the driving surface.

451. To get the best results the *heater* should be stationary; it might then be of the most suitable material, of sufficient area of surface, and of the most efficient shape, irrespective of weight. The moving portion should be the *cooler;* it should be as near as possible to the *heater,* and of the most suitable size, shape, and weight for utilising the force impinging on it. The *heater,* or driving surface, acts as if a molecular wind* were blowing from it (276), principally in a direction normal to the surface (312). By having the heater of large size, and making it of a good conductor, such as silver, gold, or copper, a very faint amount of incident radiation would suffice to produce motion.

In April, 1877, I communicated to the Royal Society a preliminary note on a new instrument which I had made in accordance with these indications, and as it was essentially different in its construction and mode of action to the radiometer, I proposed to identify it by the distinctive name of *Otheoscope* (ὠθέω, I propel). Whilst the radiometer admits of but few modifications, such an instrument as the otheoscope is capable of an almost endless variety of forms. The glass envelope is an essential portion of the machinery of the radiometer, without which the fly would not move; but in the otheoscope the glass vessel simply acts as a preserver of the requisite amount of rarefaction. Carry a radiometer to a point in space where the atmospheric pressure is equal to, say, 1 millim. of mercury, and remove the glass bulb, the fly will not move, however strong the incident radiation; but place the otheoscope in the same conditions, and it will move as well without the case as with it.

452. The instrument described in a former paper (360), and repeated here with some alterations (425, fig. 17), fulfils the essential conditions of the otheoscope, and by allowing light to shine on the blacked mica plate instead of heating it with the platinum wire and battery, the vanes may be set into good rotation.

An instrument was made of this kind, leaving out the electrical portion so as to work entirely by radiation. It is represented at fig. 25, A. A plate of mica, *a a,* is attached firmly to the support *c,* and is lampblacked on the upper side; a fly, *b b,* with polished aluminium vanes, set at an angle of 45°, is supported by a glass cap on a needle point passing through *a a,* so that when rotating the lower edge shall be about a millimetre from the mica plate. Light shining on the black surface generates molecular pressure, which reacting on the sloping vanes drives them round with considerable speed. The black disk is here the driving surface, the glass envelope not being active.

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* This movement of the molecules may be compared to the movement of the oxygen and hydrogen molecules when water is decomposed by an electric current. In the water connecting the two poles there is no molar movement whatever, although eight times as much matter is passing one way as the other.

A more sensitive form of this construction is made by having a fixed copper disk lampblacked on the upper side, and making the vanes of mica for the sake of lightness, fig. 25, B. The vanes are as numerous as can be conveniently put together, and being set at an angle, the pressure from the copper plate drives them round with great speed when set in action with even a faint light. Another form consists in suspending on the fly aluminium cups, turned at an angle of 45°, with the concave side downwards, as shown in fig. 18, No. 4 (432). This form likewise proved very sensitive.

Fig. 25.

A

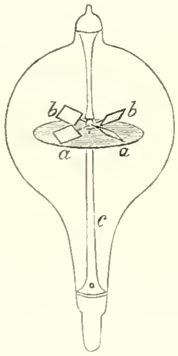
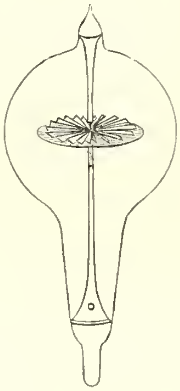


Fig. 25.

B



453. The pressure being exerted chiefly at right angles to the driving surface, it was thought that power was lost by the tangential action of the pressure in the above form of otheoscope. Other instruments were therefore made with the driving surface of thick copper, cut into 16 equal portions by radial lines stopping short of the centre. Each of the 16 sectors is then bent round to form an angle of 45° with the original plane of the disk. This driving wheel is lampblacked and fixed to the support in the centre of the bulb. Immediately above, and suspended on a needle point, is another disk cut into sectors, and bent at an angle exactly the same as the lower disk. As the upper and lower sectors face each other, being in parallel planes, a force acting at right angles to the driving surface will also strike the movable fly at right angles, and drive it round in a more direct manner. Experience shows that this form is attended with advantage.

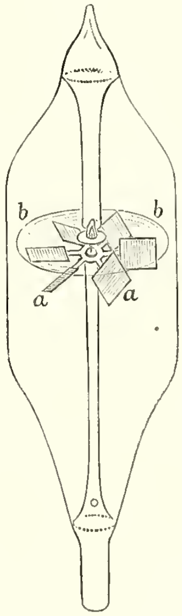
These otheoscopes are very sensitive to heat externally applied. If those shown in fig. 25 be warmed at the upper part above the movable fly, the fly is driven round in the negative direction; the pressure which acts on the vanes from above giving the opposite rotation to that produced when pressure acts from below.

If the lower part of the case is warmed, no action is seen at first; gradually, however, the heat is communicated to the fixed driving wheel, and the fly then rotates positively.

When plunged into hot water the fly revolves negatively, and then stops as soon as the temperature is equalised. On removing it from the water positive rotation sets up, and continues with considerable speed till quite cold.

454. It was suggested by Professor STOKES that a disk might be made to revolve on its axis, and the following instrument was made in accordance with the design proposed. The disk is horizontal, mounted like the fly of a radiometer, and for lightness sake is of mica, blacked above. Fixed to the bulb above the disk are four flat pieces of clear mica; each extends from the side of the bulb to near the centre and ends below in a straight horizontal edge, leaving just space enough for the disk to revolve without risk of scraping. The edge is in a radial direction, and the plane of the plate inclined about 45° to the horizon, in the same direction for them all. Exposed to the light of a candle the rotation is against the edge, the same as in the instruments last described.

Fig. 26.



It was found on experimenting that a much more sensitive instrument could be made by slightly modifying this form. Fig. 26 shows the best construction of otheoscope with rotating disk. *a a* are six vanes of copper foil, oxidised by heating to redness in the air; they are attached to arms, and are inclined at an angle of 45° to the horizon, as in the forms of driving vanes previously described. These are fixed to the support. Through the centre passes a needle point balancing a glass cup; this carries a thin clear disk of mica, *b b*, freely rotating about a millimetre above the top edges of the copper vanes. When exposed to light, the mica disk rotates with great speed against the edges. If the lower part of the envelope is heated so that the pressure from the warm glass strikes against the vanes before reaching the disk, the direction of rotation is negative owing to the deflection of the pressure on passing between the vanes. As soon as the copper vanes have become warmed, they become the driving surfaces, and the rotation changes to positive.

The upper part of the mica disk in one of these otheoscopes was divided into sectors, each of which was painted with one of the component colours of white light. When exposed to sunshire the speed was so great as to cause the colours to blend together into a neutral grey.

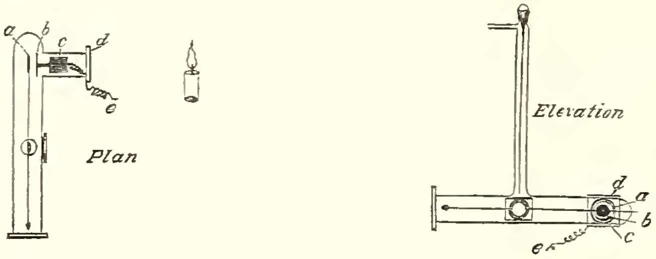
The pressure which drives the movable fly round reacts equally on the driving surface (442). By suspending the driving vanes as well as the fly on needle points, so as to allow both to revolve freely and independently (345, 360), they both rotate under the influence of light in opposite directions. Otheoscopes have been constructed of the forms shown in figs. 25 and 26, in which the upper and lower vanes or disks are capable of rotation. When exposed to light the movements take place in accordance with theory.

THICKNESS OF THE LAYER OF MOLECULAR PRESSURE.

455. Whilst experimenting with the otheoscope it was found that, for a given exhaustion, the nearer the reacting surfaces were together the greater was the speed obtained. This was in accordance with previous results (437). In a note on the theory of the radiometer which I had the honour of communicating to the Royal Society in November, 1876,* I briefly described a piece of apparatus by which I was able to measure the thickness of the layer of molecular pressure generated when radiation impinged on a blackened surface, enclosed in an atmosphere the rarefaction of which could be varied at will.

The apparatus which is represented in fig. 27 (plan and elevation), consists of a torsion balance, fitted with a glass suspending fibre and reflecting mirror, as already described in previous papers (102, 186, 198, 221, 259). To one end of the beam is attached a disk, *a*, of silver-flake mica (238), lampblacked on the face. The portion of the tube in front of this disk is opened out, and a short piece of wider tube is sealed on at right angles to the beam, as shown in the plan. The outer end of this additional tube is ground flat, and closed with a piece of clear plate glass, *d*, cemented on. In front of the blacked disk is a thin and perfectly clear disk of mica, twice the diameter of the black disk. This is connected by a rigid arm with a lead plate, *c*, curved to fit the inside of the tube, and capable of sliding freely to and fro. The clear mica screen, *b*, is easily and accurately adjusted any desired distance from the blacked disk, *a*, by gently tapping the under part of the glass tube on one side or the other of the lead weight. The space between is measured with a millimetre scale. The torsion beam is controlled by an outside magnet acting on a small magnetised needle attached to the reflecting mirror of the beam.

Fig. 27.



456. The friction of the lead plate when sliding to and fro in the glass tube, especially

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* Proc. Roy. Soc., No. 175, 1876, vol. xxv. p. 310.

at high exhaustions, produced slight electrification, which caused the two disks to be drawn together. This was remedied by connecting the lead to earth by a fine wire, *e.*

A candle was used as the source of radiation; it was set a measured distance from the blacked disk. Its rays were cut off by water and opaque screens, except at the time of trying an experiment; and all extraneous radiation, or interference from the warmth of the body, was cut off by water screens and black velvet placed around.

The scale to receive the reflected index of light was 3 feet from the mirror. A movement of the blacked disk to the extent of 1 millim. produced a deflection of 32 divisions on the scale.

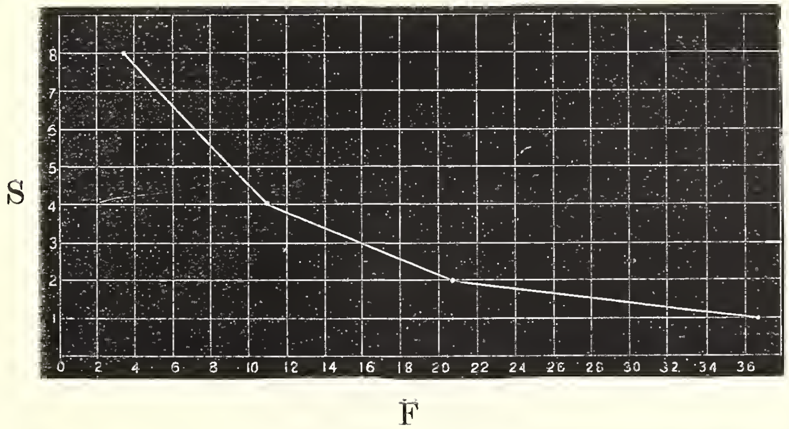
457. Preliminary experiments showed that it was not necessary to rarefy the air in this apparatus to get repulsion. By moving the control magnet sufficiently far off to secure great sensitiveness, and putting the candle 120 millims. from the blacked disk, the following results were obtained at ordinary atmospheric pressure:—

T. = Tension in millimetres of the gas in the apparatus.
C. = Distance in millimetres of the candle from the blacked disk.
S. = Distance in millimetres separating the screen and disk.
F. = Force of repulsion represented by the degrees on the scale covered by the first swing of the luminous index.

T.	C.	S.	F.
millims.	millims.	millims.	°
748	120	1	36·7
,,	,,	2	20·7
,,	,,	4	11·0
,,	,,	8	3·5

Plotted on a curve, the results in columns S and F give the line shown in fig. 28.

Fig. 28.



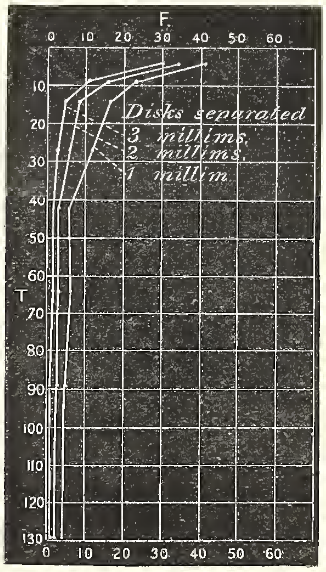
458. The repulsion being so decided at atmospheric pressure, the apparatus was

made less sensitive by the control magnet, and the candle was placed 200 millims. off. Exhaustion was then commenced, and observations were taken as recorded in the following table, each result being the mean of several observations :—

T.	C.	S.	F.
millims.	millims.	millims.	°
129	200	1	4·
,,	,,	2	2·1
,,	,,	3	1·
89	,,	1	5·
,,	,,	2	3·
,,	,,	3	1·
64	,,	1	6·5
,,	,,	2	3·
,,	,,	3	2·
42	,,	1	6·
,,	,,	2	3·
,,	,,	3	2·
27	,,	1	12·
,,	,,	2	6·5
,,	,,	3	3·
14	,,	1	17·
,,	,,	2	9·
,,	,,	3	5·
9	,,	1	23·5
,,	,,	2	16·
,,	,,	3	11·
4	,,	1	41·
,,	,,	2	34·
,,	,,	3	30·

Fig. 29 shows the curves formed when the tension in millimetres and the force of repulsion at the three distances of the screen are taken as abscissæ and ordinates.

Fig. 29.

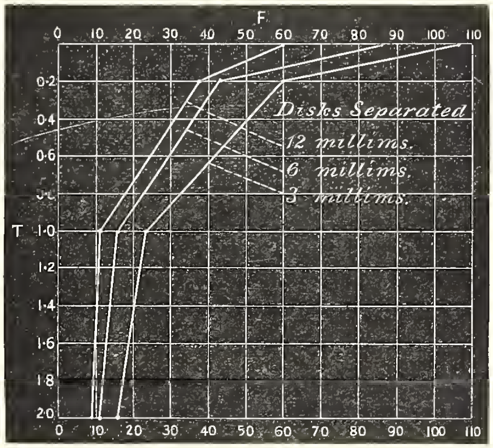


459. In the next experiments the candle was placed 400 millims. off, and the screen and blacked disk were separated 3, 6, and 12 millims. :—

T.	C.	S.	F.
millims.	millims.	millims.	°
2·0	400	3	16·
,,	,,	6	11·
,,	,,	12	9·5
1·0	,,	3	24·
,,	,,	6	16·
,,	,,	12	11·
0·2	,,	3	60·
,,	,,	6	43·
,,	,,	12	38·
0·01	,,	3	107·
,,	,,	6	86·
,,	,,	12	60·

Fig. 30 shows the curves given by these figures, using, as in the last diagram, T and F for abscissæ and ordinates.

Fig. 30.



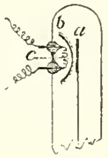
460. An examination of these tables and diagrams shows that the law of increase of the force with the diminution of the distance between the disks does not remain uniform at all rarefactions. At the lowest exhaustions the mean free path of the molecules of the attenuated gas is less than 1 millim., as rendered evident by the force of repulsion diminishing rapidly as the distance increases. At exhaustions higher than 9 millims. this condition alters ; and as the gauge approaches barometric height, the pressure tends to become uniform through considerable distances, the mean path of the molecules now being comparable with the greatest distance separating the surfaces between which they act. As the distance between the disks increases, the tangential action between them and the sides of the glass comes more into play, and tends to interfere with and augment the direct pressure between the two surfaces.

REPULSION AT ATMOSPHERIC PRESSURE.

461. The attraction or repulsion exerted between surfaces at atmospheric pressure or low exhaustions has always exhibited contradictions, the air currents caused by the heating of the glass envelope, and those generated inside by the heating of the movable indicator, being sufficient to neutralise, wholly or partly, any repulsion which was occasioned by true molecular pressure. It was thought that by special arrangements the effect of air currents might be separated from that of molecular pressure, and the following apparatus was accordingly fitted up.

It consists of a torsion balance, similar to the one last used (455, fig. 27), but instead of the adjustable mica screen and blacked disk, the experimental end of the torsion beam was arranged as shown in fig. 31. *a* is a clear mica disk suspended on the end of the torsion beam; *b* is a shallow cup of platinum, lampblacked on the convex side; *c* is a platinum wire spiral, connected with thicker platinum terminals passing hermetically through the glass. By heating the spiral the platinum cup is warmed, and the approach or retreat of the disk *a* is measured by an index ray of light reflected on to a screen in the usual manner.

Fig. 31.



The strength of the voltaic current used to heat the spiral is kept constant by the system of resistance coils and galvanometer described in a previous paper (359). Two Leclanché cells, giving a tolerably constant current, were used to heat the spiral; the heat never got up to redness. The scale was kept 3 feet from the mirror.

462. The apparatus was exhausted to dry it, and then filled with dry air which had passed through a tube packed with phosphoric anhydride. The disk was adjusted so as just not to touch the platinum cup.

When the spiral was heated, the disk was immediately attracted to the cup. The mica disk was now put at different distances from the cup. On heating the spiral, strong attraction always took place, the disk rushing across to the cup, rebounding from it, and rapidly settling in close contact. A wet finger[*] placed on the opposite side to the cup, when it had got cold, drew the disk away.

463. Exhaustion was now proceeded with, observations being taken at intervals. The control magnet was moved so as to let the disk lightly touch the cup; the scale was moved till the index ray of light stood at −140. The magnet then brought the torsion beam back till the index ray of light stood in the centre of the scale, at zero. As now adjusted, the mica disk is several millimetres from the cup, and can move

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[*] Wetted to avoid electrification by friction.

towards it till the index ray of light marks −140, when the disk and cup touch; a positive movement to the right signifying repulsion, and a negative movement to the left signifying attraction. Electrical contact was made by pressing down a key. The current was kept on during the whole time of each experiment.

464. *Pressure* 260 *millims.*—When the current is turned on the index ray moves to +2; it then swings back to −140, the disk touching the cup, and being held there for some time after contact is broken.

465. *Pressure* 240 *millims.*—On making contact the movements of the index are the same as at 260, the first movement, however, being +3 divisions.

466. *Pressure* 210 *millims.*—The same as above, but the first movement being +4.

467. *Pressure* 160 *millims.*—On making contact the first movement of the index is +4·5° repulsion. It then swings back to −139, the plate just failing to touch the cup by one division. There was no permanent attraction, as in former cases, but the beam swung quite freely.

468. *Pressure* 125 *millims.*—The index moved on making contact +6°, repulsion; it then returned to −136°, attraction, being 4 degrees short of contact.

469. *Pressure* 110 *millims.*—When contact was made the first movement was +9, repulsion; the return movement was to −135°, attraction, or 5 degrees short of contact.

470. *Pressure* 90 *millims.*—The first movement was +15, repulsion; the return movement was −131°, or 9 degrees short of contact.

471. *Pressure* 60 *millims.*—The first movement was +22, repulsion; the return movement was to −115, or 25 short of contact.

472. *Pressure* 55 *millims.*—The first movement of the index on heating the spiral was +34, repulsion; the return swing was to −94.

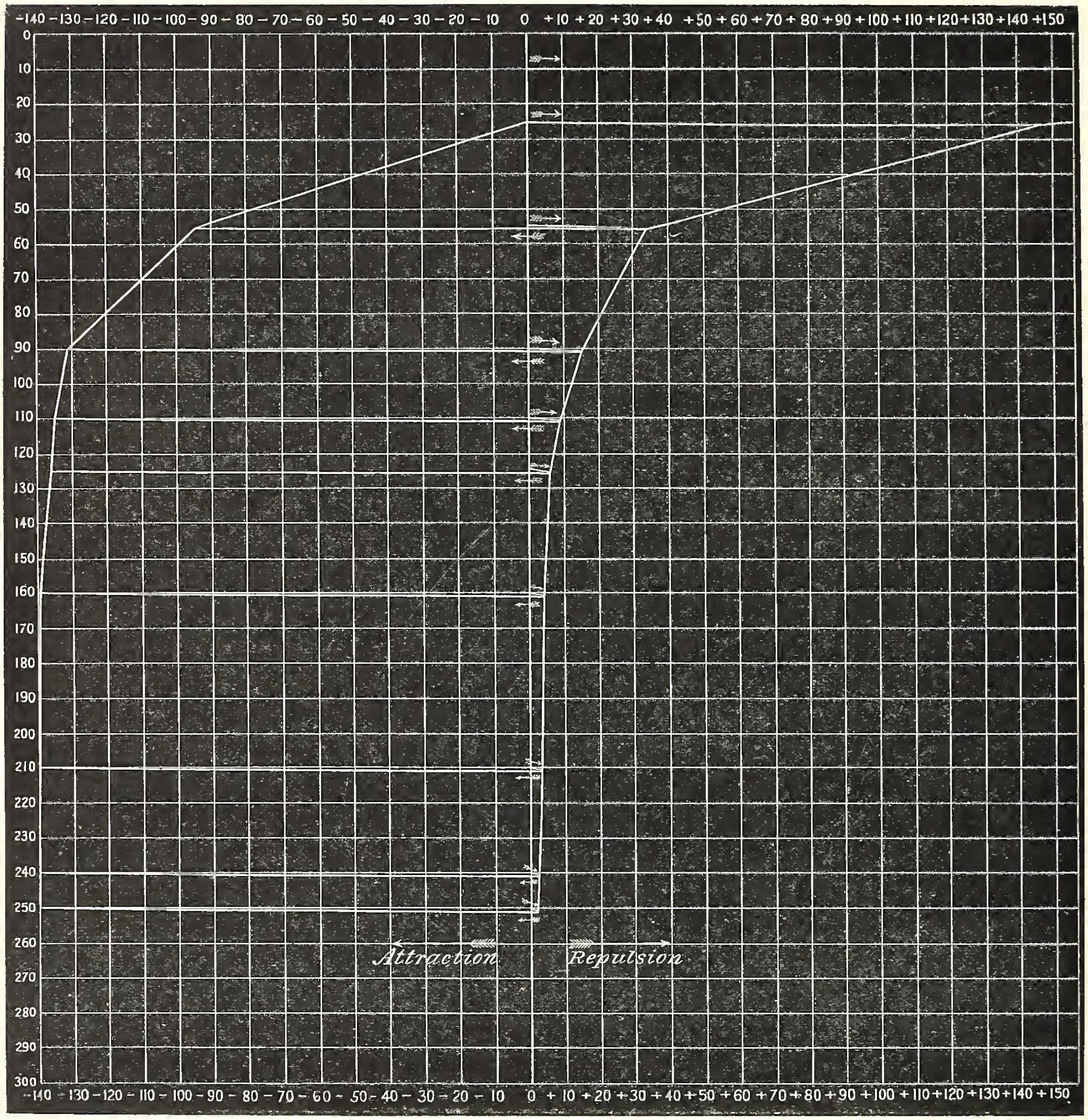
473. *Pressure* 15 *millims.*—On heating the spiral the index ray showed repulsion, going to +145. It then oscillated a little to and fro, but did not show any attraction.

474. *Pressure* 10 *millims.*—As soon as battery contact was made, strong repulsion ensued, the index ray going off the scale (250°+), and the mica disk being driven against the side of the tube and remaining there. This being the case, no further experiments could advantageously be tried with the apparatus.

Between each of the latter experiments the index took a long time to return to zero, owing to the mass of heated metal inside, which had to get quite cold.

475. Fig. 32 shows the results of the above experiments drawn as curves. The attraction, which is very strong at pressures between atmospheric and 210 millims., begins to decline after that degree of exhaustion is passed, until it disappears at 15 millims. At the same time, the repulsion, which begins to be apparent at 250 millims., increases as the attraction diminishes. It is probable that the attraction is the effect of air currents caused by the permanent heating of the surface in front of the mica disk, and that if the first action of the molecular pressure, which I assume is practically instantaneous, could be still further separated from the more slowly acting air currents, we should succeed in getting decided repulsion at still higher pressures.

Fig. 32.



476. An experiment tried three years ago with the bar photometer (described in Part III. of this research, par. 135*) bears so closely on these results that I condense the following description from my note book.

The bulb was filled with dry air at atmospheric pressure, and the movement of the bar being observed by an index ray of light on a graduated scale, a candle was uncovered at different distances.

With the candle six inches from the bulb, the index first showed repulsion 4 divisions, and then strong attraction, going quite off the scale, more than 500 divisions.

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* Phil. Trans., 1876, Vol. 166, part II., p. 333.

With the candle 1 foot off, there was at first repulsion 10 divisions, immediately followed by attraction 100 divisions.

With the candle 2 feet off, there were 10 divisions repulsion, and 85 divisions attraction.

With the candle 4 feet off, the first movement was repulsion 35 divisions, and then attraction 10 divisions.

At 6 feet off the candle only gave repulsion 50 divisions, and no attraction.

At 8 feet off the candle gave repulsion 10 divisions, with no subsequent attraction.

The bar of the photometor was of pith, one-half plain, and the other half lampblacked. The repulsion is much stronger on the black than on the white half. The warm bulb attracts much more powerfully than radiation repels, therefore by increasing the distance of the candle from the bulb, heating is avoided and the action of radiation becomes apparent.

477. The candle was now placed 10 inches from the bulb of the photometer. A glass cell full of water, and having parallel sides, was interposed, near the candle, and a thick sheet of plate glass was put close to the bulb. The light was screened off till an experiment was to be tried, and the exposure was continued only as long as was requisite to get the result. The index ray of light and the scale were the same as before.

Tried in air (pressure 745 millims.) there were 10 divisions repulsion with no subsequent attraction, showing that the water and glass screens prevented the heating of the bulb.

478. The bulb was then exhausted with the SPRENGEL pump, observations being taken from time to time.

At 330 millims. pressure there was repulsion	=12°
,, 225 ,, ,, ,, ,,	11
,, 115 ,, ,, ,, ,,	9
,, 85 ,, ,, ,, ,,	5
,, 80 ,, ,, ,, ,,	3
,, 72 ,, ,, ,, ,,	2
,, 68 ,, ,, ,,	no movement
,, 60 ,, ,, ,,	,,
,, 45 ,, ,, ,,	,,
,, 40 ,, ,, ,,	repulsion 3°
,, 35 ,, ,, ,,	no movement
,, 25 ,, ,, ,,	,,
,, 10 ,, ,, ,,	,,
,, 2 ,, ,, ,,	repulsion 2°
,, 1 ,, ,, ,,	,, 25°

In this series I obtained no attraction. The gradual diminution of action to 72 millims. pressure, its absence (with one exception) till the gauge was within 2 millims. of barometric height, and its rapid increase after that, may, however, be regarded as fair reasons for suspecting that the repulsions at high and at low pressures are not entirely due to the same cause.

MEASUREMENT OF THE FORCE OF REPULSION.

479. The following experiments were undertaken with the view of getting measurements of the actual amounts of force exerted by radiation in causing repulsion.

A horizontal torsion balance was employed similar in construction to the one described in a previous paper* (209, 210, 211). A somewhat stiffer torsion fibre was employed, and the part corresponding to the pan of the balance was a clear mica disk 16 millims. in diameter. A similar disk was fastened to the tube in which the beam oscillated, in such a position that when in equilibrium the pan should be 1 millim. above the fixed disk. This fixed disk was lampblacked on the upper surface, and had beneath a thin platinum spiral connected with terminals sealed in and passing through the glass, as in apparatus fig. 31 (461). When this spiral is heated by an electric current, the blacked mica disk fixed above it becomes heated, and the molecular pressure thereby generated between it and the mica pan causes the latter to rise. The glass thread attached to the beam is now twisted by means of the graduated circle, and the number of degrees through which the thread has to be twisted, in order to bring the beam back to equilibrium, is noted. This gives a measurement of the pressure exerted in torsional degrees. To convert these degrees into grains it is only necessary to ascertain through how many degrees the glass thread has to be twisted, in order to balance a known weight placed on the pan. A piece of iron weighing 0·01 grain was used, and it was found that to restore the beam to equilibrium with this weight on the pan, 3 complete revolutions and 34°5 additional, or 1114·5 degrees of torsion, were required. The force with which the glass fibre tends to untwist itself, being directly proportional to the number of degrees through which it has been twisted, the value in grains of any number of torsional degrees is readily calculated. A ray of light reflected from a mirror in the centre of the beam is used as an index, being brought to a definite mark on a scale for zero.

The battery power was kept constant by the means already described (359, 461).

480. In the following table the degrees of exhaustion are given in millionths of an atmosphere, and the force of repulsion is given in fractions of a grain, calculated from the degrees of torsion.

---

[*] Phil. Trans., 1876, Vol. 166, part II., p. 371.

Exhaustion.				Grain.
2237· M,	force of	molecular	pressure	=·000126
1316· M,	,,	,,	,,	=·000206
424· M,	,,	,,	,,	=·000368
259· M,	,,	,,	,,	=·000511
153· M,	,,	,,	,,	=·000718
94· M,	,,	,,	,,	=·000987
64·7 M,	,,	,,	,,	=·001086
32·9 M,	,,	,,	,,	=·001140
26·0 M,	,,	,,	,,	=·001076
20·0 M,	,,	,,	,,	=·000987
13·9 M,	,,	,,	,,	=·000727
12·1 M,	,,	,,	,,	=·000646
9·3 M,	,,	,,	,,	=·000682
9·1 M,	,,	,,	,,	=·000619
6·0 M,	,,	,,	,,	=·000520
1·3 M,	,,	,,	,,	=·000269
0·7 M,	,,	,,	,,	=·000224

481. In fig. 33 these results are plotted as a curve, taking the abscissæ in units of the hundred-thousandth of a grain, and the ordinates in millionths of an atmosphere. The maximum action would not be far from 40 millionths, increasing slowly up to that exhaustion, and diminishing suddenly after that point.

482. In the Proceedings of the Royal Society for November 16, 1876,* I gave a similar curve of the variation of the force of repulsion in air at different degrees of exhaustion. The source of radiation was a candle, and the body repelled was blackened mica. In that curve the maximum action is very near 40 millionths of an atmosphere, rising slowly and sinking rapidly.

483. Another similar curve, obtained in quite a different way, is given in the fifth part of this research (par. 334).† The instrument here experimented with was a cup-shaped aluminium radiometer, and the source of light was a candle. In this case also the maximum action occurred close upon an exhaustion of 40 millionths of an atmosphere, increasing slowly and dying away rapidly.

484. In the same paper (par. 383)‡ I have given a third curve of the variation of the action of a candle on a blacked mica surface in an air vacuum. Observations are wanting between 59 and 14 millionths of an atmosphere, but by continuing the curve passing through the other points, it is seen that here, likewise, the maximum would not be very far from 40 millionths of an atmosphere.

485. These agreements must be more than accidental coincidences. Care must, however, be taken not to consider them as expressions of more than a partial law, for in the diagram containing the curve last referred to (483, 383), is another curve

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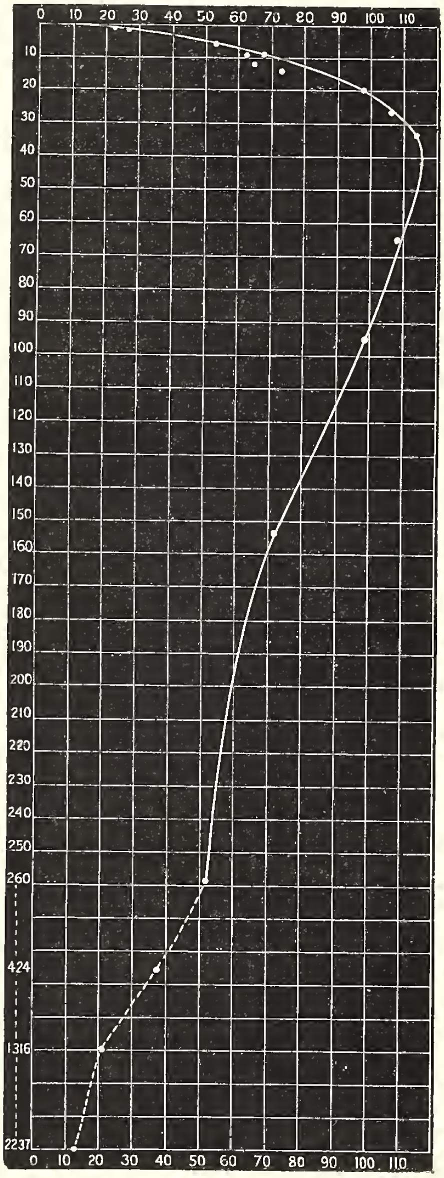
* Vol. xxv. p. 305.

† Phil. Trans., 1878, part I. (The Bakerian Lecture), p. 301.

‡ *Loc. cit.*, p. 316

taken under somewhat similar circumstances to the curve in fig. 33 (480), a hot platinum wire being used as an internal source of radiation, which shows a still increasing power of repulsion up to as high an exhaustion as 0·4 millionths of an atmosphere.

Fig. 33.



In concluding this series of papers "On Repulsion resulting from Radiation," it is a pleasure for me to state that during the six years they have been in progress I have been materially aided by my assistant, Mr. C. H. GIMINGHAM, whose extraordinary mechanical dexterity and skill in glass manipulation have been called almost daily into service.

V. THE BAKERIAN LECTURE.—*On the Illumination of Lines of Molecular Pressure, and the Trajectory of Molecules.*

By WILLIAM CROOKES, *F.R.S., V.P.C.S.*

Received December 5,—Read December 5, 1878.

[PLATE 14.]

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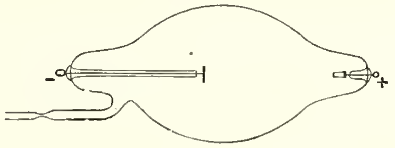
THE INDUCTION CURRENT THROUGH RAREFIED GASES. DARK SPACE ROUND THE NEGATIVE POLE.

486. When the spark from a good induction coil traverses a glass tube containing a rarefied gas, certain phenomena are observed which vary greatly with the kind of gas and the degree of exhaustion. There is one appearance, however, which is constant in all the gases which I have examined, and within very wide limits of pressure, viz.: the well-known dark space round the negative pole. I have long been impressed with the idea that this dark space coating the pole was in some way related to the layer of molecular pressure causing movement in the radiometer, and the following experiments were instituted with the object of testing this hypothesis.

Fig. 1.



A glass bulb (fig. 1) was furnished with platinum wire terminals sealed into the glass, ending outside in loops and inside in aluminium poles; the positive pole being a wire and the negative pole a disk about 10 millims. diameter, bare in front and covered with mica at the back. The bulb being full of dry air and connected with the SPRENGEL pump, was exhausted. An induction coil capable of giving sparks 68 millims. long in air when actuated by 3 GROVE's cells, was connected with the terminals, the disk being always negative except when otherwise stated.

487. At a pressure of 20 millims. of mercury a soft velvety halo of violet light commenced to form on the edges of the disk; as the exhaustion proceeded, this glow flickered over different parts of the surface, till at a pressure of 15 millims. it remained steady all over the front of the disk.

Plate 14, fig. 2, shows the appearance at a pressure between 15 and 20 millims.: *a* shows the glow creeping over the disk as the exhaustion increases; *b* is the appearance on the mica side of the disk; and *c* shows the edge of the disk. By close examination edge-wise it can be seen that the glow and the metal are not in contact, but are separated by a very minute interval.

At a pressure of 7·5 millims. the glow is thicker, and the black space easily visible. At 1·6 millims. the appearance is as shown in Plate 14, fig. 3. The dark space is half a millim. in thickness, and the glow is very bright. As the exhaustion proceeds the appearance round the negative disk goes through the stages shown in Plate 14, figs. 4, 5, 6, and 7, the glow diminishing in intensity and the dark space widening out; the outline still being distinctly shown. Plate 14, fig. 7, shows the appearance at a pressure of ·078 millim., the thickness of the dark space being 8 millims.

488. At any particular pressure the dark space separating the glow from the metal disk can be slightly altered by varying the power of the contact breaker. When it is screwed so as to increase the intensity of the spark the dark space slightly contracts, and when the intensity of the spark is diminished the dark space slightly expands, the variation in size being about 1 millim. For the same intensity of spark and degree of exhaustion the dark space is always uniform in size.

489. Another bulb was now made of a different construction, to ascertain if the size of the dark space varied with the distance separating the poles. Each pole consisted of a flat metal disk, one being fixed and the other capable of sliding along a glass tube fitting loosely into a wider tube. By tapping the tube, the adjustible pole can be brought close to the fixed pole or removed some distance from it, contact between the disk and the outer loop being always preserved by wires in the sliding tubes.*

As the exhaustion proceeded the phenomena observed were similar to those already described. It was found that the dimensions and thickness of the dark space did not vary, whatever was the distance separating the fixed and sliding pole.

490. The battery power actuating the coil was also varied between 1, 2, 3, and 4

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* The device of making one of the poles of a vacuum tube extensible so as to alter the distance between the two poles has already been adopted by Mr. SPOTTISWOODE. Experiments with such a tube are described by Messrs. DE LA RUE and MÜLLER in the Phil. Trans. for 1878, p. 163.

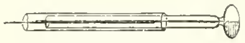
cells, but I could detect no alteration in the size of the dark space so long as the pressure remained constant. The only effect was to cause an increase or decrease in luminosity.

Air was let in and exhaustion recommenced several times. At the same exhaustion the dimensions of the dark space were always the same. It is indeed easy to tell the degree of exhaustion by measuring the thickness of the dark space. On afterwards taking the pressure in the usual way the agreement is seen to be close enough for ordinary observations.

When only one pole of the coil is connected with one pole of the tube the same sized dark space is seen round the pole as when the two poles are connected, but the light is faint, and the room requires to be darkened for accurate observations.

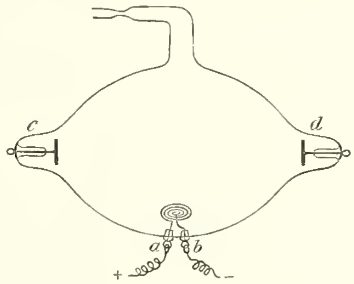
491. The sliding pole was now removed; one-half was coated with lampblack as shown in fig. 8, and it was replaced. Exhaustion was recommenced, and at a convenient pressure the coil was attached. The dark space round the pole was now seen to have lost its uniform thickness ; it was 1 millim. thick over the metallic half and 1·5 millims. thick over the lampblack half. On continuing to exhaust, the dark space became larger ; that over the lampblack keeping about the same proportion in excess of that over the bare metal. As the exhaustion increased the size of the dark space over the platinum gained a little more upon that over the bare metal.

Fig. 8.



492. A tube was made, as shown in fig. 9 ; *a b* are the two ends of a long platinum wire, coiled into a close spiral, and sealed in the bulb. The coils of the wire are not in contact, and the ends are sealed in separately, so that the spiral can be made red hot by connecting the wires *a* and *b* with a battery. *c* and *d* are two flat aluminium poles. An apparatus was connected with the pump whereby different gases could be introduced into the bulb and experimented with at different pressures.

Fig. 9.



493. The following experiments were tried with air :—At a pressure of 3·25 millims. the dark space round the negative pole was 1·75 millims. from the pole. The size and appearance were the same whether the negative pole was *c* or *d*, or whether the spiral *a b* was made negative. The induction spark was passed between *a b* (negative) and *c*

(positive); the dark space, as already mentioned, was 1·75 millims. from the platinum spiral pole; the wires *a b* were now connected with 3 GROVE's cells through a contact key, the induction spark passing at the same time. On pressing the key the spiral *a b* became ignited to bright redness, and the dark space immediately expanded from 1·75 to 2·5 millims. in thickness. On cutting off the igniting current the dark spheroid slowly contracted to its former dimensions.

494. At a pressure of 1·5 millims. the dark spheroid surrounded the platinum spiral to a distance of 4·5 millims. Igniting the spiral with the battery caused the dark space to expand considerably, but the amount of expansion could not be accurately seen, owing to its outline becoming very indistinct after it had expanded to about 7 millims. semi-diameter.

495. The poles *c* and *d* were connected with the induction coil, the spiral *a b* being still attached to the 3 GROVE's cells. The whole tube was filled with light, and I then made battery contact with the spiral, igniting it to redness. No space could be seen round the hot wire, and no alteration was observed in the illumination of the tube.

496. Hydrogen gas was now passed into the tube, and after several times exhausting and refilling with hydrogen the following experiments were tried: the spiral *a b* being always the negative pole of the induction coil, and also being connected with the 3-cell GROVE's battery.

At a pressure of 5 millims. the dark spheroid extended 1·75 millims. from the pole. On making the pole red hot the dark space expanded to 2·5 millims. in thickness.

At a pressure of 3·75 millims. the dark spheroid was 2·25 from the pole, expanding to 3·75 millims. when the pole was ignited.

At a pressure of 2·75 millims. the expansion of the dark space on heating the pole was from 4 millims. to 6·5 millims.

The phenomena in hydrogen are therefore similar to those in air, but the dark space round the negative pole is larger for the same degree of exhaustion.

497. Carbonic acid was then passed into the tube, and after several fillings and exhaustions the following experiments were tried:—

At a pressure considerably less than that at which the dark space would appear in air or hydrogen, there was no appearance of it in carbonic acid.

At a pressure of 4 millims. the dark space was ·75 millim. thick. At a pressure of ·62 millim. its thickness was 2·5 millims., and at a pressure of ·27 millim. its thickness was 4 millims. Igniting the negative pole caused expansion of the spheroid, but not to so great an extent as if the gas had been air or hydrogen.

ILLUMINATION OF LINES OF MOLECULAR PRESSURE.

498. These experiments appear to prove three things:—

a. Setting up an intense excitement in a disk of metal by electrical means produces a molecular disturbance which affects the surface of the disk and the surrounding gas.

When the gas is dense the disturbance extends a short distance only from the metal; but as rarefaction proceeds the layer of molecular disturbance increases in thickness. In air at a pressure of ·078 millim. this molecular disturbance extends for at least 8 millims. from the surface of the disk, forming an oblate spheroid around it. Had the pole been a point the spheroid would have become a sphere 16 millims. diameter.

b. The diameter of this dark space varies with the exhaustion; with the kind of gas in which it is produced; with the temperature of the negative pole; and, in a less degree, with the intensity of the spark. For equal degrees of exhaustion it is greatest in hydrogen and least in carbonic acid as compared with air.

c. The shape and size of this dark space do not vary with the distance separating the poles; nor, except very slightly, with alteration of battery power, or with intensity of spark. When the power is great the brilliancy of the unoccupied parts of the tube overpowers the dark space, rendering it difficult of observation; but on careful scrutiny it may still be seen unchanged in size, nor does it alter even when, with a very faint spark, it is scarcely visible. On still further reduction of the power it fades entirely away, but without change of form.

499. I consider, therefore, I am justified in assuming that if the molecular disturbance communicated to the surrounding gas by the metal were to be further reduced in intensity, the spheroid would still remain the same for the same pressure, although invisible. If instead of exciting the disturbance in the metal pole by electrical means, I effect the same by heating it to redness, the spheroid of molecular disturbance in the surrounding gas would probably be there; and if the disturbance is only produced by allowing light to shine on the metal plate, we may still imagine the molecular disturbance to extend around the plate, although vastly reduced in intensity.

500. The vibrations communicated by the induction spark to a metal plate are doubtless different in kind as well as in intensity from those caused by heating the plate, for the electrical excitement induces luminosity in the gas, although the metal pole is only slightly heated; while igniting the metal to bright redness in the absence of electrical excitement from the induction coil causes no luminosity in the surrounding gas.

501. Numerous experiments have been tried with the object of seeing if this *visible* layer of molecular disturbance was the same as the invisible layer of molecular pressure or stress, the investigation of which has formed the subject of many papers presented to the Royal Society*. Small exploring flies similar to the one described in a former paper (417)† were moved about in various parts of the bulb, both inside and outside the spheroid, but the results were most contradictory, and were difficult to obtain at all owing to the electrification of the fly.

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* Phil. Trans., Vol. 164, p. 501; Vol. 165, p. 519; Vol. 166, p. 325; Vol. 166, p. 355; Vol. 169, p. 243; Vol. 170, p. 87.

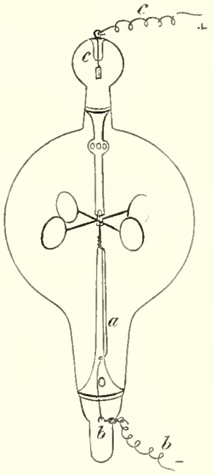
† Ibid., Vol. 170, p. 103.

ELECTRICAL RADIOMETERS.

502. Successful observations were ultimately obtained by making radiometers with metallic vanes, in the following manner :—

The needle point on which the fly rotates is connected with a copper wire, *a*, fig. 10, passing down the central tube and attached to the platinum terminal, *b*, sealed through the glass. The other terminal, *c*, passes through the top of the bulb, and is of aluminium wire inside. The vanes are of aluminium, lampblacked on one side, and connected by aluminium arms to a hard steel cup ; the whole fly is therefore in metallic connection with the lower terminal.

Fig. 10.



503. At a moderately low pressure, the coil being attached and the fly being made the negative pole, a faint halo appears on the bright side of the vanes, and there is a tendency to negative rotation. As the exhaustion proceeds, the dark space appears and gradually widens out. The space on the black side is nearly double the thickness of that on the bright side, but is fainter (491) ; on exhausting still more, the dark space widens out until it forms a large, irregular shaped, oval round each vane. Whilst the dimensions of this oval are small, the fly does not move definitely one way or the other, but when the dark space on the lampblacked side has grown till its luminous boundary touches the glass, positive rotation commences, and continues with increasing speed as the exhaustion increases. The difference in size between the dark spaces on each side of the vanes being but small, that from the bright side soon impinges against the glass, and prevents the positive rotation from acquiring much speed.

504. Another instrument was therefore made similar to the one last described, but having one side of each metallic vane covered with a thin piece of mica. It was connected with the coil, and experiments tried as exhaustion proceeded.

At a pressure of 4·5 millims. a halo of a velvety violet appearance forms on the metallic side of two of the vanes only, the others being dark. The movement is of a negative tendency.

At a pressure of 1·4 millims. the bright velvety light has spread over all four vanes on their metallic sides. With the screw of the coil arranged to give a faint spark the movement is negative. If a Leyden jar is inserted, or the contact screw turned to give strong sparks, the movement is positive. Inserting the jar makes the halo fainter. The best pressure for negative rotation is 1·14 millims.

When the pressure is ·51 millim., the dark space on the metallic side extends nearly to the glass, and the whole inside of the bulb is luminous; the rotation is now always positive with the coil. Introducing a piece of wet string into the circuit causes the vanes to go negatively.

As the exhaustion proceeds, the dark space widens out and flattens itself against the glass, the positive rotation getting faster until the maximum is reached. Exhaustion beyond that point causes the speed to diminish, but I cannot get negative rotation again by any amount of over exhaustion.

505. Two radiometers of this kind were made and sealed off, one at a pressure of ·19 millim., showing good positive rotation, and the other at a pressure of 1·14 millims, showing negative rotation. Connecting the two in series, that is to say, the vanes forming the negative pole of one with the positive of the other, and passing the induction current through, caused them both to rotate at the same time, one positively and the other negatively.

506. The vanes of the radiometer were now made of aluminium cups (334) slightly favourably presented to the side of the glass. They were almost hemispheres, bright on both sides. The other parts of the radiometer remained as before.

When the exhaustion is low and the induction coil is connected, the fly rotates negatively. This is caused by the electrified air flying off the edges of the cups, and driving them round like the well-known electrical fly. Luminosity appears on the cups at a pressure of 3·5 millims., and as the exhaustion proceeds it spreads over each side and is separated by a dark space. This widens out, retaining as much as possible the shape of the cup, passing successively through the appearances shown in Plate 14, fig. 11, *a*, *b*, *c*, *d*, and *e*. The luminous margin to the dark space is concentrated at the concave side of the cup, forming a luminous centre, and widens out at the convex side. As soon as the luminous boundary of the dark space reaches the side of the glass positive rotation commences. On continuing the exhaustion, the dark space becomes flattened against the glass, and the rotation becomes more rapid. At higher exhaustions, the dark spaces which surround each cup in the form of an irregular ellipsoid drawn in at the focus of the cup, touch one another. They now act as if they were solid elastic bodies, and become flattened out along the lines of contact (the distance between two cups not being large enough for complete ellipsoids) and form broad longitudinal lines of light down the bulb between each pair of cups, the rest of the bulb being comparatively dark. These lines are nearer the concave than the convex side, and turn with the fly.

507. The train of reasoning carried out in pars. 498, 499, and 500, is therefore seen

to be well supported by experiment, and justifies the theory that the induction spark actually illuminates the lines of molecular pressure caused by the electrical excitement of the negative pole. The thickness of the dark space is the measure of the length of the path between successive collisions of the molecules. The extra velocity with which the molecules rebound from the excited negative pole keeps back the more slowly-moving molecules which are advancing towards the pole. The conflict occurs at the boundary of the dark space, where the luminous margin bears witness to the energy of the discharge. When the exhaustion is sufficiently high for the length of path between successive collisions to be greater than the distance between the fly and the glass, the swiftly-moving rebounding molecules spend their energy, in part or in whole, on the sides of the vessel, and a production of light accompanies this sudden arrest of velocity.

CONVERGENCE OF MOLECULAR RAYS TO A FOCUS.

508. In the sixth part of my researches on "Repulsion resulting from Radiation," which was presented to the Royal Society in June last, I gave at par. 415 a theoretical explanation of the movement of radiometer flies with curved surfaces, on the supposition that the lines of molecular pressure acted in a direction normal to the surface. In the figures illustrating this theory (fig. 14, A and B, pars. 415 and 416) I drew the lines of pressure radiating outwards from the convex surface, and predicted their converging to a focus and thence diverging at the concave surface. These figures were drawn long before the experiments just described were commenced, and it is a corroboration almost amounting to absolute proof that the theory was correct, to see how well those old drawings represent the actual lines of pressure as illuminated by the induction spark.

509. The convergence of the lines of force to a focus, demanded closer investigation than was possible when the cup was in rapid rotation. A bulb was therefore made similar in general character to the electrical radiometers just described, but having the cup-shaped negative pole fixed instead of movable. Exhaustion was proceeded with, and the successive appearances noted were like those shown in fig. 11. On further continuing the exhaustion, the dark space spreads out still further, and the focus of light converging from the concave side falls on the luminous boundary. Inside this focus the rays can be seen converging, and are of a violet colour. Outside the line bounding the dark space the rays diverge from the focus, and are of a lighter colour than those inside; the whole appearance being strikingly similar to the rays of the sun reflected from a concave mirror through a foggy atmosphere.

GREEN PHOSPHORESCENT LIGHT OF MOLECULAR IMPACT.

510. When the exhaustion approaches 30 M,[*] a new phenomenon makes its appearance. The dark space has spread out so much that it nearly fills the bulb; the

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[*] M signifies the millionth of an atmosphere.

violet light by which the focus was rendered visible has become so faint as to be difficultly traced, but with care it can be seen converging to a focus beyond the focal point noticed at lower exhaustions. At the part of the bulb on which the rays impinge, a faint spot of greenish-yellow light is observed, sharp in outline. On exhausting to 14 M, and making the cup the negative pole of the coil, the projection from the cup is represented by a brilliant green spot of light about 7 millims. diameter, and the focus can scarcely be traced. The rest of the bulb is nearly dark, but at those parts furthest removed from the negative pole the faintly luminous boundary of the dark space can still be seen. A little blue light is seen round the positive pole extending somewhat into the bulb. On reversing the poles and making the cup positive, the bulb becomes beautifully illuminated with greenish-yellow light.

511. This phosphorescent light only appears in its full intensity when the dark space surrounding the negative pole extends to the surface of the bulb. At lower exhaustions, it can be detected when specially sought for outside the luminous boundary of the dark space, but it is faint and not easily noticeable. The colour depends on the kind of glass used. Most of my apparatus are made of soft German glass, and this gives a phosphorescent light of a greenish-yellow colour. English glass phosphoresces of a blue colour; uranium glass becomes green; a diamond became brilliantly blue.

512. The greenish-yellow phosphorescence of the soft German glass only takes place under the influence of the discharge from the negative pole.* As the exhaustion increases the light gets stronger. At 10 M the diffused violet light surrounding the positive pole is faint, and the green light on the glass is getting stronger. At 4 M

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[*] While this paper was passing through the press, my attention has been drawn to two Memoirs by H. EUGEN GOLDSTEIN, communicated to the Berlin Royal Academy of Sciences, May 4, 1876, and November 23, 1876, in which some of the results announced in this paper have been anticipated. Referring to the green phosphorescent light, H. E. GOLDSTEIN writes:—

" The author has specially examined the green light which appears in tubes of common glass at certain degrees of pressure and intensity of the discharge. The luminosity of the sides of the tube is not a phenomenon of fluorescence but of phosphorescence, and can change from green to orange.

" The negative light which produces this phosphorescence is, as was already assumed by HITTORF, a rectilinear radiation, which extends from the negative pole into surrounding space. Still there are essential differences between the diffusion of this remarkable motion and the likewise rectilinear movement of the light, some of which differences are here brought forward.

" HITTORF observed that a body placed between the side of the glass and a point-like cathode, throws a shadow in the phosphorescent light of the latter.

" Well defined, though not very sharp shadows of small objects may be obtained not merely from a point-like or linear negative pole, but also from extended negative surfaces placed at a small distance from the opaque object.

" A surface which merely radiates light, *e.g.*, an ignited body, under similar conditions throws a scarcely visible expanded penumbra.

" The negative light is therefore a rectilinear radiation, which is propagated preferably in a manner almost normal to the producing surface.

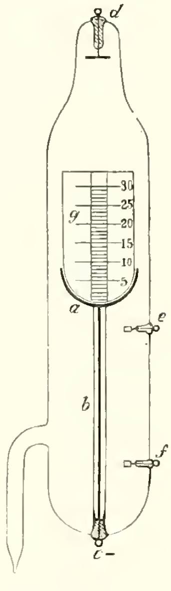
" If between the cathode and the green luminous side of the tube there is introduced a solid body, its the green light is still better, and there is no violet light on the positive pole. At ·9 M the green phosphorescence is about at its maximum. When the exhaustion reaches ·15 M the spark has a difficulty in passing, and the green light only appears in flashes occasionally, and then only in patches over the tube. At ·06 M the vacuum is almost non-conducting, and a spark can only be forced through by increasing the strength of the coil and well insulating the tube and wires leading to it. Beyond that exhaustion nothing can be observed.

513. It is difficult to experiment at these high vacua. After the induction spark has been forced through a few times, gas is liberated from the metallic poles or from the surface of the glass, conduction through the residual gas begins, and the appearance of light commences. Further exhaustion again impairs the conductivity. The greatest care must be taken to keep the conducting wires away from the tube except just where they touch the terminals; they should be well insulated and no metal should approach the tube. With all these precautions, however, it is difficult to prevent a spark passing through the side of the tube, when the vacuum is at once spoiled.

FOCUS OF MOLECULAR ENERGY.

514. To investigate still further the phenomena attending the concentration of force to a focus, another apparatus was made, as shown in fig. 12. *a* is a hemicylinder, 22 millims. diameter, accurately made of polished aluminium; it is supported in position by the glass tube *b*, through which passes a fine copper wire connecting

Fig. 12.



shadow is thrown upon the side, since it excludes such rays of the cathode as impinge upon it from reaching the side. If the solid body after some time is removed, the shadow disappears, but an image of the body remains, distinguished from the surrounding luminous surface by its greater brightness, and exactly reproducing the shape of the former shadow."

the hemi-cylinder with the platinum terminal *c*. At the other end of the tube a flat aluminium disk forms the terminal. Other aluminium terminals are sealed in at *e* and *f*. In the axis of the tube, resting on the centre of the hemi-cylinder, and at right angles to its axis, is a flat glass plate *g*, on which is etched a millimetre scale.

515. On exhaustion, the hemi-cylinder *a* being made the negative pole through the wire *c*, the usual phenomena are observed. The dark space gradually increases in size, and as it spreads from the concave surface towards the centre it forms a denser and more luminous glow of light. As the exhaustion proceeds this glow becomes smaller and brighter, until it forms a narrow brilliant line along the axis of the hemi-cylinder. A bright spot forms on the scale at 11 millims. (the centre of curvature of the hemi-cylindrical pole), showing that the gaseous molecules from the negative pole are projected in a direction normal to the surface. The distance between the convex surface of the hemi-cylinder and the luminous boundary of the dark space is also about 11 millims. The pressure at this point is 106 M.

516. As the exhaustion becomes higher the rays projected from the concave surface of *a* are seen on the divided scale to cross each other, and the focus lengthens out and becomes ill-defined, as if the molecules from the centre of the pole were projected to a greater distance than those from the edges, giving an appearance of spherical aberration. It is difficult to fix the exact focal point, but it gradually creeps up from 11 millims. to about 19 millims.

517. When the focus of light has reached 20 millims. it is very faint. The green-yellow phosphorescent light has been visible for some time, and now, the pressure being 19·3 M, a faint concentration of green light is seen to occur at 11 millims. on the scale. This shows that the cause to which the green phosphorescence is due is not the same to which the luminosity previously noticed in the tube is due ; for the two phenomena are now existing simultaneously.

518. In another tube, similar to the one shown at fig. 12, but without the flat glass plate *g*, the appearance at an exhaustion of one millionth of an atmosphere when the induction current passes through, is shown in the coloured lithograph, Plate 14.

[Having received permission from Mr. De La Rue to test some of these high vacuum tubes with his large chloride of silver battery, I gladly availed myself of it, and tried some experiments, among others, with this tube. The exhaustion was the millionth of an atmosphere. Not knowing what the effect of so large a battery might be, cells were put on gradually. Up to 4800 cells scarcely any action was visible. At 6300 cells a current equal to 0·00048 weber passed through the tube, and the green phosphorescence commenced to be visible on the glass. With 7760 cells the current passing through was equal to 0·00095 weber, and the appearance of the tube was similar to that produced by an induction coil giving about a one-inch spark. When 9920 cells were used, the current through was =0·00143 weber, and the green phosphorescent curves on the tube were about as bright as when an 18-inch spark from an inductive coil was passed through. The chromo-lithograph (Plate 14) represents this very well. With 11,000 cells the current through was the same as with 9920 cells, and the phosphorescence was only a little brighter. The resistence of the tube was found to equal 7,642,600 ohms, and the tube potential was 10,600. I have to thank Mr. De La Rue for not only letting me use his battery, but for kindly taking these measurements for me.—W. C., August 12, 1879.]

NATURE OF THE GREEN PHOSPHORESCENT LIGHT.

519. Other differences are observed. The focus of the green phosphorescent light remains at the centre of curvature, appearing independent of the degree of exhaustion, provided this is good enough to render it visible; whilst the bluish focus of light lengthens out with the exhaustion. The green focus is not to be seen in the body of the tube, but only where the projection causing it strikes the glass; whilst the concentration of light first observed is only visible in the space of the tube, and produces no special effect when it touches the sides.

The bluish focus at low exhaustions, proceeding from the negative hemi-cylinder, is only seen when the pole *d* is positive, no light on the concave side being seen. When the exhaustion is high, and the green phosphorescent focus is bright, it makes little difference whether the positive pole is above the concave surface, at *d*, or below it, at *e* or *f*; in either case the projection from the concave surface of *a* is visibly focussed on the phosphorescent surface of the glass (526, 527, 549).

520. Spectrum observations show another difference equally decided. When the luminosity observed in the focus of the hemi-cylinder at a low exhaustion (say 100 M) is examined in the spectroscope, it shows the lines due to the residual gas, whether it be nitrogen, hydrogen, or carbonic acid; but when the exhaustion is at a very high point (say 1 M) the blue light has disappeared, and no lines are detected in the green phosphorescence whatever be the gas whose residue causes it. The spectrum of the greenish-yellow light, so beautifully illuminating the tube, is continuous, most of the red and the higher blue rays being absent.

521. The green phosphorescence commences at a lower exhaustion in hydrogen than it does in air, and the phenomena known to be due to hydrogen gas also disappear sooner than in the corresponding case with air. In all gases, however, when a high exhaustion is reached the subsequent phenomena are the same.

522. The viscosity of a gas is almost as persistent a characteristic of its individuality as its spectrum. For many years I have been carrying on a research on the variations of viscosity of different gases at high exhaustions. In the Proceedings of the Royal Society for November 16th, 1876,[*] I gave in a preliminary note a diagram of the variation of viscosity of air, hydrogen, and other gases, at exhaustions varying from 250 M to 0·1 M. A comparison of the indications on that diagram with the data given in the present paper will show that when the

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[*] Vol. xxv., p. 305.

exhaustion is such that the spectral and other luminous appearances in the tube are characteristic of the special gas under examination, the viscosity is little different from what it is at full atmospheric pressure. When, however, the spectral and other characteristics of the gas begin to disappear, the viscosity also commences to decline, and at an exhaustion at which the green phosphorescence is most brilliant, the viscosity has rapidly sunk to an insignificant amount.

523. Another special characteristic of a gas at exhaustions between 250 **M** and 35 to 40 **M**, is the molecular pressure generated, when light falls on a black surface immersed in it. In the diagram just referred to, the variation in the force of repulsion is also given. This is seen to decline rapidly and almost to die out along with the viscosity: the phosphorescent phenomena become more brilliant as these special characteristics of the gas disappear.

PROJECTION OF MOLECULAR SHADOWS.

524. In ordinary vacuum tubes, illuminated by the induction current, the luminous phenomena follow the tube through any amount of curves and angles; a hollow spiral becomes illuminated just as well as if the tube were in a straight line. Not so, however, the phenomena of green phosphorescence observed at these high exhaustions. The molecular ray which gives birth to green light absolutely refuses to turn a corner, and radiates from the negative pole in straight lines, casting strong and sharply-defined shadows[*] of anything which happens to be in its path. In a **U** tube with poles at each end, one leg will be bright green and the other almost dark, the light being cut off sharply by the bend of the glass, a shadow being projected on the curvature. I can detect no trace of polarisation in the green phosphorescent light on the surface of the glass, except, of course, when it emerges at an angle through the side of the glass tube.

525. The projection from the negative pole of a shadow rendered visible by a sharply-defined image on the side of the glass, seemed worthy of more close examination. A tube was accordingly made, as shown in fig. 13. In the centre, dividing the tube into nearly equal parts, is a screen of thin mica, *a a*, loosely fitting into a groove blown round the tube. A flat plate of uranium glass, *b*, about half a millimetre thick, is rivetted to the mica on one side. *c* is a star-shaped piece of aluminium foil attached to a platinum terminal, and *d* is a similar star made of mica. At each end of the tube are two terminals, *e* and *h*, being flat aluminium disks, and *f* and *g*, aluminium points.

With this apparatus experiments were carried on during exhaustion. When the exhaustion is moderate (say 1 or 2 millims.), so as to show stratifications and the ordinary phenomena of vacuum tubes, the luminosity extends from one pole to the other. Thus, if *e* and *g* are the two poles, the light extends the whole length of the tube; if, however, *e* or *f* is one pole and *c* the other, the luminosity only occupies the upper part of the tube, and if *e* and *f* are the two poles, the light keeps close to the top. The whole appearance shows that both poles are at work in producing the phenomena.

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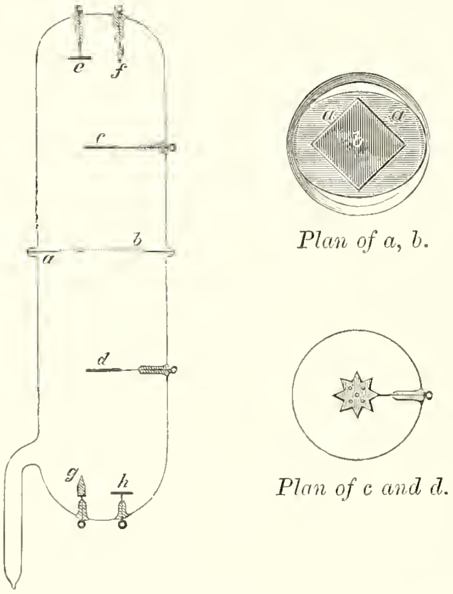
[*] See note, page 143.

526. When, however, the vacuum is sufficiently high for the dark space round the negative pole to have swollen out to the dimensions of the tube, there is little difference in the phenomena of green phosphorescence and projection of the shadow of *c* on *b*, whichever is the positive pole, provided *e* be made the negative. The appearances are almost the same, and the shadows projected from the negative pole *e* are just as sharp and intense whether I make *f* or *h* the positive pole. The positive pole, in fact, seems to have little or nothing to do with the phenomena. (519, 527, 549.)

Fig. 13.



Plan of a, b.

Plan of c and d.

527. The best and sharpest shadows are cast by the flat disks *e* and *h*. The shadows thrown by the pointed poles *f* and *g* are faint and undecided in outline. An aluminium ring scarcely makes any shadow; a spherical pole, owing to the rays from it diverging more, gives faint and broad shadows; a square pole acts the same as a disk. Using the upper flat pole *e* as the negative, the shadow of the star *c* is thrown distinctly on the uranium plate *b*, where it is seen magnified about two diameters, but perfectly sharp in outline; either *f*, *g*, or *h*, and even the star itself may be made the positive pole without affecting the appearance of its shadow on *b*. (519, 526, 549.)

528. The whole upper part of the tube which is in the line of direct projection from the negative pole, glows with an intense yellowish-green fluorescent light. The uranium plate is still more brilliant, and of a greenish colour. Where the shadow of the star falls on it, no phosphorescence whatever is visible. The mica plate *a*, where uncovered at the side of the uranium plate, gives no phosphorescence, and no

shadow is therefore seen on it. When the lower pole, *h*, is made negative, so as to project the shadow of the mica star *d*, no shadow is seen on the mica plate, neither is any seen on the uranium plate above the mica. The thin film of mica entirely prevents the uranium glass from becoming fluorescent under the influence of the negative pole. Other experiments have, however, shown that the mica star gives just as sharp and intense a shadow as the aluminium star, provided a suitable screen is used to receive it on.

529. If the aluminium star is made the positive pole, any one of the others being the negative pole, it casts an enlarged and somewhat distorted image of itself all over the upper part of the tube. This image is not sharply defined.

The sharpness of the shadows cast by the negative pole is slightly affected by the intensity of the current; when the spark is very strong, the shadow widens out a little.

530. I have already advanced the theory that the thickness of the dark space surrounding the negative pole is the measure of the mean length of the path of the gaseous molecules between successive collisions (507). The electrified molecules are projected from the negative pole with enormous velocity, varying, however, with the degree of exhaustion and intensity of the induction current (498). In the dark space they are few in number in comparison to what they are at the luminous boundary. When the exhaustion is so high that the mean path of the molecules stretches right across the tube, their velocity is suddenly checked by the glass walls, and the production of light is the consequence of this sudden arrest of velocity. The light actually proceeds from the glass, and is caused by fluorescence or phosphorescence in or on its surface, and not by an evolution of light by the molecules themselves, crowding together and striking each other on the surface of the glass. Had this been the case—had the molecules themselves been the lamps—they would shine equally well whatever were the arresting surface, and their light would have shown the spectral characteristics of the gas whose residue they constituted. But no light is caused by a mica or quartz screen, however near it may be brought to the negative pole; and generally speaking the more fluorescent the material of the screen, the better the luminosity.

531. The theory best supported by experiment, and the one which although new is not at all improbable in the present state of our knowledge respecting molecules, is that the greenish-yellow phosphorescence of the glass is caused by the direct impact of the molecules on the surface of the glass. The shadows are not optical but *molecular* shadows, only they are revealed by an ordinary illuminating effect. The sharpness of the shadow, when projected from a wide pole, proves them to be molecular. Had the projection from the negative pole radiated in all directions, after the manner of light radiating from a luminous disk, the shadows would not be perfectly sharp, but would be surrounded by a penumbra. Being, however, projected material molecules in the same electrical state, they do not cross each other, but travel on in slightly divergent paths, giving perfectly sharp shadows with no penumbræ.

532. It was suggested by Professor STOKES, as an alternative to the above theory,

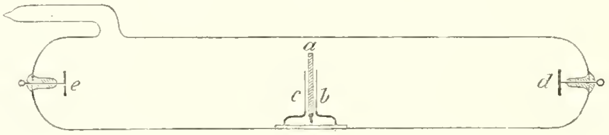
that the phosphorescence of the glass might possibly be excited, as we have hitherto known phosphorescence to be excited, by radiation, instead of being excited by the actual impact of the individual molecules on the molecules of the glass. The difficulty of this supposition was, where could the radiation come from? The sharpness of the shadows preclude the idea that it comes from the immediate neighbourhood of the negative electrode, which is far from being a mere point. It might, however, be supposed that the incandescent molecules were projected with such enormous velocity from the negative, that the space around it was kept comparatively clear; that there were comparatively few molecules within this space, but that, adjacent to the glass, there was a thin layer of them densely packed. Thus, if the molecules be thought of as lamps giving out, not the green light observed, but the invisible ultra-violet radiation which is capable of exciting the green phosphorescence, it is conceivable that the absence of phosphorescence within the sharply-bounded shadows might be accounted for by the absence of the thin overlying phosphorogenic stratum.

PHOSPHORESCENCE OF THIN FILMS.

533. An experiment was suggested by Professor STOKES which seemed calculated to be decisive, or almost decisive, between these different views. If the phosphorescence were really produced by impact, it must be confined to a stratum of almost infinitesimal thinness, but if it were produced by radiation from a thin layer of densely-packed gaseous molecules, glowing with phosphorogenic light, the fluorescence should extend a measurable distance into the glass; and if an excessively thin film of the glass were taken (thin enough to show the colours of thin plates), and placed in front of a thicker plate of phosphorescent glass, the full amount of fluorescence should not be excited on it, but some of the phosphorogenic rays would be likely to pass through, and the thin film should not cast a black shadow. By taking very thin films of different glasses, and also thin plates of quartz, the result would be more conclusively shown, for it is not likely that thin layers of all these media possess a metallic opacity to the phosphorogenic rays.

534. To experiment on these points, an apparatus was made in the following manner:—In the centre of a tube, furnished with flat aluminium terminals, *d*, *e*, is supported a plate of uranium glass, *a*. About half a millimetre on one side is a film of uranium glass, *b*, and on the other side a similar film of German glass, *c*. The two films are thin enough to show brilliant colours of thin plates when viewed by

Fig. 14.



reflected light, and are only large enough to partially cover the uranium plate. The plate and the two films are supported on a light aluminium frame.

535. When the tube is exhausted to within a few millionths of an atmosphere, and the pole *d* is made negative, the uranium film *b* casts a strong shadow on the uranium plate *a*. No phosphorescence can be detected on the part of the plate which is beneath the films, but where the plate is uncovered it shines with its characteristic greenish light.

536. On making contact with the coil, the film *b* flashes out suddenly, all over its surface, with a yellowish-green phosphorescence, which instantly disappears except at the part where it is stuck to the aluminium support at its back. The plate does not seem to become phosphorescent quite suddenly, but the phosphorescence remains permanent as long as the coil is kept at work.

537. When the contact hammer of the coil is screwed so as to make the spark excessively faint, the thin film remains more luminous than the plate; but if the screw is turned so as to get an intense spark, the luminosity of the film sinks, and that of the uncovered part of the plate increases; the plate now being three or four times brighter than the film.

538. If an intense spark be used and the contact made and broken suddenly, so as to send one spark at a time through the tube, only the film becomes visible, flashing out with a brilliant yellowish-green light, the plate, even in its uncovered part, remaining dark.

539. Exactly similar phenomena take place if the pole *e* be made negative, so as to experiment with the film of German glass, *c*. The only difference is in the colour of the phosphorescent light, which is much yellower with German than with uranium glass. In other tubes, English glass, which phosphoresces with a bluish light, has been found to behave in the same way. A very thin film of quartz casts a perfectly opaque shadow, but it does not become phosphorescent. The same thing occurs with mica.

540. These experiments are conclusive against the phosphorescence being caused by the radiation of phosphorogenic ultra-violet light from a thin layer of arrested molecules at the surface of the glass, for were this the cause the film could under no circumstances be superior to the plate.

The momentary phosphorescence of the film, and its rapid fading out, prove more than this. The molecular bombardment is too much for the thin film. It responds to it at first, but immediately gets heated by the impacts and then ceases to be luminous, except where it is attached to the aluminium support, which conducts the heat away. The plate, however, stands the heating without getting hot enough to lose its power of phosphorescing. The experiment demonstrates for the first time the transference to a finite distance, from molecule to molecule, of even that short period vibration on which phosphorescence depends, and it further exhibits experimentally the decadence of phosphorescence by transformation into a disturbance of longer period, which is quickly propagated from molecule to molecule, and carried away as heat.

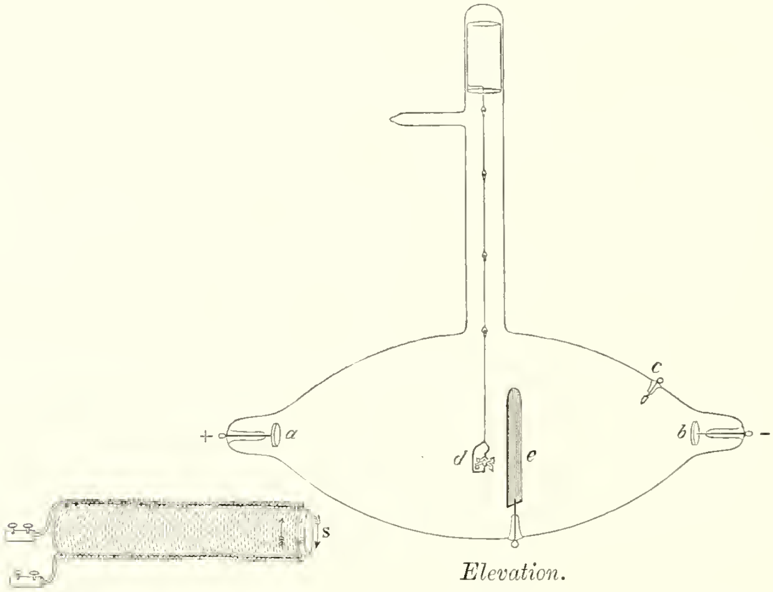
The cessation of phosphorescence by the heating of the phosphorescent body was observed by Professor STOKES. In his paper on the "Change of the Refrangibility of Light," in the Phil. Trans. for 1852, at page 532, it is mentioned that glasses and certain sulphides, which were phosphorescent when cold, lost their sensitiveness on heating, but recovered it again on cooling.

MECHANICAL ACTION OF PROJECTED MOLECULES.

541. It was noticed that when the coil was first turned on, the thin glass film was driven back at the moment of becoming phosphorescent. This seemed to point to an actual material blow being given by the molecular impact, and the following experiment was devised to render this mechanical action more evident.

A large somewhat egg-shaped bulb (fig. 15, elevation) is furnished at each end with flat aluminium poles, *a* and *b*; a pointed aluminium pole is inserted at *c*. At *d*, a little indicator is suspended from jointed glass fibres, so as to admit of being brought into any position near the middle of the bulb, by tilting the apparatus. The indicator consists of a small radiometer fly 8 millims. in diameter, furnished with clear mica vanes 2 millims. across, and delicately supported on a glass cup and needle point. A screen, cut out of a flat aluminium plate 12 millims. wide and 30 millims. high, is supported upright in the bulb at *e*, a little on one side of its axis, being attached to the bulb by a platinum wire passing through the glass, so that if needed the screen *e* can be used as a pole.

Fig. 15.



Elevation.

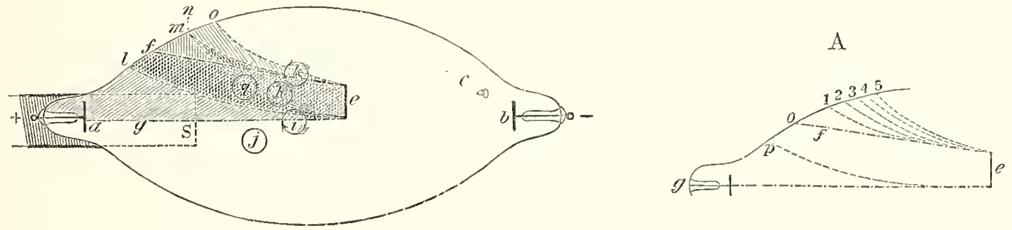
542. This apparatus was designed with a double object. The indicator fly is not blacked on one side or favourably presented, therefore if immersed in a full stream of projected molecules, there will be no tendency for it to turn one way rather than the

other. If, however, I tilt the bulb so as to bring the indicator half in and half out of the molecular shadow cast by the screen, I should expect to see the fly driven round to the right or to the left by the molecules striking one side only, thus confirming the observation on the movement of the thin glass film under the molecular impact (541).

543. The other subject which I had in view was the following. It is well known that a movable conductor carrying a current of electricity is deflected under the influence of magnetic force, and experiments tried very early in this research, and repeated with the apparatus already described, showed that the stream of molecules projected from the negative pole obeyed in a very marked manner the power of a magnet. It was hoped that the form of apparatus now under experiment would throw some light on this action of magnetism on molecules.

544. The bulb being exhausted to the necessary high degree, the pole *b* is made negative, so as to cast the shadow of the screen *e* across the tube, where it can be traced as a broad band along the lower part of the glass. Fig. 16, plan, shows the appearance, the shadow of *e* projected by the pole *b* being enclosed within the lines *f e*, *g e*, shaded diagonally.

Fig. 16.



Plan.

The indicator fly is first brought into position *h*, where it is entirely screened from the molecular stream; no movement takes place. The apparatus is slightly tilted, till the fly comes into position *i*, half in and half out of the shadow; very rapid rotation takes place in the direction of the arrow, showing that impacts occur in the direction anticipated. The apparatus being further tilted, so as to bring the indicator quite outside the shadow into position *j*, no movement takes place. When the indicator is brought to the other side of the shadow into position *k*, the rotation is very rapid in the direction of the arrow, opposite to what it is when at *i*.

MAGNETIC DEFLECTION OF LINES OF MOLECULAR FORCE.

545. An electro-magnet is placed beneath the bulb, shown at S in fig. 15, elevation, and by dotted lines at S in fig. 16, plan. A battery of from 1 to 5 Grove's cells is connected with the magnet. The current is made to pass in such a direction that the pole under the bulb (marked S) is the one which would point towards the south were the magnet freely suspended.

546. The induction current being turned on, the shadow of *e* is projected straight along the tube in the position *f e*, *g e*; the edge of the shadow *f* is called the zero position, marked 0 on fig. 16, A. The electro-magnet is now excited by 1 cell. The shadow is deflected sideways to the position shown by 1, *p*, the edge *f e* now moving to 1, and the edge *g e* moving to *p e*. The lines *f e* and *g e*, when under no magnetic influence, are marked along the bottom of the bulb by perfectly straight lines, but when deflected by the magnet they are curved as at 1 *e*, *p e*.

On increasing the number of cells actuating the electro-magnet, the deflection of the shadow likewise increases. Thus, the following is the deflection of the edge *f e* under varying magnetic force :—

No. of cells.	Position on drawing. (Fig. 16, A.)	Distance deflected, measured from zero, or *f*.
		millims.
0	*f e*	0
1	1	6
2	2	10
3	3	13
4	4	16
5	5	17·7

547. The width of the deflected shadow varies in like manner with the magnetic force, as seen by the following table :—

No. of cells.	Width of shadow, measured along the side of the bulb.
	millims.
0	39
1	24
2	17
3	13
4	10
5	7·5

With 3 cells the position of the shadow is shown at *l m* in fig. 16, plan, shaded from left to right by diagonal lines, and with five cells the shadow is in position *n o*, indicated in the figure by horizontal lines. The position of the shadow when not under magnetic influence is represented by diagonal lines running from right to left.

On reversing the battery current passing round the magnet, the above-named deflections are obtained in the opposite direction.

548. The indicator fly is brought into position *h* (fig. 15, plan), and the induction spark passed, the pole *b* being negative. The shadow entirely envelopes the indicator, and no movement is produced. The current from 3 cells is then sent round the electro-magnet. This twists the shadow round to position *l e*, *m e*, and thereby brings half the indicator into the stream of molecules. Instantly the fly rotates with great speed in the normal direction, the same as it did at *i*. The indicator is now brought into position *q*, where it is entirely immersed in the shadow whilst deflected by the magnet. No movement occurs until battery contact is broken with the electro-magnet, when the shadow instantly returns to its normal position, and the further half of the fly being thereby left uncovered it rotates rapidly in the same direction as it did at *k*.

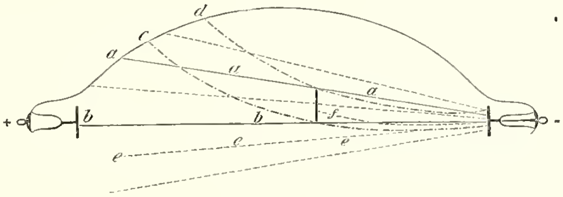
549. The pole *b* being negative, no difference in any of the phenomena here recorded could be detected, whether the positive wire was connected with the pole *a* or the pole *c*, or even when the screen itself, *e*, was made the positive pole. (519, 526, 527.)

550. The position of the magnet was now moved from under the positive pole *a*, to under the negative pole *b*, its axis still remaining parallel to that of the axis of the bulb. The shadow was now deflected when the magnet was excited with two cells, to a greater extent than it was with five cells in the former position.

THE TRAJECTORY OF MOLECULES.

551. The indicator in the apparatus just described (541) is supported by a thin fibre of German glass, which becomes highly fluorescent in the molecular stream. By moving it about in different parts of the bulb the shape of the shadow can be determined as it passes through space. Fig. 17 shows the trajectory of the molecules forming the shadow

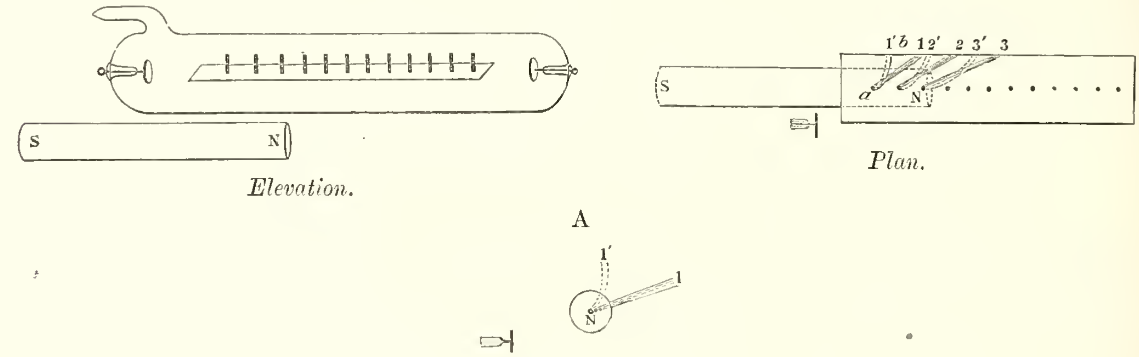
Fig. 17.



when under the influence of the electro-magnet actuated by 3 cells. The normal position being represented by *a a*, *b b*, the deflected shadow is shown by *c c*, *d d*. On tracing *c d* back past the screen to the negative pole the trajectory of the molecules forming it is found to be a curved line. The lines *b b* and *a a* are straight, but the corresponding lines, *c c* and *d d*, are curved. Moreover, the line *c c* starts from the negative pole in a contrary direction to that which it ultimately takes. This shows that the line of molecules whose straight path forms the boundary line of shadow *b b* is not the same line of molecules which forms the curved line *c c*. The action of the magnetic force is to twist the whole of the trajectories of the molecules round in a direction at an angle to their normal path, and to a greater extent as they are nearer the magnet, the direction of twist being that of the electric current passing round the electro-magnet. The line *c c* is therefore not the line *b b* twisted out of its place, but is another line, *e e*, which is bent round from *e* to *c* by the magnet, the line *b b* passing away somewhat in the direction *f f*.

552. A long glass tube, fig. 18, was made with a flat fluorescent screen running along its whole length; this screen had 12 vertical glass pegs fixed into it at intervals of 25 millims. apart. The negative pole of the coil was not quite in the axis of the tube, but a little on one side, so as to cast sloping shadows of the pegs across the fluorescent plate. When the magnet was placed in the position S N, but not excited, the shadows fell as at 1, 2, 3; but when the current was turned on the shadows twisted round to 1′ 2′ 3′, shadow No. 1 bending in at *a* as well as out at *b*, and the others acting in a similar way to a less extent.

Fig. 18.



Elevation.

Plan.

A

553. When the north pole of the magnet is placed vertically under one of the pegs, as at fig. 18, A, the shadow is twisted from position 1 to 1′.

554. These phenomena of magnetic deflection are obtained with permanent magnets as well as with electro-magnets. A copper wire helix without the iron core acts like an electro-magnet, but fainter, when an electric current is passed through it. The shadows experimented with in the tubes and bulbs already described are easily deflected by bringing a magnet near.

ALTERATION OF MOLECULAR VELOCITY.—LAW OF MAGNETIC DEFLECTION.

555. It has been shown (546) that the position of the edge of the shadow is affected by varying the magnetic power used to deflect it. It became of interest to see if by keeping the magnetic power constant, the position of the edge of the shadow could be altered by any circumstance affecting the intensity of the spark, such as intercalating a Leyden jar in the circuit, or screwing the contact breaker one way or

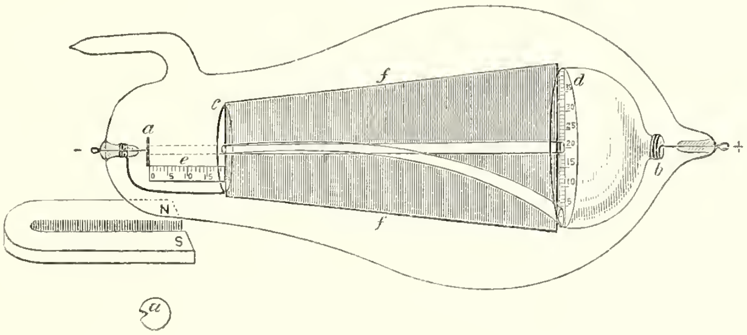
the other. If the molecules are projected from the negative pole with different velocities we might expect that under a constant magnetic deflection the higher velocities would show the flatter trajectories. With the apparatus shown at figs. 15 and 16, these variations in the trajectory of the molecules were not obtained in a decided manner, although indications of an alteration of curve by intensifying the spark were apparent.

556. Another apparatus was accordingly made in order to test this point, and also to obtain a more definite relation between the dimensions of the dark space round the negative pole, the commencement of the green phosphorescence, and the magnetic deflection under varying conditions of pressure in different gases.

I have spoken of shadows being deflected by the magnet as a convenient way of describing the phenomena observed; but it will be understood that what is really deflected is the path of the molecules driven from the negative pole and whose impact on the phosphorescent surface causes light. The shadows are the effect of a material obstacle in the way of the molecules.

In the apparatus now about to be described, a ray of light was used instead of a shadow. Fig. 19 shows the arrangement.

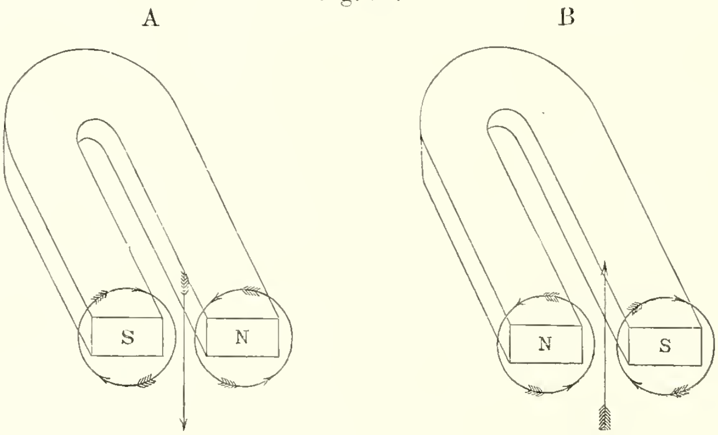
Fig. 19.



The poles are at *a* and *b*. The negative pole *a* is a flat aluminium disk with a notch cut in it. The pole *b* is a ring of aluminium; *c* is a mica screen with a small hole in the middle about 1 millim. in diameter; *d* is a flat plate of German glass with a millimetre scale engraved on it vertically; *e* is a mica scale of millimetres. The scale *e* is to measure the thickness of the dark space as the exhaustion proceeds; the hole in *c* is to enable a spot of light to be thrown on the scale *d* from the pole *a*; the notch in the pole is to enable me to see if the spot of light projected on *d* is an image of the pole *a*, or of the hole in *c*; the scale on *d* is to enable me to measure the deflection of the ray proceeding from *a*, through *c*, to *d*, when bent by the magnet; *ff* is a vertical screen of mica in the plane of the movement of the ray, covered with a phosphorescent powder. On this the path of the ray traces itself in a straight line when the magnet is absent, and curved when the magnet is present.

557. The magnet used is a compound horseshoe magnet capable of supporting about 5 lbs. It is placed underneath the negative pole, as shown in fig. 18 at N S. The experiments already tried with the straight electro-magnet show that the molecular ray is deflected in a spiral to the right or to the left according to the pole presented to it. When two opposite poles, as in a horseshoe magnet, are placed in a line with the normal direction of movement of the molecules, one tends to twist it spirally to the right, and the other to the left (fig. 20, A, B). The resultant direction is that the ray is bent in a curve, up or down, along a plane at right angles to the plane of the magnet, and the line joining the poles, according to whether the N pole is to the right or to the left. The magnet is so arranged that it can be put in position or moved away as often as necessary, with the certainty of always being replaced in exactly the same position.

Fig. 20.



The first observations were taken with air at different pressures. Exhaustion = 411·7 millionths of an atmosphere. The thickness of the dark space in front of the negative pole, measured along the scale *e*, is 6·5 millims. The bulb is filled with blue and violet light; no green phosphorescence is visible.

558. Exhaustion = 161·8 M. Thickness of dark space = 8·5 millims. Some green phosphorescence is seen near the negative pole, but no projection of an image is yet seen on the screen *d*.

559. Exhaustion = 102·6 M. Thickness of dark space = 12 millims. A spot of green light is now seen projected on the scale *d*, bright in the centre and shading off at the edges. The spot is about 5·5 millims. diameter. The centre of the spot, the magnet being away, is at 18·3 millims. on the scale. On placing the magnet in position, the spot of light is bent vertically down to 5 millims. on the scale, or a distance of 13·3 millims.

The magnet would easily bend the light much more if brought nearer, but it was not thought advisable to get a greater deflection for fear of the light coming off the scale in other gases.

560. Exhaustion = 55·2 M. Thickness of dark space = 15 millims. The green phosphorescence is increasing in the bulb. The spot of light on the screen is better defined, but no image of the notch can be seen in it. The magnet brings the spot down to about the same position as in the last experiment.

561. Exhaustion = 30·5 M. Thickness of dark space = 16 millims., but it is getting very faint. The spot of light on the screen is much more brilliant and is also sharply defined. The magnet brings the spot down to 6·8 on the scale, showing a deflection of 11·5 millims. The effect of greater exhaustion is therefore to flatten the trajectory of the molecules.

562. Exhaustion = 26·3 M. Thickness of dark space about 25 millims., but it is too ill-defined to measure accurately. The bulb is very green, and only a little blue light is visible near the positive pole. The spot of light keeps steady at 18·3 on the scale; it is very brilliant, but somewhat ill-defined in outline. When the magnet is put into position the spot sinks to 8·5, showing a deflection of 9·8 millims.

When the spot of light is bent down to division 8·5 on the scale, its position can be varied about 1 millim. by altering the intensity of the induction spark.

563. Exhaustion = 9·7 M. The dark space has now disappeared, and the whole tube, except the parts in shadow, glows with a bright greenish-yellow light. The spot of light is now no longer of any particular shape, but may be likened to a *splash* of green light on the scale. It is also unsteady in position, the centre varying between 18° and 20° on the scale. The magnet makes it more steady and brings it down to 16°, giving a deflection of about 3 millims.

Exhaustion = 8·9 M. The spot of light is too unsteady to enable its position to be determined with accuracy. The magnet brings it down between 2° and 3°.

564. It being found difficult to get a higher vacuum than this in so large a bulb, carbonic acid was let in, and after exhausting and refilling several times with pure carbonic acid, the following experiments were tried.

Exhaustion = 456·5 M. Thickness of dark space = 4·5 millims. No spot of light is to be seen on the scale, and no green phosphorescence in the bulb.

565. Exhaustion = 264·4 M. Thickness of dark space = 6·5 millims. Traces of green phosphorescence are visible, but there is no spot of light on the scale.

566. Exhaustion = 206·6 M. Thickness of dark space = 8·5 millims. The projected spot of light can just be detected with difficulty on the screen, at about 18° on the scale.

567. Exhaustion = 59·2 M. Thickness of dark space = 12·5 millims. Green light is appearing in the tube and the spot of light on the screen is brighter. The magnet causes it to sink 14°.

568. Exhaustion = 28·4 M. Thickness of dark space = 13·5 millims. The spot of light on the screen is brighter, and a well defined circle, no notch being seen on its edge. The magnet brings it down 11°.

569. Exhaustion = 10·2M. Thickness of dark space is about 21 millims., but it is

very ill-defined and appears broken in the centre, leaving two curves of light at the sides. The spot of light is unsteady and ill-defined. The magnet brings it down 9·5°. By increasing the intensity of the spark as much as possible, the deflection caused by the magnet is diminished, the spot only sinking 6·5 millims. instead of 9·5 millims.

570. The apparatus was now filled with electrolytic hydrogen, and after exhausting and refilling with the gas several times, the following experiments were tried :—

Exhaustion = 427 M. Thickness of dark space = 7 millims. There is on the screen an exceedingly faint spot of green light, which is deflected by a magnet, but it is not bright enough to allow observations to be taken.

571. Exhaustion = 147 M. Thickness of dark space = 12 millims. The spot of light on the screen is now brighter and better defined. The magnet being brought into position lowers the spot 13 millims. The light in the tube shows hydrogen, both by the colour and to the spectroscope ; the green phosphorescence is spreading.

572. Exhaustion = 27 M. Thickness of dark space = 15·5 millims. The spot of green light is much brighter and very well defined. The magnet lowers it 11·5 millims.

573. Exhaustion = 14·8 M. Thickness of dark space = 17 millims. The spot of light is bright and round. The magnet brings it down 10·5 millims. The tube is of a brighter green than usual, and there is less hydrogen light about the poles.

574. Exhaustion = 10·25 M. No dark space is to be seen. The light on the screen is no longer well defined, but is a mere splash of very bright green light. The rest of tube is of a brilliant green colour. The magnet lowers the spot of light about 3 millims.

575. Exhaustion = 7·25 M. The exhaustion is getting rather too high for the green light, which is not quite so bright. The sparks will not always pass through. The magnet has very little action on the spot of light.

576. These experiments prove several important points. In par. 559, when working with an air vacuum, it is recorded that the spot of green light is visible on the screen at a pressure of 102·6 M when the thickness of the dark space is only 12 millims. from the pole. Assuming, as I do, that this is a measure of the free path of the molecules before collision, it follows that some of the molecules sufficient to cause green phosphorescence on the screen, are projected the whole distance from the pole to the screen, or 102 millims., without being stopped by collisions. It is probable that this would have occurred at a still lower exhaustion, for on reference to par. 566, it is seen that the green spot was detected on the screen when the mean path in carbonic acid was 8·5 millims., and it was seen with hydrogen (570) when the mean path was only 7 millims.

577. If we suppose the magnet permanently in position, and thus exerting a uniform downward pull on the molecules, it is seen that their trajectory is much curved at low exhaustions, and gets flatter as the exhaustion increases. A flatter trajectory corresponding to a higher velocity of the molecules, it follows that the molecules move quicker the better the exhaustion. This may arise from one of two causes : either the

initial impulse given by the negative pole is stronger, or the collisions are less frequent. I consider the latter to be the true cause. The molecules which produce the green phosphorescence must be looked upon as in a different state from those which are arrested by frequent collisions. These impede the velocity of the free molecules and allow longer time for the magnetism to act on them; and although the deflecting force of magnetism might be expected to increase with the velocity of the molecules, Professor STOKES has pointed out that it would have to increase as the square of the velocity in order that the deflection should be as great at high as at low velocities.

Comparing the free molecules to cannon balls, the magnetic pull to the earth's gravitation, and the electrical excitation of the negative pole to the explosion of the powder in the gun, the trajectory will be quite flat when no gravitation acts, and gets curved under the influence of gravitation ; it is also much curved when the ball passes through a dense resisting medium ; it is less curved when the resisting medium gets rarer, and, from par. 562, it is seen that intensifying the induction spark, equivalent to increasing the charge of powder, gives greater initial velocity, and therefore flattens the trajectory. The parallelism is still closer if we compare the evolution of light seen when the shot strikes the target, with the phosphorescence produced in the glass screen by molecular impacts.

578. In carbonic acid the mean free path of the molecules is shorter for the same degree of exhaustion than in air, and the velocity of the molecules measured by the flatness of the trajectory under magnetic influence is also lower than what it is in air at similar pressures.

579. Although in many cases, especially at a moderate exhaustion and using a feeble spark, the image on the screen was a perfectly well-defined circle, I could detect no image of the notch cut in the negative pole. The hole in the mica was quite small enough to have given a good image of the negative pole inverted on the screen had it been shining by ordinary light, but the rays being corpuscular and the particles not crossing, no image of the pole is formed, but only the image of the hole in the mica.*

580. Attempts to obtain continuous rotation of the ray of molecular light by means of a magnet have hitherto failed. The stream of molecules does not obey AMPÈRE's Law as it would were it a perfectly flexible conductor joining the negative and positive pole. The molecules are projected from the negative pole, but the position of the positive pole, whether in front, at the side, or even behind the negative pole, has no influence on their subsequent behaviour, either in causing phosphorescence, producing mechanical effects, or in their magnetic deflection (519, 526, 527, 549). The magnet seems to give them a spiral twist, greater or less, according to its power, but diminishing as the molecules get further off, and independent of their direction.

---

* With another apparatus, in which the pole and screen are nearer together and the mica with the hole in it capable of being moved to and fro, I have succeeded in seeing an ill-defined image of the negative pole. This only occurs at low exhaustions and soon disappears, giving way to a sharp image of the hole.

FOCUS OF MOLECULAR HEAT OF IMPACT.

581. In the experiment described at pars. 536 to 540, the heating effect of the molecular bombardment is assumed to be the cause of certain phenomena. It was thought that by concentrating the molecular impacts to one point the heat produced might be rendered apparent. The experiments tried in the apparatus shown in fig. 21 prove that this supposition is correct. A polished aluminium cup *a* is made the negative pole in a properly exhausted tube. The focus is seen very sharp and distinct, as at A, and of a dark blue colour. The light, although blue in the centre of the tube, when it spreads out and strikes the tube at the end, illuminates it beautifully with the yellowish-green light. By means of a magnet the focus was deflected to the side of the tube, as shown at B, the path of the rays being beautifully curved. On the tube the appearance was that of a sharply-defined oval of a yellowish-green colour with a dark spot in the centre. To ascertain if heat were developed here I touched it with my finger and immediately raised a blister. The spot where the focus fell was nearly red hot.

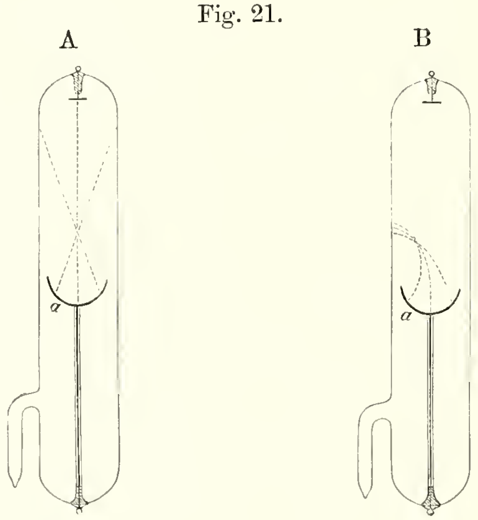


Fig. 21.

582. Another apparatus was now made, as shown in fig. 22. A nearly hemispherical cup of polished aluminium *a* is made one pole in a bulb, and a small disk of aluminium *b* is made the other pole. At *c* a strip of platinum is held by a wire passing through the glass, and forming another pole at *d*. The tip of the platinum strip is brought to the centre of curvature, and the whole is exhausted to a very high point. On first turning on the induction current, the cup being made the negative pole, the platinum strip entered into very rapid vibration. This soon stopped, and the platinum quickly rose to a white heat, and would have melted had I not stopped the action of the coil.

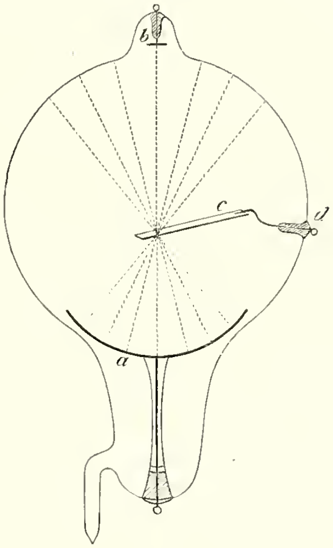
The same phenomena of ignition take place if the platinum strip itself is made the positive pole.

583. Experiments in a similar tube in which a piece of charcoal is the body to

be ignited, show that the ignition takes place at a less high exhaustion in hydrogen gas than it does in air.

I have great pleasure in expressing my continued obligation to the great skill in glass-blowing and manipulation possessed by my friend and assistant, Mr. C. H. GIMINGHAM, whose dexterity in executing complicated forms of apparatus has rendered easy a research which otherwise would have been full of difficulties.

Fig. 22.



584. I hope I may be allowed to record some theoretical speculations which have gradually formed in my mind during the progress of these experiments. I put them forward only as working hypotheses, useful, perhaps necessary, in the first dawn of new knowledge, but only to be retained as long as they are of assistance; for experimental research is necessarily and slowly progressive, and one's early provisional hypotheses have to be modified, adjusted, perhaps altogether abandoned in deference to later observations.

AN ULTRA-GASEOUS STATE OF MATTER.

585. The modern idea of the gaseous state of matter is based upon the supposition that a given space of the capacity of, say, a cubic centimetre, contains millions of millions of molecules in rapid motion in all directions, each having millions of encounters in a second. In such a case the length of the mean free path of the molecules is excessively small as compared with the dimensions of the vessel, and properties are observed which constitute the ordinary gaseous state of matter, and which depend upon constant collisions. But by great rarefaction the free path may be made so long that the hits in a given time are negligible in comparison to the misses,

in which case the average molecule is allowed to obey its own motions or laws without interference; and if the mean free path is comparable to the dimensions of the vessel, the properties which constitute gaseity are reduced to a minimum, and the matter becomes exalted to an ultra-gaseous or molecular state, in which the very decided but hitherto masked properties now under investigation come into play.

The phenomena in these exhausted tubes reveal to physical science a new world—a world where matter may exist in a fourth state, where the corpuscular theory of light may be true, and where light does not always move in straight lines, but where we can never enter, and with which we must be content to observe and experiment from the outside.

VI. *On the Sensitive State of Electrical Discharges through Rarefied Gases.*

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Received April 2,—Read May 8, 1879.

[PLATES 15–20.]

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I.—*Definition and description of the sensitive state.*

It has frequently been remarked that the luminous column produced by electric discharges in vacuum-tubes occasionally displays great sensitiveness on the approach of a finger or other conductor to the tube. The exact effect of such approach varies considerably with the circumstances of the discharge. In many instances the luminous column is repelled; in others, and especially when the finger is brought into actual contact with the glass, the column is severed; and in the latter case, in addition to the luminosity previously present, there often appears proceeding from the interior of the tube, at the point where the finger rests, the blue haze which usually characterises the negative end of a discharge. In some cases the discharge is so powerfully affected that the well-known green or blue fluorescence appears on the side of the tube opposite to the point touched.

The degree of sensitiveness varies between wide limits. Discharges frequently occur in which close observation is necessary to detect any trace of it, while others may be produced so sensitive that the magnetic action of a powerful electro-magnet is hardly more marked than the action which is due to it as a conductor. The condition in question does not appear to be confined to any particular gaseous medium or to any special form of tube; and it is in fact probable that in almost any tube sensitiveness may be produced by adopting suitable precautions.* This state of sensitiveness may be exhibited by stratified discharges, but it is more commonly associated with those which show no clear traces of stratification. It is not however universally present in either kind of discharge.

The state in which the discharge is affected by the presence or approach of a conductor will be called the *sensitive state;* while that in which it is not so affected will be called the *non-sensitive state.* The former is the subject of the present inquiry.

The attention of the authors of this paper was drawn to this form of vacuum discharge, partly by the striking character of the phenomena which it presents and partly by their own earlier observations which led them to connect it with an interruptedness or intermittence in the current. Although there are a great variety of methods by which the sensitive state can be produced, they all agree, as will presently be seen, in causing a rapid intermittence in the current. The experiments of Mr. De La Rue and others have furnished strong evidence for concluding that vacuum discharges are never continuous in the same sense that the flow of electricity along a metallic conductor is continuous, but that they are always disruptive and periodic with an extremely high rate of periodicity. Such being the case, it appeared desirable to subject to special examination a form of discharge in which periodicity (though much less rapid) is produced by artificial means, and where the periodicity is therefore capable of being varied at will. Moreover, the usefulness of an investigation into the phenomena of discontinuous discharges is not wholly dependent on the theory which regards all vacuum discharges as discontinuous; for, as the mathematician is often obliged to reason from the discontinuous to the continuous—from the polygon to the curve—so the physicist may ultimately be best able to arrive at the causes of the phenomena accompanying the ordinary vacuum discharges (whether these be absolutely continuous or only relatively so) by a study of the phenomena which rapidly intermittent discharges present. And the importance of this method is greatly increased when we consider that no one of the usual phenomena of ordinary vacuum discharges is necessarily absent from or incompatible with the discontinuous discharge; and that if the theory of the discontinuous character of the ordinary vacuum discharge be correct, this method offers us the prospect of constructing a continuous chain of phenomena, extending from the single flash of a Leyden-jar discharge to the steadiest striated column which can be produced by a battery-current.

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* Subsequent experiments have removed all doubt on this point. So long as the tube permits any traces of a luminous column to appear, this column can be rendered sensitive.

II.—*The sensitive state is due to a periodic intermittence in the discharge of considerable rapidity and regularity, the quantity of electricity in each individual discharge being sufficiently small to permit the discharge to be instantaneous.*[*]

In order to establish this proposition we shall first describe all the modes known to us of producing sensitive discharges, and shall show that in all cases the circumstances of the discharge give great probability to the idea that it is discontinuous or intermittent in the manner above described. We shall then give direct experimental evidence that in all these cases such intermittence actually exists in the resulting discharge. And lastly we shall examine the cases in which we know that intermittence actually exists in non-sensitive discharges, and shall show that such intermittence has not the requisite characteristics stipulated for above, but is of a wholly different type from that which we have described as the cause of sensitiveness in vacuum discharges.

The simplest method of producing the sensitive state is by conducting through the exhausted tube a current from a Holtz machine or some other constant source of electricity of high tension, and interposing a small air-spark in the path of the current from one of the terminals of the machine to the tube. The following plan will sufficiently describe the apparatus used. So soon as the air-spark is made to intervene (see Plate 15, fig. 1), the discharge in the tube becomes sensitive; and this sensitiveness may in general be increased by increasing the length of the air-spark, until the discharge becomes visibly intermittent, so as no longer to appear to the eye as a steady continuous discharge. Although this is by no means the only way in which the sensitive discharge can be produced, it is the one which is the most generally convenient for the purposes of experiment; and it may on that account be regarded as the typical mode of production. That the discharge within the tube should be intermittent when the supply of electricity is rendered intermittent by the air-spark is so obvious as not to need discussion, and though we cannot positively assert (without experimental evidence of the fact) that the pulses of free electricity of which the discharge is composed pass into the tube instantaneously, it is difficult to imagine any circumstances under which we should less expect the individual discharges to have a prolonged or continuous character than when, as in the present case, they are due to charges of free electricity that come suddenly to the terminal of the tube after leaping in the form of a disruptive discharge across the air-spark interval.

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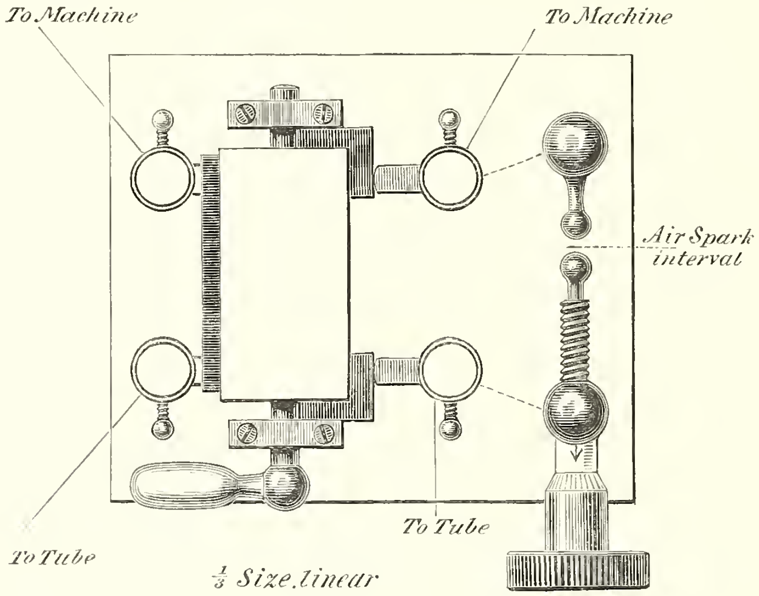
[*] In speaking of the sensitive state as being due to an intermittence in the discharge, the authors of this paper are treating of discharges lasting for a finite time. It was, naturally, in this form that sensitiveness was first known to them; but in the course of the inquiry they have discovered, as will hereafter appear, that each individual discharge is sensitive, and that the sensitiveness of the intermittent discharge is due solely to the sensitiveness of its individual members. The effect of the intermittence is only to change through rapid iteration the instantaneous phenomena presented by each discharge into an appearance which to the eye is steady and continuous.



Another method of producing a discharge which shows indications of sensitiveness is by using an induction coil in connexion with a large condenser, such as that described in a paper "On Stratified Discharges, V." (Proceedings of the Royal Society, xxvii., p. 60). If the discharge be allowed to pass through the tube while the coil is at work, a certain amount of sensitiveness will usually be visible. (See Plate 16, fig. 6.) But if the coil be stopped and the current allowed to flow from the condenser through the tube without disturbance from the entrance of the coil-discharges, the discharge will be found to have lost all its sensitiveness. The explanation of this we believe to be that so long as the coil was at work its pulsations produced sudden periodic elevations of tension in the condenser, which caused corresponding discharges through the tube. It is probable that in these cases the periodic discharges are only superposed upon the continuous discharges which we see upon stopping the coil, and which are caused by the mean tension in the condenser, so that it would be more strictly accurate to say that the discharge is subject to intermittent or periodic variations of intensity than to say that it is itself intermittent. But, as all the phenomena agree with the supposition that the discharge is partly sensitive and partly non-sensitive, and that the sensitive portion is wholly caused by the pulsations produced by the coil, it is better to look upon it as a superposition of two discharges, and to regard the sensitive one as intermittent.

It is not, however, universally the case that the discharge from the condenser is non-sensitive when the coil is not acting. When the tension in the condenser becomes low, it is very usual to find sensitiveness in the discharge. This phenomenon, however, does not in any way make against the view that sensitiveness arises from intermittence; for the very last stages of the discharge from the condenser are intermittent even to the eye, and it is not strange therefore that the stage immediately preceding should also be intermittent, though with a much more rapid period. That such is the true explanation of the phenomenon can also be shown directly, either by means of a rapidly revolving mirror, or by interposing a telephone in the circuit, as will be more fully described hereafter.

Again, certain tubes appear to render the discharge from a continuous source sensitive without the necessity of artificially producing intermittence in the current. Such cases will, however, be found to offer no difficulties. They are merely instances of tubes which at higher tensions of the condenser give the phenomenon already described as being so common when the tension is low. No doubt, if the tension in the condenser, or perhaps, more strictly speaking, if the quantity of electricity discharged (strength of current) were still further increased, the sensitiveness could in all cases be made to vanish.*

Again, a sensitive discharge may be produced by connecting one terminal of the machine with one terminal of the exhausted tube, and the other terminal of the machine with the outside of the tube, as in Plate 15, figs. 2 and 3. If the current be then permitted to pass between the terminals of the machine by leaping a considerable distance (say half an inch) in air, so that the discharges in the tube are caused partly by conduction from one terminal of the machine, and partly by induction due to the rapid alternations of high and low tension in the wire from the other terminal of the machine to the outside of the tube, the resulting discharge will be found to be sensitive. The arrangement here described, and which may be represented by the subjoined diagram, was devised by Mr. P. WARD, in order to show the phenomena of "CROOKES' Lines."* When the sparks passed to P, the outer surface of the glass near P was traversed, over an area of an inch or more in diameter, by irregular bright lines branching out from P as a centre. From the point of the inner surface adjacent to P, CROOKES' radiating lines shot out in various directions across the tube, and made themselves visible by fluorescence on the opposite side in the neighbourhood of R. The fluorescence figure, usually in the form of a cross, could be made to revolve about its centre by the action of a magnet; the direction of the rotation being reversed when the polarity of the magnet was reversed. In this case the principal discharges in the tube take place when the passage of a spark from the one terminal of the machine to the other has caused a sudden relief of tension in the wires leading to the tube, so that the discharge has the same intermittence as the disruptive discharge between the terminals of the machine; in other words, it has the same intermittence as a long air-spark, and direct experiment with a revolving mirror has shown that the luminous phenomena in the tube opposite to the wire are synchronous with the passage of the sparks from one terminal of the machine to the other.

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* It is worthy of remark, that the correlative proposition has been found to be in general true of the tubes examined by the authors of this paper for the purposes of this investigation. Though the greater number of them exhibit no trace of sensitiveness when the current is produced by the full power of a large twelve-plate HOLTZ machine, yet they can be made to show sensitiveness (without the introduction of an air-spark) by throwing some of the plates out of work, so as to reduce the quantity of electricity in the current.

Again, rapid intermittence and sensitiveness in what would otherwise be a continuous discharge may be produced by the use of a "wheel-break," such as is described (see Plate 15, fig. 3) in a paper "On Stratified Discharges, III." in the Proceedings of the Royal Society for February 15, 1877. This instrument consists of a wheel platinized at the edge, on which a platinum spring rests. In the circumference of the wheel a number of slots are cut, and filled with ebonite plugs, so as to interrupt the current. If the wheel-break be interposed in the circuit of a HOLTZ machine when producing a luminous discharge in a vacuum-tube, and the break be worked at a considerable speed so as to cause the current to be interrupted some 400 to 2000 times per second, the discharge becomes highly sensitive. This instrument is used as a shunt, viz., so as to divert from the tube the current given by the machine by permitting it to pass through a continuous metallic circuit during the time that the platinum wire rests upon the metallic portion of the circumference. In this way the current is never actually broken, and one great advantage of this arrangement is that it simply produces intermittence in the discharge through the tube without interfering with it in any other way.

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* See Proceedings of the Royal Society, December, 1878.

Another method of producing a sensitive discharge is by the use of a very rapidly vibrating break, such as that described in a paper "On Stratified Discharges"*

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* The following is an extract from the paper in question:—

"In the stratified discharges through rarefied gases produced by an induction-coil working with an ordinary contact-breaker, the striæ are often unsteady in position and apparently irregular in their distribution. Observations made with a revolving mirror led me to conclude that an irregular distribution of striæ does not properly appertain to stratification, but that its appearance is due to certain peculiarities in the current largely dependent upon instrumental causes.

"Having reason to think that the defects in question were mainly due to irregularity in the ordinary contact-breaker, I constructed one with a steel rod as vibrator, having a small independent electro-magnet for maintaining its action. The natural vibrations of the rods which were tried varied from 320 to 768 per second; while under the action of the battery-current and electro-magnet they varied from 700 to 2500, or thereabouts, per second. The amplitudes of the vibrations were exceedingly small, in fact not exceeding ·01 of an inch; and it is to this fact, coupled with the extreme rapidity and consequent decision of make and break, that I mainly attribute the steadiness of the results.

"With a contact-breaker of this kind in good action several phenomena were noticeable; but first and foremost was the fact that, in a large number of tubes (especially hydrocarbons) the striæ, instead of being sharp and flaky in form, irregular in distribution, and fluttering in position, were soft and rounded in outline, equidistant in their intervals, and steady in proportion to the regularity of the contact-breaker. These results are, I think, attributable more to the regularity than to the rapidity of the vibrations. And this view is supported by the fact that, although the contact-breaker may change its note (as occasionally happens), and in so doing may cause a temporary disturbance in the stratification, yet the new note may produce as steady a set of striæ as the first: and not only so, but frequently there is heard, simultaneously with a pure note from the vibrator, a strident sound, indicating that contacts of two separate periods are being made; and yet, when the strident sound is regular, the striæ are steady. On the other hand, to any sudden alteration in the action of the brake (generally implied by an alteration in the sound) there always corresponds an alteration in the striæ.

"It is difficult to describe the extreme delicacy in action of this kind of contact-breaker, or 'high break,' as it may be called. The turning through 2° or 3° of a screw, whose complete revolution raises or lowers the platinum pin through ·025 of an inch, is sufficient to produce or to annihilate the entire phenomenon. A similar turn in a screw forming one foot of the pedestal of the brake is enough to adjust or regulate the striæ; and a slight pressure of the finger on the centre of the mahogany stand, apparently rigid, or even on the table on which the contact-breaker stands, will often control their movements.

"The discharges described above are usually (although not always) those produced by breaking contact; but it often happens, and that most frequently when the strident noise is heard, that the current produced by making contact is strong enough to cause a visible discharge. This happens with the ordinary as with the high break; but in the latter case the double current presents the very remarkable peculiarity that the striæ of one current are so arranged as to fit exactly into the intervals of the other; and, further, that any disturbance affecting the column of striæ due to one current affects similarly, with reference to absolute space, that due to the other, so that the double column moves, if at all, as a solid or elastic mass. And this fact is the more remarkable if we consider, as is easily observed in a revolving mirror, that these currents are alternate, not only in direction, but also in time, and that no one of them is produced until after the complete extinction of its predecessor. And it is also worthy of note that this association of striæ is not destroyed even when the two currents are separated more or less towards opposite sides of the tube by the presence of a magnetic pole. There seems, however, to be a tendency in that case for the

(Proceedings of the Royal Society, June 10, 1875). If such an instrument be used with an induction coil, the discharge, though often beautifully stratified, is intensely sensitive. The lowest limit of rapidity with which we have produced *stratified* sensitive discharges in this way is 240 breaks per second.

We will now consider the direct experimental evidence that the current is intermittent in all cases in which the discharge is sensitive. This evidence is derived from the revolving mirror and the telephone. If the body of the tube containing the discharge be hidden by an opaque screen which contains a narrow longitudinal slit, and the image of this slit be observed in a rapidly revolving plane mirror, a series of bright and black bands appear whenever the discharge is sensitive, showing that there are

striæ of one current to advance upon the positions occupied by those of the reverse current, giving the whole column a twisted appearance. But as there is no trace, so far as my observations go, of this association of alternate discharges when produced by the ordinary break, we seem led to the conclusion that a stratified discharge, on ceasing, leaves the gas so distributed as to favour, during a very short interval of time, a similar stratification on the occurrence of another discharge, whether in the same or in the opposite direction.

".The column of striæ which usually occupy a large part of the tube from the positive towards the negative terminal have hitherto been described as stationary, except as disturbed by irregularities of the break. The column is, however, frequently susceptible of a general motion or 'flow,' either from or towards the positive pole, say a forward or backward flow. This flow may be controlled, both in velocity and in direction, by resistance introduced into the circuit, or by placing the tube in a magnetic field. The resistance may be introduced in either the primary or the secondary circuit.

" When the striæ are flowing they preserve their mutual distances, and do not undergo increase or decrease in their numbers. Usually one or two remain permanently attached to the positive electrode; and as the moving column advances or recedes, the foremost stria diminishes in brilliancy until, after travelling over a distance less than the intervals between the two striæ, it is lost in darkness. The reverse takes place at the rear of the column. As the last stria leaves its position, a new one, at first faint and shadowy, makes its appearance behind, at a distance equal to the common interval of all the others: this new one increases in brilliancy until, when it has reached the position originally occupied by the last stria when the column was at rest, it becomes as bright as the others. The flow may vary very much in velocity; it may be so slow that the appearances and disappearances of the terminal striæ may be watched in all their phases, or it may be so rapid that the separate striæ are no longer distinguishable, and the tube appears as if illuminated with a continuous discharge. In most cases the true character of the discharge and the direction of the flow may be readily distinguished by the aid of a revolving mirror. In some tubes, especially in those whose length is great compared with their diameter, the whole column does not present the same phase of flow; one portion may be at rest while another is flowing, or even two conterminous portions may flow in opposite directions. This is seen also in very wide tubes, in which the striæ appear generally more mobile than in narrow ones. But in all cases these nodes or junction-points of the flow retain their positions under similar conditions of pressure and current; and it therefore seems that, under similar conditions, the column in a given tube always breaks up into similar flow-segments.

" These nodes will often disappear under the action of a magnetic pole. Thus if the first segment, measured from the positive terminal, be stationary and the second be flowing backwards (*i.e.*, from − to +), a magnetic pole of suitable strength, placed at the distant end of the latter, will stop its flow, and the whole column will become stationary throughout. An increase in the strength of the magnet, or a nearer approach of it to the tube, will produce a general forward flow of the column."

intervals between the luminous discharges, during which the tube is dark.[*] If the ordinary non-sensitive discharge be observed in a similar way there are no such dark bands, and no available speed of the mirror suffices to show any break in the uniformity of the luminous image of the slit. The occurrence of these dark bands shows conclusively that the discharge in which they appear is intermittent and discontinuous, and it may be stated as a general proposition that a revolving mirror has thus invariably shown intermittence in the discharge whenever a sensitive column has been examined by it through a slit of proper dimensions, whatever may have been the particular means by which the sensitiveness has been produced.

We have said that the telephone, as well as the revolving mirror, may be made to show that there is intermittence in the discharge in vacuum tubes so long as there is sensitiveness. The evidence it gives to us is as follows: When a telephone is in circuit with a non-interrupted, non-sensitive discharge, a rushing sound is frequently although not always heard. In a very large number of cases there is absolute silence. But as soon as the discharge becomes sensitive the silence is broken by a shrill sound, or if the rushing sound of which we have spoken previously existed, there is a sudden change in the character of the sound, which usually becomes musical. The pitch of the note is always high, and naturally varies with the circumstances of the discharge. When the air-spark is again abolished the note ceases, or gives way to the rushing sound mentioned above.[†] Occasionally, in the sensitive discharge, the rushing sound and the musical note are heard simultaneously. This may arise from rapidly succeeding periods of interruptedness and non-interruptedness; but it is not a phenomenon of any particular importance at the present stage of our inquiry.[‡]

It will have been noticed that at the outset we defined the intermittence capable of causing sensitiveness to be such as would permit only so small a quantity of electricity to pass at each individual pulsation as not to interfere with the instantaneous character of the discharge. This is necessary in order to exclude such cases of intermittence or fluctuation of current as are observable within a single-coil discharge when a comparatively slow break is used. Some observations were published ("On Stratified Discharges, II.," Proceedings of the Royal Society, May 18, 1876) by one of the authors of this paper, relative to the appearance of the coil discharge when viewed in a large revolving mirror moving through one revolution between each contact made by the break.* The rapidity with which the contacts followed each other was necessarily slow in comparison with the rapidity of intermittence of which we have been speaking; but, what is of more importance, the duration of the discharges, consequent upon the nature of the contacts, was far greater. Moreover, it was found that each discharge was of a very complicated character. The discharge opened with a sudden rush,

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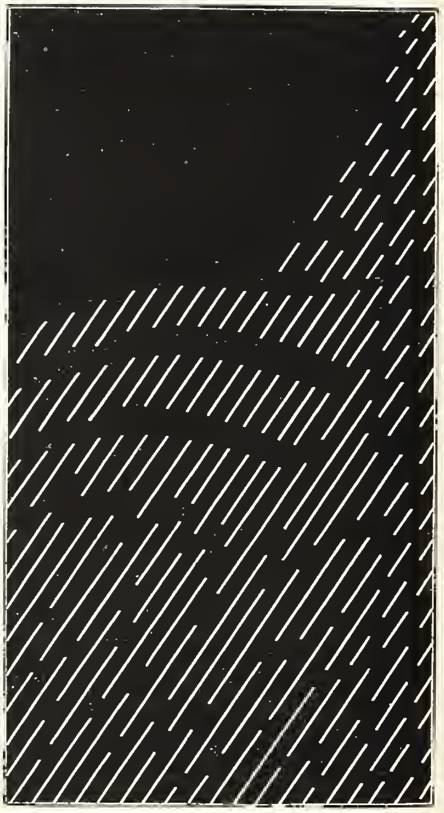
[*] This experiment is more fully described in § 10.

[†] It may fairly be objected that this test shows only the intermittence of the current in the circuit external to the tube, and does not directly tell us anything of the discharge through the tube. We shall, however, show later on how the telephone may be made to give us more direct evidence of the intermittence of the discharge through the tube. But to render clear the nature of this evidence would require that the theory of the sensitive discharge should be more fully developed than is possible at this early stage.

[‡] When the air interval is increased beyond striking distance, a silent discharge takes place across the interval; and the discharge through the tube is the same in character as that produced without an air-interval; it is also non-sensitive. And instead of the shrill whistling which is heard with an air-spark proper, nothing is in this case audible but a soft rushing sound. A similar result is obtained when an arc is formed across the air-spark interval, which is found to be the case under certain circumstances which will be referred to later on.



* The break here used was a mercurial one, the plunger of which worked on a cam attached to the axle of the mirror, so that the action of the contact-breaker was regulated by the motion of the mirror, and the image was always formed in the same position in the field of view. An opaque screen with a slit in it was usually placed on the tube so as to face the mirror. In order to convey a clearer idea of the phenomena alluded to in the text than could otherwise be obtained without reference to the paper itself, one of the diagrams is reproduced. This represents the appearance (in the mirror) of a carbonic acid tube with the slit attached. This tube, viewed by the eye, shows flake-like fluttering striæ, with a slight tendency to flocculency near the head of the column. The commencement of the discharge is at the right hand, and the negative terminal at the top. The drawing fairly represents the appearance of the upper part or head of the column of striæ during one complete coil-discharge. When the battery-surface exposed is small, the whole consists of, first, three or four columns of striæ of decreasing length, and afterwards of an almost unbroken field of striæ. Each of the initial columns is perfectly stratified; and the same disposition of striæ prevails throughout the entire discharge. The striæ which fill the main part of the field present a proper motion nearly uniform, but slightly diminishing towards the end. These striæ are for the most part unbroken, but are occasionally interrupted at apparently irregular intervals. When the battery-surface is increased, the elementary striæ are more broken, and near the head of the column the interruptions occur as in the figure. The separation of the earlier part of the discharge into striated columns divided by intervening rifts does not, excepting in the case of the first, extend far towards the positive terminal. Nevertheless, even as far as the positive terminal itself, there seems at times to be a fuller development of discharge than is subsequently maintained.

The first rift in the discharge, following the first outburst, is sometimes distinguishable even as far as the positive terminal; and perhaps in those cases indicates a real cessation of the discharge. This is corroborated by the fact that a similar interruption is then perceptible in the glow surrounding the negative terminal; but after this the negative glow retains its unbroken character throughout the entire discharge.

usually followed by an apparently complete cessation of discharge; it then recommenced, and continued to exhibit during its entire period of existence variations of a very irregular kind. It is, in fact, probable that during the period subsequent to the first outburst and the momentary cessation of discharge that often follows it, the state of the tube is much the same as it would be during a continuous discharge, in consequence of the absence of any regularity of pulsation. The tube is always conducting a discharge, although perhaps the consequent disturbance is more violent in one part than in another. There is no alternation of a sharply-defined discharge and electrical emptiness. So important is the distinction between what may be called irregularly fluctuating and periodically intermittent discharges, that in the case of a fairly rapid coil discharge it is frequently found that, while the striæ in the main luminosity give but very slight traces of sensitiveness, the faint ghostly striæ which are seen to project far beyond the end of the positive column proper show very great sensitiveness (see Plate 16, fig. 6).* And the experiments above referred to with the revolving mirror show that this faint column of luminosity is due solely to the first outburst of which we have spoken; so that such outbursts form a sensitive system, while the fluctuating discharges that follow are non-sensitive. As the action of the break gets more rapid it is found that the fluctuating portion of the discharge becomes less and less developed, and the discharge more and more nearly consists of the first portion only. It is a matter of observation that as we attain to still higher rapidity of break-action this becomes more and more the case, and that at very high speeds the discharge reduces itself to this first portion alone; and thus we understand how it is that with a very high-speed break we obtain the most intensely sensitive discharges from a coil.

It will thus be observed that all the methods of producing the sensitive state agree not only in the intermittent character of the discharge, but also in the shortness of duration of the individual discharges themselves,† and this it is which has induced the authors of this paper to consider brevity of duration as much an essential feature of the individual discharges that produce the sensitive state, as rapidity or regularity of interval between these discharges. The latter characteristics are, in fact, of more importance for maintaining the persistence of the sensitive state during a finite interval of time than for actually producing it, since careful experiments fail to show any inferior limit of rapidity of the periodicity necessary to produce such a discharge. In truth, there is no impossibility in producing by a single flash a discharge having the characteristics of sensibility. If a charged Leyden jar[*] ("On Stratified Discharges, IV.," Proceedings of the Royal Society, March 22, 1877) be employed with a suitable tube, the instantaneous discharge that passes through the tube on the jar being connected with it will show all the symptoms of sensitiveness during its passage through the tube.

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* These faint sensitive striæ projecting beyond the luminous column in a coil discharge, are the only case of a sensitive discharge of which the authors of the present paper are aware which is not produced by one or other of the methods previously described.

† These remarks may seem to be inapplicable to the case of the Holtz machine with the wheel-break; for here the intervals during which the tube forms part of the circuit are usually equal to those during which it is not so. But it does not follow that the electricity is passing through the tube during the whole of the time that the tube is in circuit. For it will not commence to pass until there has been such a charging up of the terminals as is sufficient to effect a discharge.

This direct evidence of the connexion between intermittence of discharge and the sensitive state will, we believe, derive further support from the accordance of the various phenomena with the theory of the sensitive state which will be developed in the course of this paper, and which rests upon the hypothesis that the sensitive state is due to such an intermittence—a hypothesis which the authors of this paper venture to hope is sufficiently demonstrated by the experimental facts already stated.

In conclusion, the authors of this paper desire it to be understood that, in claiming intermittence as an essential of the sensitive discharge, they do not intend to exclude all ideas of intermittence from the non-sensitive discharge. The experiments of Sir W. GROVE,[†] Mr. DE LA RUE, and others, have shown almost to demonstration that there is intermittence in the most stable and non-sensitive discharges. But such intermittence must be of an order almost incomparably higher than that of which we are speaking. Though sensitiveness has been found to co-exist with intermittence at the highest speeds attainable by the mechanical or chemical means above referred to, and even in many cases to increase with the rapidity of intermittence, it is by no means necessary that it should continue to do so when this rapidity is increased, say, many hundred or thousand fold. Indeed, with the rapidity given by a short and rapid air-spark, the sensitiveness is not so marked in its features as with a longer and less rapid air-spark. Nor, as will be seen, does our view require that very great rapidity of intermittence should necessarily produce very great sensitiveness. On the contrary, so far from its being an advantage that the discharge should be divided into a very large number of small discrete discharges, it would seem that the most favourable condition for the production of sensitiveness is that the individual discharges should be as large (and therefore as few in number) as is compatible with an absence of the prolonged or fluctuating phases observed in the ordinary coil discharge.[‡] And therefore, in order to avoid any misconception, the terms *interrupted* and *uninterrupted* will be used to describe the condition of the currents causing the sensitive and non-sensitive state respectively, instead of the terms *intermittent* and *continuous*, which might otherwise have been adopted.

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[*] This experiment will be found more fully described on p. 208, and properly belongs to a later portion of the investigation, and is referred to here only in passing.

[†] Phil. Mag., 1858, July–December, p. 18.

[‡] It will be found that this conclusion, which was arrived at from observation, is a necessary consequence of the theory of sensitiveness which the authors of this paper desire to put forward.

III.—*The effect produced on a sensitive discharge by the approach of a conductor, is directly due to the relief given by its presence to the instantaneous electric tension within and around the tube, caused by the individual discharges in their passage through the tube.*

In the case of the sensitive or interrupted discharge we have seen that there are discrete pulses of electricity passing between the terminals. It is, therefore, not an improbable supposition that the pulse of positive electricity leaving the positive terminal in the form of free electricity, and consequently exercising electric induction in every direction round it, may cause that electric tension or (as Professor MAXWELL would call it) displacement, which, so far as we know, is the universal accompaniment of free electricity. Such induction would cause an electric tension upon the outer surface of the glass and in the space beyond; but, inasmuch as the glass is a non-conductor, it could not cause any redistribution of the electricity within it or on its surface. It is somewhat difficult to ascertain the effect on the space around the tube, occupied as it is with the mobile and rapidly-moving particles of air; but there can be little doubt that the induction from the discharges would only occasion redistribution of the electricity in this part of the field to a very slight extent. If, however, we place a conductor on or near the glass, the induction can readily occasion a redistribution of electricity in the conductor. The state of the electric field in the neighbourhood of the conductor would then be different to that at any other part of the tube; and this would in its turn react upon the discharge or upon the gaseous matter which exists within the tube. We shall now show that the observed effect is due to such a redistribution of electricity in the conductor; and shall subsequently proceed to examine the cause of this redistribution and its *modus operandi* in producing the observed effect.

In order to demonstrate that the phenomena of sensitiveness are primarily due to a redistribution of electricity in the conductor caused by alterations in the electric condition of the interior of the tube, it is only necessary to observe that a non-conductor, however highly charged, does not affect the sensitive discharge. Nor will a conductor of small size, in contact with the tube (such as a piece of tinfoil), affect the discharge so long as it is insulated. But if the tinfoil be connected to earth, or to a distant conducting body, it at once produces a marked effect on the sensitive discharge.

A consideration of these experimental facts leads to the following conclusions: (1) that the effect is due to a redistribution of electricity in the conductor; and (2) that such effect is periodic. If a continuous electric state in the external body were the necessary condition, the observed effect could be produced by charged non-conductors. But as this is not the case, we must look to the facility of change in electric state afforded by conductors for an explanation of their effect on the sensitive discharge. This conclusion is supported by the fact previously stated that a small piece of tinfoil placed upon the tube produces no effect so long as it is insulated. Such a piece of tinfoil would give but little scope for redistribution of electricity—at all events, in such a way as to affect the space around it. But if it be connected metallically with a distant conducting body, so that positive or negative electricity can be driven from it to a sensible distance from the tube, the case is different; and if this be done it is at once found to affect the sensitive discharge.

If, then, the effect on the sensitive discharge is caused by the facility for a redistribution of electricity within the portion of the electric field occupied by the conductor, it follows that there must be a varying electric action upon it from the discharge in the tube. And that such is the case may be shown by connecting a ring of tinfoil placed round the glass with the earth, and interposing a telephone in the circuit between the tinfoil and the earth. So soon as the current becomes interrupted by an air-spark, a sound is heard in the telephone corresponding with the sound of the air-spark causing the intermittence. This shows conclusively that at each pulsation there is an electrical redistribution within the system composed of the earth, the wire, and the tinfoil. And as this continues indefinitely, without producing any charge upon the tinfoil, it is clear that there must, during the complete period of each pulsation, be a flow of one kind of electricity from the tinfoil, followed by its return or a similar flow of the opposite kind of electricity from the tinfoil.

Now, since this periodic redistribution is effected by the pulsations of electricity within the tube, it follows that we may regard the conductor as affording a kind of relief to the tension in the electric field round the portion of the tube with which it is in contact. And that such is the nature of its action can be shown by numerous experiments, such as the following: If, instead of connecting the tinfoil to earth, a wire leading to earth be placed almost in contact with it, sparks will be seen to pass, showing the state of electric tension to which the tinfoil is from time to time subjected. And further, if the effect upon the sensitive discharge be examined closely, it will be found to be stronger in proportion as the circumstances in which the tinfoil is placed are more favourable for giving more complete relief. So long as the wire does not touch the tinfoil, or permit sparks to pass from or to it, the tinfoil is unable to affect the sensitive discharge. But if the wire be near enough for sparks to pass, the sensitive discharge will be affected, but to a decidedly less extent than would be the case were the wire in actual contact with it. In the latter case the relief which it could afford would be perfect; in the former it would be imperfect, since the sparks would not be able perfectly to equalize the potentials of the tinfoil and the wire. Again, the effect of the tinfoil when in electrical connexion with the conducting body will be found to vary in proportion to the capability of the system to relieve the tension of the tinfoil. If the conducting body be large or distant, the effect will be nearly the same as though the tinfoil were joined to earth; if, on the other hand, it be small and near to the tube, the effect will be much less. The same remarks apply to cases in which, though the tinfoil is not absolutely disconnected from earth, yet its connexion is electrically very imperfect. Thus, if the wire from the tinfoil be allowed just to touch the finger, it will give less effect than if it be tightly held, and similiarly for other imperfect connexions. Even in the case of the telephone in circuit between the tinfoil and the earth, if the telephone be not too far from the tube a marked diminution in loudness will be observed when the connexion between the telephone and the earth is severed, so that the displaced electricity, no longer able to escape to the earth, is compelled to re-arrange itself in the telephone and the wire connected with it.

There is no need to place the conducting body in actual contact with the glass in order to affect the sensitive discharge. If the body be near the tube it will produce nearly the same effect as though it were in contact, as in Plate 15, fig. 5; if it is farther off it will produce less; and, as in the former case, if it is large and capable of permitting displaced electricity to remove to a considerable distance from the tube, it will produce a much greater effect than if the contrary be the case. In short, it may generally be stated that the completeness of the effect of a conducting body upon the sensitive discharge depends on the completeness of the capability it possesses of permitting a displacement of electricity within it (caused by the electric disturbances in the tube) without any material alteration of its potential.

Such experiments as those of the telephone in circuit with the tinfoil and the earth may not unnaturally suggest the idea that these effects may be due to electro-magnetic and not to electro-static induction. But careful examination negatives this supposition. In the first place it is not necessary that the tinfoil should have any considerable length in the direction of the tube, which is presumably the direction of the current through it. A narrow ring of tinfoil round the tube will give all the results above described. Again, if a coil of insulated wire be placed near or round the tube, no difference is produced by joining the ends of the wire. In fact, no difference of effect is observable whether the conductor be made up of insulated wire or be a mere mass of metal; the whole seems to owe its effect to its being a conductor of considerable size. No form of conductor possesses greater effect than a narrow ring of tinfoil connected to earth by a single fine copper wire perpendicular to the tube, and no conducting system could be devised more unfavourable to the production of electro-magnetic inductive currents by the original current in the tube. Moreover, the effect is far too great to be attributable to electro-magnetic induction in circuits so unfavourable to its action. In short, every feature of the case, upon examination, tends to render it less probable that any of the electric phenomena in the external conducting system are due to electro-magnetic induction; and the supposition has been noticed here rather with the view of showing that it has not escaped consideration, than because it was considered to have any valid claims to notice.

The arrangement in which the tinfoil is connected to earth is that which affords the most complete relief to the periodic changes of electric tension produced by the periodic pulsations of electricity through the tube. And, considering that the earth may be regarded as an infinite conducting body of potential zero, it seems justifiable to use the term *complete relief* in connexion with its effect. But in order to render the relief complete, it is not necessary that the potential of the relieving conductor should be zero; it is the relief to the periodic alternations of electric tension which produces the effect. If a Leyden jar be connected with the tinfoil, its effect on the sensitive discharge is the same whether it be charged or not. In short, any conductor of sufficient size connected with the tinfoil will produce the same effect as the earth, at whatever potential it stands.

It is so important to establish clearly that the effect of a conductor is independent of the potential at which it stands, and is due only to the facility for redistribution of electricity which it affords, that we shall here refer to an experiment which very clearly demonstrates such to be the case. Let a HOLTZ machine be taken as the source of the current, and an air-spark be interposed at some part of the current outside the tube, say, for example, in the part between the tube and the positive terminal of the machine, and let a ring of tinfoil be placed round the tube and connected with the earth and the relief-effect duly noted. It will be found that no effect is produced either upon the current* or the relief-effect by connecting either of the terminals of the machine to earth. That no effect should be produced upon the current is not surprising, as the whole effect, like all other electro-dynamic effects, must be due to the difference of the potentials of the two terminals and not to their absolute values. But it is very remarkable that the relief-effect is the same in both cases, inasmuch as in one case the tinfoil is of the same potential as the positive terminal of the machine, which is of higher potential than any other part of the whole circuit (including the tube), and in the other case it is of the same potential as the negative terminal of the machine, which is necessarily of lower potential than any other portion of the circuit. No stronger proof could, we think, be given that the effect is completely independent of the potential of the relieving system.

From this follows a remarkable result, the accuracy of which is, however, fully confirmed by experiment. It is a matter of common occurrence in physics, that where a cause having one period acts upon a subject having a different period, the result is, on the average, the same as though the former remained constant at its mean value. Applying such considerations to the present case, we see that if the tinfoil be joined to a conductor whose electric state periodically changes, but whose period is not the same as that of the discharge in the tube, we may expect the effect to be the same as though it remained constant at its mean value; that is to say, it will be the same as though the tinfoil were connected with the earth (supposing the conductor to be of sufficient capacity), since it is a matter of indifference at what potential the relieving conductor actually stands. Thus to whatever conductor of sufficient capacity the tinfoil be connected, the effect will be the same as though it were connected to earth, unless the conductor have an electrical period identical with that of the discharge in the tube. The effect produced by joining the tinfoil to earth therefore naturally becomes our standard effect, with which all others may be compared. We shall in future call this effect the *to-earth* or *relief-effect.*

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* So far as the current is concerned, the actual experiment when made by the authors of this paper would not support this statement, owing to a peculiarity in the source of electricity then used. This source was a large HOLTZ machine, and it was discovered that there was a large excess of positive electricity in the currents produced by it, so that when the terminals were metallically connected, sparks of a considerable length could be obtained from them. The obvious consequence was that the current through the tube was stronger when the negative terminal was connected to earth than when the positive terminal was so connected. But when allowance was made for this instrumental peculiarity the experiment fully supported the above statement. The peculiarity itself was doubtless due to an escape of negative electricity into the air within the machine.

We have said that experiment fully confirms the theoretical result that any conductor of sufficient capacity not having the same electrical period as the discharge in the tube will give the *to-earth* or *relief*-effect. And this is so. If the tinfoil be connected with either terminal of the HOLTZ machine when the current is obtained therefrom,* or to an independently working coil or to a wire carrying a current with an independent air-spark, the relief-effect will be produced exactly as though the tinfoil had been connected with earth. It must, of course, be understood that if the changes of tension in the conductor be sufficiently violent, they will of themselves produce inductive discharges in the tube, just as they would in the case of a tube with no discharge through it, and the total effect will be the sum of the relief-effect and these inductive discharges. But allowing for such independent effects of the variations in the electric state of the conductor, we have not found any exception to the rule enunciated above, viz., that any conductor of sufficient capacity whose electric state does not suffer variations of like period with the interrupted discharge in the tube will, when connected with the tinfoil on the tube, produce the relief-effect, that is say, the same effect as though the tinfoil had been connected with the earth.

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* There is sometimes, however, a slight heightening of the relief-effect when the tinfoil is joined to the terminal of the HOLTZ machine nearest to the air-spark. This is due, no doubt, to a slight periodic variation of tension there, caused by, and synchronous with, the pulsations of the air-spark, and its period is, therefore, the same as that of the discharge in the tube, so that it presents no exception to the general rule enunciated above.

IV.—*The relief-effect (when the intermittence is effected near the positive terminal)** *assumes the form either of repulsion or of discharge from the interior surface of the glass. These two effects are identical in nature, and the form actually assumed depends, in the same tube, solely on the intensity of the action which calls it forth.*

We shall in the present section suppose that the air-spark or wheel-break, or other contrivance used to cause intermittence, is placed between the positive terminal of the tube and the source of electricity. This being so, the most cursory examination of vacuum-tubes enables us to distinguish two types of effects when a conductor is made to approach to the tube. In the one case the luminous column is repelled by the conductor; in the other the luminosity appears to leap towards the conductor, usually in two tongues, which approach, but do not necessarily touch the glass, one on each side of the place where the conductor most closely approaches the glass, while between the two tongues there appears a luminous haze similar to that which ordinarily surrounds the negative terminal. We shall call these respectively the *repulsion-form* and the *discharge-form* of the *relief-effect* (Plate 16, fig. 7).

Further examination shows that these two forms of the relief-effect† include all cases. It is not uncommon to find both forms present at once. The luminous column is not always wholly interrupted; the two tongues which branch from it towards the conductor being often connected at their bases by the remains of the luminous column. This connecting piece usually shows signs of having been repelled by the conductor, and though not remarkable in other respects, it is in most cases less bright than the remainder of the luminous column. The *discharge-form* again is often destitute of one or other of its members; the two tongues may be very feeble, and the haze may be so faint as to be practically invisible. The magnitude of the repulsion-effect also varies greatly, depending, as will be seen, upon all the causes which are at work in producing the sensitive discharge. In many cases the sensitiveness is only indicated by a slight depression of the luminous column when the finger is placed on or very close to the glass; in others, the finger cannot be brought anywhere near the glass without the luminous column being strongly repelled. But these variations do not affect the truth of the statement that the effects of bringing a conductor near to the tube consist in all cases of one of the above described types or of a mixture of the two; and the doubtful cases can be shown to be cases in which the faintness of the effect creates a difficulty in recognizing their true features rather than cases in which a foreign element is present in the appearances themselves. We shall presently examine the causes which increase sensitiveness, but we wish first to examine the relationship in which the two forms of the relief-effect stand to one another.

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* In thus examining the effects produced when the air-spark is in the "positive circuit" (*i.e.*, between the positive electrode and the source of electricity), before dealing with the matter in its more general form, the authors of this paper are following the actual course of their investigations. It has been thought advisable so to do because the class of effects examined in this section fully merit a separate treatment, and are more conveniently dealt with in this way than in connexion with the more general theory.

† By using the term *relief-effect*, the authors of this paper do not mean anything further or other than the effect, on the luminous discharge, of a neighbouring conductor, whose condition is such that its potential remains more or less completely unchanged by the electric disturbances in the tube. No theory of the *modus operandi* of these effects upon the luminous discharge is intended to be conveyed in the term. It was selected because the electric strain in the tinfoil was *relieved* by the process of joining it to a larger conductor, and has been retained because no more convenient term has been found.

Take a tube exhibiting a sensitive discharge, and find a spôt where on contact with the finger the discharge-effect is manifested. Instead of placing the finger on the glass, commence with the finger at so great a distance from the spot that no visible effect is produced on the sensitive discharge. Now slowly approach the finger to the spot chosen on the tube. The first symptom of sensitiveness will in all cases be a slight repulsion of the luminous column; this will, on nearer approach of the finger, become more decided, and at length a slight haze will become visible between the repelled luminous column and the spot on the glass. This will, if the process be continued, rapidly develop into the ordinary discharge-effect. The change from the state of greatest repulsion to that of the discharge-effect is usually extremely rapid, so much so as to give rise to the idea that there is an absolute discontinuity at this point; but if care be taken to get a steady current, and the approach be very gradual, it can be shown that there is a continuous graduation from the one state into the other.

A more accurate method of trying the experiment consists in fixing a piece of tinfoil or thin metal, bent to the curvature of the tube, at the end of a glass rod, and connecting it to earth by a thin wire running along the rod. If this be made to work through a socket in a stand so as to admit of complete control, it will be found that the luminous discharge may be made to pass continuously through all gradations from almost imperceptible repulsion to the discharge-effect merely by diminishing the distance of the tinfoil from the tube. Actual contact is, as may easily be imagined, the most favourable arrangement for the production of discharge-effect, but it is by no means essential to it, nor is there indeed any discontinuous or even very rapid change in the effect when contact commences. The experiment fully bears out the principle previously enunciated, that the more complete the relief the greater the effect on the sensitive column, understanding for the future that the discharge-effect must rank higher than the repulsion-effect, or, in other words, must be considered as an intensified form of it (Plate 16, fig. 8).

A very slight modification of the arrangement last described will serve for an experiment which yet more clearly demonstrates that the intensity of the relief-effect depends on the completeness of the relief afforded to the static induction caused by the free electricity in the tube, and that for this purpose the discharge-effect must be taken as an intensified form of the repulsion-effect. If the tinfoil, instead of being connected to earth, be connected to any system of conductors insulated from earth, it will be found that by increasing the capacity of the system we can cause the relief-effect produced to vary from slight repulsion to the full to-earth effect. A very striking way of exemplifying this is by taking a thin copper wire, some 18 inches or 2 feet in length, and fixing its two ends on two glass rods. Coil the wire round one till it is all wound round it, and place the other so that the end of the wire is in electrical contact with a piece of tinfoil placed on the tube at some spot where the discharge-effect is strongly manifested when the tinfoil is put to earth. The capacity of the system in such shape is so small that it will only produce a slight repulsion. If now the rod which is in contact with the tinfoil be held stationary, and the other be removed from it, the wire being allowed to uncoil itself as the rods separate, it will be found that the repulsion gradually increases and passes almost imperceptibly into the discharge-effect. The removal of the other end of the wire from the tube has simply had the effect of increasing the capacity of the system to give relief to the tinfoil when under the action of the static induction of the electricity in the tube, and this, after causing the effect on the column to increase from faint to strong repulsion, further intensifies it by making it pass into the still stronger form of discharge-effect.

There are other ways of passing continuously from the repulsion-effect to the discharge-effect; and these not only support our view of the identity of the two effects, but by comparison with the results just obtained throw some light upon the causes of these phenomena. If the finger be passed along a tube containing a discharge of considerable sensitiveness, it will usually be found that from the positive terminal to a certain distance from it the discharge-effect is produced, but beyond that the repulsion-effect. If we stop at any point where the discharge-effect is produced and vary the effect by substituting for the finger some conducting system capable of giving only imperfect relief, we shall find that the nearer we are to the positive terminal the more difficult it is to reduce the effect to mere repulsion; in other words, the relief-effect tends to assume its more intense form the nearer we approach to the positive terminal.

A third method of causing the repulsion-effect to pass continuously into the discharge-effect is by increasing the violence of the interruptions which render the current sensitive. If a very small air-spark be introduced into the "positive circuit" of an uninterrupted discharge, the luminous column will be rendered slightly sensitive, but such sensitiveness will be manifested solely in the form of the repulsion-effect. If the air-spark be increased so that the pulsations become more violent but less frequent, this sensitiveness will increase, and the relief effect near the positive terminal will assume the discharge form, while the rest of the luminous column will only show repulsion. If the air-spark be still further increased, it will be found that the limits of the discharge-effect gradually extend themselves further and further along the tube until the whole column is capable of manifesting it; and this state of things remains until the interruptedness of the discharge becomes so great by reason of the length of air-spark, and the consequent slowness of the intermittence, that the discharge becomes flickering and unsteady, and ceases to be, in the proper sense of the term, a sensitive discharge.

It appears, then, that we can pass continuously from the repulsion to the discharge form of the relief-effect in any one of three ways :

(1.) By keeping the relieving system fixed at the same spot on the tube, and varying the completeness of relief, the discharge remaining the same ;

(2.) By keeping the relieving system constant, and varying its position on the tube, the discharge remaining the same ;

(3.) By keeping the relieving system constant and fixed in position, and varying the interruptedness of the discharge.

It is probable that there is a fourth way, viz.: by varying the quantity of the discharge, all other things remaining the same. The only phenomenon bearing on this point at present observed is that when the condenser is allowed to run down, the discharge-effect appears where previously there had only been repulsion. This might be supposed to show that decrease of quantity tends to give the discharge-effect, but it is probable that the reverse is the case, and that the phenomenon in question is due to the increased interruptedness—the greatly longer period of the intermittence of the discharge—which accompanies the falling of the tension in the condenser, and which, as we have seen, would tend to give the discharge-effect. The importance of slowness of intermittence in giving the discharge-effect is shown by the fact that on working the coil with a very high-speed break the sensitive discharge thereby produced only gives repulsion-effects, while so great is the tendency of the coil discharge when a slower break is used to give the discharge-effect, that it will often appear in a luminous discharge when the condenser and coil are used under circumstances which, from the size of the air-spark, would lead us not to expect it, but will vanish when the coil is stopped and the condenser allowed to work alone ; thus showing that its appearance was due to the pulsations of the coil making themselves felt in the discharge, and by their slowness and violence causing the discharge-effect in places where the more rapid and equable intermittence of the air-spark would not have produced it. No doubt some portion of the above effect is due to the greater violence of the coil pulsations, but on the whole it tends to support the other experiments which show that the discharge-effect is intimately connected with great interruptedness of current.

It must not be thought that all tubes give both forms of relief-effect. So far as the observation of the authors of this paper has extended, all tubes give a repulsion-effect ; but it is comparatively rare to find tubes which present both the forms. Speaking generally, the typical effect, especially with tubes of high resistance, is a repulsion not very great in amount. A further discussion of the signification of those forms of relief-effect will be necessary when we consider the analogues to them which exist in ordinary vacuum discharges.

In accordance with what has previously been stated, when the air-spark is on the positive side, the same effects are obtained by connecting the tinfoil to the negative terminal, or to any independent source of electricity of sufficient magnitude, as by connecting it to earth. In such cases, therefore, the question whether the effect will assume the discharge or the repulsion form will be decided by the same circumstances that control it in the case of the connexion to-earth ; viz., the position of the tinfoil relatively to the positive terminal, and the nature of the discharge, or, as the latter may in all probability be expressed, its magnitude and degree of interruptedness.

We therefore have, as has before been pointed out, a standard phenomenon at every point of a tube illuminated by a sensitive discharge, viz., the *to-earth* or *relief*-effect. We can get other effects connected with this, and to a great extent dependent upon it, by varying the completeness of the relief afforded. These effects form a continuous series from zero up to the complete relief-effect ; and this series can easily be passed through continuously by some such process as that above described, in which two glass rods and a connecting wire were employed to form the relieving system. Hence, if at a point of the tube we obtain by any process an effect on the sensitive luminous column, we can at once ascertain whether it belongs to the category of relief-effects or not. If it does not, there must exist some foreign element other than the mere facility for granting electrical relief in the influences brought to bear upon the tube. We now proceed to examine these special or non-relief effects.

V.—*On the special or non-relief effects produced on the sensitive luminous discharge by connecting it with the air-spark terminal.*

In the present section we shall at first suppose the intermittence to be caused by an interruption in the circuit between the source of electricity and the positive terminal of the tube. For convenience we shall express this fact by calling the positive terminal *the air-spark terminal.*

If we connect a wire with a piece of tinfoil placed upon the tube, and connect the wire with any independent conducting system, we shall obtain, as we have seen, more or less complete forms of the relief-effect. Both the wire and tinfoil will, in the majority of cases, repel the luminous column. But if the wire be connected with the positive terminal a sudden change takes place in the phenomenon. Instead of the luminous column being repelled by the wire, the course of the latter along the tube (supposing it partly to rest upon the tube) will be marked by a bright line of luminosity on the inner surface of the glass as though it had attracted the luminous column instead of repelling it. And the effect of the presence of the tinfoil is changed in a no less remarkable manner. Instead of the former repulsion, a tongue of luminosity will be seen apparently starting from the actual inner surface of the glass under the tinfoil and stretching toward the negative terminal of the tube, while the luminous column on the positive side of the tinfoil is usually depressed or repelled, and is often well nigh severed in two. If the tinfoil be in the form of a ring round the tube the appearance of the phenomena is very striking. In many cases the luminous column extending from the positive terminal is brought to an abrupt termination and ends in a sharply defined head, somewhat rounded at the extremity. Around this there is a well marked hollow cone of luminosity, springing from the inside of the tube immediately beneath the tinfoil, bright and sharply defined on the outside but hazy and blue on the inside, which is turned to and in fact surrounds the termination of the positive column above described. This hollow cone does not come to an apex on its external surface, but passes into a luminous column which stretches away towards the negative terminal of the tube, and supplies the place of the former luminous column, which it resembles in all respects (Plate 16, fig. 10). When the air-spark is considerably increased the truncated luminous column is very much altered; but as the various forms presented by the phenomena will require a close examination at a later stage we shall not detail them here.

The one thing to which it is necessary to call attention at the present stage in regard to these effects is their complete dissimilarity to those we have described as relief-effects. As we have shown in a previous section, each part of a tube in which a sensitive discharge is passing has a standard relief-effect called the *to-earth* effect, produced by connecting it with a conducting system of practically infinite capacity. A continuous gradation of relief-effects from zero to this *to-earth* effect can be obtained by limiting the capacity of the relieving system so that it can no longer be subjected to the influence of the electrical disturbances that take place in the tube without its potential undergoing material alteration. Now, if we compare the appearances described in this section with any member of this chain, we see they are wholly dissimilar, so that we at once learn that these effects are not due to the relieving capacity of the system with which the tinfoil is now connected, but to some special electrical interference to which the new arrangement has given rise. It is on the ground of this dissimilarity that we shall call these effects *special* or *non-relief* effects; and it must be borne in mind that these effects are only obtained when we connect the tinfoil with the air-spark terminal or make use of an arrangement electrically equivalent to this.

If the negative be the air-spark terminal, there will be found to be an equally great contrast between the effects obtained by joining the tinfoil to earth and to the air-spark terminal. As before, the relief-effects can be obtained by joining the tinfoil to any conducting system of sufficient capacity, whatever be its potential, provided that it does not undergo variations of potential co-periodic with those within the tube. The special or non-relief effects can, on the other hand, be only obtained by connecting the tinfoil with the negative terminal. It will not, however, be necessary here to examine or contrast these effects, as they will subsequently be shown to fall under a general rule; and it suffices here to state that the contrast above referred to is found to exist.

VI.—*On the nature of the electrical actions by which the relief and non-relief effects are respectively caused or accompanied.*

It is clear, as we have seen, that tinfoil placed upon the tube and connected to earth produces its effect on the luminous discharge by permitting the electric forces caused by or arising from the discharge within the tube to produce electric displacement within it and the system connected with it. And as it is immaterial how the actual displacement is produced, we may, without affecting the result, suppose that there is a suitable charge of electricity sent into the tinfoil from some source without the tube. This charge will, of course, be the same in quantity, but opposite in sign, to that which passes out of the tinfoil to the earth. We may thus discard all thoughts of relief or non-relief, and consider both states as being due to charges sent into the tinfoil from external sources at the proper times. And in the case of the relief-effect it is evident that these charges will be co-periodic with the electrical changes in the tube since they are caused by them.

In considering this subject we are at once struck with the fact that the non-relief-effect has been, up to this stage of the inquiry, produced only by connecting the tinfoil electrically with the air-spark terminal. It is evident that the peculiarity possessed by this source of electricity does not arise from its high or low potential, for we have seen that the potential does not affect the power of the conducting system to influence the luminous discharge; nor, as we have also seen, would mere fluctuations of potential in the wire produce any other than the relief-effect, or interfere with or destroy it, unless they were of like period with the pulsations in the tube. Hence the special effect in question must be due to the fact that connexion with the air-spark terminal interferes in a special manner with the electrical state of the tinfoil, and the peculiarity must consist in the fact that the period of the charges (and consequent variations of potential) is such that they keep time with the pulsations in the tube.

Now the displacement of electricity, which would otherwise give the standard relief-effect, may be interfered with either by exaltation, or by reversal, or by diminution. The last alternative is excluded by the experimental fact that these special or non-relief-effects do not resemble any of the series of relief-effects which we have described, and which range from zero to the complete to-earth effect.

To decide between the remaining alternatives we may adopt a similar course. Like the relief-effect, the special or non-relief-effect may be obtained in other than the complete form we have already described. In that form a piece of tinfoil placed upon the tube is connected with the air-spark terminal. If, instead of resting on the tube, the tinfoil (still connected with the air-spark terminal) be fastened to an insulated rod and made to approach the tube, a series of special effects are produced which pass from zero to the complete non-relief-effect. Now if this complete non-relief-effect were merely an intensification of the relief-effects we should expect to obtain the ordinary relief-effects, or some close approximation to them, at some part of this chain of special effects. But such is not the case. From the commencement to the end the special effects are wholly dissimilar to the relief-effects, showing clearly that the special effects cannot be due to an alteration in intensity of the same kind of electrical pulses that produce the relief-effects, but must be due to pulses of an opposite character.

This is, of course, not the only proof that the electrical pulses that produce the two effects are of opposite kinds. In fact, an examination of the appearances of the two effects is almost enough to convince the observer that it is impossible that one can be a mere heightening of the other, so radical is the dissimilarity, especially in their more marked forms. But the proof given above is the simplest and most direct experimental proof, and it is sufficient at the present stage, even when taken alone. It will, however, be subsequently found that there is strong confirmation of the correctness of this view.

We conclude, therefore, that the displacements of electricity in the tinfoil caused by joining it with the air-spark terminal are the same in period but opposite in character to those which produce the relief-effect. But it is evident, taking the positive terminal to be the air-spark terminal, that the former consist of periodic pulsations of positive electricity at the moments when the air-spark passes.* Hence in the case of the non-relief-effect, the tinfoil receives a pulse of positive electricity each time a spark passes. This must be, as we have seen, of the opposite character to that which causes the relief-effect. Hence to every pulsation coming from the air-spark there corresponds in the case of the relief-effect a pulsation of negative electricity to the tinfoil; that is to say, a pulsation of positive electricity *from* the tinfoil.

This conclusion is exactly what we might have expected on the assumption that when the positive is the air-spark terminal the positive electricity passes intermittently through the tube in pulses of the same period as the air-spark. If each time the air-spark passed there were a sudden charging-up of the positive terminal and the portion of the circuit metallically connected therewith, causing a sudden discharge of positive electricity into and through the tube, this positive electricity, on reaching the part near to the tinfoil, would, by its induction, tend to drive away positive electricity from the tinfoil, and to attract negative electricity to it. This is what we find to be the case, and similar reasoning will apply to the case when the negative is the air-spark terminal. Therefore we shall assume that in the relief-effect there is a pulse from the tinfoil of the same character as the excess of electricity within the part of the tube opposite to it, or as it may be more simply stated, a pulse of electricity to the tinfoil of the opposite character to the electricity which is discharged from the air-spark terminal; and that in the non-relief-effect there is a pulse to the tinfoil of the same kind as the electricity which is discharged from that terminal.

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* If it is necessary to give an experimental proof for a conclusion so obvious, it is sufficient to connect this portion of the circuit to earth, placing a telephone in circuit, when the air-spark note will distinctly be heard, showing that there are periodic variations of potential accompanying each spark.

The first takes place at the moment that the electricity in the tube passes the spot where the tinfoil is; the second keeps time with the charging-up of the terminal which leads to the discharge, and is therefore most intense at the moment of discharge. And, further, we have the experimental fact that under ordinary circumstances these effects are of a diametrically opposite kind.

This is, perhaps, the most suitable place for explaining the connexion between intermittence and sensitiveness in vacuum discharges. The effects observed in sensitive discharges have been traced in the preceding part of the paper to the effect of static induction from the free electricity that passes into the tube at each pulse. Now, in order that this static induction outside the tube may be strong, it is clearly necessary that the quantity of free electricity within the tube while the discharge is passing should be great. In other words, the current must pass in large pulses and not in driblets. This is exactly what intermittence effects. The electricity is penned back until it has accumulated in large quantities, and then in one pulse it bursts into and sweeps through the tube. In the so-called continuous current, also, the electricity doubtless passes into the tube in pulses, but as these pulses are perhaps many thousand times more numerous in any given time than is the case with the sensitive discharge, the quantity in each pulse is proportionately smaller, and is thus insufficient to produce a static induction which is of sufficient magnitude to be capable of causing in its turn discharges within the tube. Hence the so-called continuous current is really a sensitive current of infinitely small sensitiveness. The difference between the two is much the same as between the effect of letting the superfluous waters of a river overflow a weir each moment, and of penning them back for a whole day and then suddenly letting them free.

VII.—*Examination and interpretation of the special or non-relief-effect when the positive is the air-spark terminal.*

For the purposes of this section we shall take first the form of special or non-relief-effect already described. It is a very marked form, but by no means an unusual one, and indeed it is probable that it can be obtained in a more or less complete form in most vacuum tubes; and it possesses, as we shall see, the important property of being the typical form from which all others can be derived by modifications of known and definite kinds corresponding to peculiarities in the circumstances of the tube or the discharge.

If we place round the tube a narrow ring of tinfoil, and connect it with the positive terminal (where the air-spark is supposed to be) by a wire passing at a sufficient distance from the tube to prevent its directly affecting the luminous column, the following appearances will be noticed :—

(1.) The column which starts from the positive terminal will be found suddenly to terminate at the tinfoil ring in a bright column of small diameter occupying the céntre of the tube. This column is usually striated, and ends in a stria with rounded head.

(2.) On the side of the tinfoil towards the negative end of the tube a conical column of luminosity is seen to start from the inside of the tube immediately beneath the tinfoil, and to stretch towards the negative terminal. This cone, in fact, forms the base of the new positive column.

(3.) On examination, this luminous cone is found to be hollow, the interior having an ill-defined and hazy surface in strong contrast with the somewhat sharp and regular outline of the exterior (Plate 16, fig. 10).

Let us consider what is the explanation of these appearances. We know that strong pulsations of positive electricity pass to the positive terminal of the tube and the tinfoil, keeping time with the passage of the air-sparks. These pulsations, when they arrive at the terminal, are of sufficient intensity to cause a discharge to pass through the tube, and the pulsations that reach the tinfoil must be of exactly the same strength as those that go to the terminal. Such pulsations must drive off positive electricity in corresponding pulsations from the interior parts of the tube contiguous to the tinfoil. These latter pulsations are similar to the discharges that take place from the positive terminal, and they seek relief in the same manner, viz.: by rushing towards the negative terminal of the tube. In this process they form the hollow luminous cone mentioned above. These discharges of positive electricity from the inner surface of the tube leave behind them an excess of negative which would be held prisoned by the positive charge in the tinfoil if that were permanent; but just as the latter was generated by the momentary charging-up due to the passage of the air-spark, so it is released by the relief given to such charging-up by the discharge through the tube. On such discharge taking place, the negative on the interior of the tube is set free, and in its turn satisfies the positive electricity of the discharge that meets it in its passage from the positive terminal. Thus we naturally get the termination of one positive column on the side of the tinfoil nearest to the positive terminal, and a complete discontinuity between it and the second, which starts in a hollow cone from the edge of the tinfoil nearest to the negative terminal.

To show more clearly that this is the true interpretation of the phenomena, and that the effect of the arrangement is thus to substitute for the original discharge two independent discharges occupying different parts of the tube, take two or three such rings separated from each other by spaces somewhat less than the diameter of the tube and connect them as before with the positive terminal. Each of these will be found to be the base of a hollow cone similar to that above described; and each such cone will form the base of a luminous column having all the features of a positive column, and terminating sharply behind the next tinfoil-ring or at the borders of the usual negative dark space near to the negative terminal of the tube. These sectional positive columns will be small in diameter and bright, and will or will not be striated according as their length is sufficient to permit of striæ or not. If they are not striated the hollow cone will end in a blunt rounded head of considerable brightness (Plate 16, fig. 9, and Plate 17, fig. 11). Each of these hollow cones will have its hazy irregular internal surface turned towards the bright end of the sectional positive column immediately behind it, *i.e.*, towards the positive terminal of the tube. If now we bring the finger or a conductor to the side of the tube these columns will all display sensibility, but each will move independently of the others and of the remainder of the positive column and behave as though it started from the tinfoil ring at its base, still preserving, however, its position relative to the sectional columns on each side of it. If a magnet be used, it deflects all these columns, but each column still moves independently and as though it had its tinfoil ring as base. The magnet also shows, as might be expected, that the discharge through each of the columns is in the same direction, viz. : the positive passes from the hollow cone towards the bright termination of the column of which it forms the base or commencement.

Now, taking a tube in which the above phase is clearly marked, let us gradually increase the air-spark interval, so as to render the electrical pulsations more violent. The sharply-defined head of the terminated column will be found to flatten and widen out, growing less bright in the centre till it becomes almost an annular ring of light. As the air-spark interval is still further increased, this annular ring widens and gets less definite, until at last the column on the positive side of the tinfoil assumes somewhat the form of a hollow cone pointing towards the positive terminal of the tube. The base of this cone is however not quite close to the tinfoil,* and is formed, as above stated, of the last phase of the annular luminous ring formed out of the head of the terminated positive column (Plate 17, fig. 12).

If the air-spark interval be too small these phenomena will not be obtained, but only a more or less faint foreshadowing of them mixed with diffused luminosity. This is not to be wondered at, as in such cases the electrical pulsations to the outside of the glass are probably insufficient to produce within it discharges of sufficient intensity to give complete satisfaction to the discharge from the positive terminal, and hence we have an effect produced by the superposition of an ordinary luminous discharge and one interrupted in the way above described. But careful examination of various tubes has convinced the authors of this paper that the forms above described are the truly typical forms of the special or non-relief-effect when the positive is the air-spark terminal.

VIII.—*Examination and interpretation of special and relief effects in general in the case of interrupted discharges.*

We have hitherto confined our examination of these effects to the single case in which the air-spark or other cause of the interruptedness of the current is situated in the positive portion of the external circuit: that is to say, between the positive terminal of the tube and the source of electricity, and of this case we have only attempted to interpret one of the two classes of effects. But, as we have already remarked, it is not only in this case that we get sensitiveness, or that the relief and non-relief-effects in the luminous discharge are observable. All these phenomena are similarly manifested when the air-spark or cause of intermittence is situated in the negative portion of the external circuit, although, as we shall see, the characteristics of the two effects are quite changed. We shall now proceed to examine the appearances in the four cases, viz.: the relief and non-relief-effects when the air-spark is in the positive and the negative respectively, and to trace the connexions and the resemblances which exist between them.

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* The authors of this paper believe that such is always the case, but the phenomenon is complicated by the presence of a reverse current which accompanies the use of a large air-spark in the positive circuit.

To do this satisfactorily it is necessary to examine the various forms which these effects assume as we vary the intensity of the causes which give rise to them continuously from zero up to the highest limit practically attainable with the instruments used. This variation may, as we have seen, be made in several different ways, of which the following are the most convenient:—

(1.) We may vary the air-spark from zero up to the limit of striking distance; or if the interruptedness of the current be arrived at by other means, some equivalent mode of altering the character of such interruptedness may be adopted.

(2.) We may vary the distance of the tinfoil which produces the effect upon the tube from the greatest distance at which an effect is observable to actual contact.

(3.) We may move the tinfoil up and down the tube so as to be at a greater or less distance from the air-spark terminal.

Supposing, then, the tinfoil to be in the form of a narrow ring surrounding the tube, and taking first the *non-relief-effect* with the air-spark in the *positive*, we obtain the following results by these respective methods:—

(1.) As we increase the air-spark from its initial value, zero, there appears first of all a brightening of the central portion of the positive column in the part of the tube close to the edge of the tinfoil on the same side of it as the positive terminal of the tube. As the air-spark gets bigger this becomes a more or less marked division of the column into two parts, the details at the place of division shadowing forth the effect we have already described of the hollow cone and bright termination of the positive column. As the air-spark is still further increased this effect grows clearer and clearer, till it comes to its greatest perfection. Increasing the air-spark still more, the terminal stria or head of the column gets wider, or more nearly of the same diameter as the tube, and the brightness is greatest round its edges, so that it in reality becomes annular. It then grows shadowy and gets very close to the tinfoil, and finally disappears or becomes from its position so difficult to observe that its presence cannot be satisfactorily ascertained, so that the column on the positive side of the ring appears hollow, as it is on the negative side.

(2.) As the ring contracts from a large size, so that we get at first only the inductive effects from a comparatively distant conductor, the first appearances are those we have described for a small air-spark, and the others come in turn as the ring is decreased in size.

(3.) As the ring is moved away from the air-spark terminal the stria-effect is turned into the double hollow-cone-effect, and to reproduce the former effect the ring has to be widened.

Next, taking the air-spark in the *negative* and observing the *relief-effects,* they are found to be as follows :—

(1.) An increase of the air-spark from its initial value, zero, produces exactly the same series of effects as that just described under (1), with minor differences in the definition of the various phases. The bright and sharply-defined termination of the positive column of pointed or annular shape according as the air-spark is smaller or larger is often very beautifully shown.

(2.) Increasing the ring was equivalent to decreasing the air-spark.

(3.) Moving away from the positive made a pointed termination become annular. But this phenomenon is generally very difficult to observe.

This identity of effect in the two cases is the strongest confirmation possible of the correctness of the conclusions previously arrived at. We know that the special effect when the positive is the air-spark terminal is caused by pulses of positive electricity passing to the tinfoil co-periodic with the elevations of potential at the positive terminal produced by the passage of charges of electricity across the air-spark interval. And we find that such pulses of electricity in the tinfoil produce the hollow-cone discharges in the tube, and these form the base of a subsequent positive column while the original positive column stops short within the hollow cone and its discharge apparently finds satisfaction in the negative electricity set free there. Applying the knowledge thus obtained to explain the exactly similar effects which we obtain by connecting the tinfoil to earth and making the negative the air-spark terminal, we see that we must ascribe the pulses of positive electricity which give these effects to the inductive action of the negative charges which rush through the tube from the negative terminal at each passage of electricity across the air-spark interval. These pulses set free positive electricity within the tube, which rushes from the glass in its normal form of the hollow-cone discharge to satisfy the negative discharge which by its inductive action originated it, and the negative electricity on the inner side of the tube, set free again by reason of this satisfaction of the original discharge, passing from the interior of the hollow cone, continues the discharge through the tube. It will be observed, therefore, that the interior of this luminous hollow cone is taken to act the part of a negative terminal in both cases.

One difference between the two cases must be noticed here. The positive pulses in the latter case are directly due to the inductive action of the *actual electric variations within the tube near the tinfoil,* and will thus be proportional to the extent of these variations. In the former case they are directly due to the elevations of potential at the positive terminal which cause the discharge. In this case, therefore, they will be of equal strength throughout the tube; in the other case they will be so only if the extent of the electrical variations is the same throughout the tube, which we shall hereafter show is in all probability not the case. And hence we find, on comparing the observations given above, that while in the *special effect* the intensity increases *cæteris paribus* with the distance from the air-spark terminal, in the *relief-effect* it is stronger the nearer it is to that terminal. We shall show hereafter that the instantaneous variations of potential produced by the discharges are greater nearer the air-spark terminal than at a greater distance from it, so that it is natural that the relief-effects should be most marked in the neighbourhood of that terminal, while it is not strange that the non-relief-effects, in which the forces at work outside the glass are the same whatever part of the tube be selected, should be most marked at the parts of the tube where the electrical actions in opposition to which they act are the weakest.

Having thus demonstrated the identity of these two classes of effects there remain the other two, viz.: the *relief-effect* when the *positive* is the air-spark terminal, and the *special* or *non-relief-effect,* when the *negative* is the air-spark terminal. Both these should be produced, according to the theory, by pulses of negative electricity rushing to the tinfoil, caused in the one case by the inductive action of the positive discharge within the tube, and in the other case coming directly from the terminal of the tube which is in metallic connexion with the air-spark interval.

To examine these effects a tube was taken, and a narrow ring of tinfoil placed round it which was connected with the negative terminal, and the air-spark interval was placed in the part of the external circuit leading from the machine to the negative terminal of the tube. The effect was to constrict the luminous column by the formation of a dark space between the part of the interior of the tube directly under the tinfoil and the luminous column (Plate 17, fig. 13). The section of this dark space through the axis of the tube was in the form of a segment of a circle of which the centre would be in or near the tinfoil or within the glass immediately beneath it, so that the whole figure was that of a section of an anchor-ring by a co-axial cylinder. As the air-spark was increased the dimensions of this space also increased, till nothing was left of the luminous column within the part of the tube surrounded by the tinfoil but a very bright central line (Plate 17, fig. 14).

It will be observed that this phenomenon is precisely what would be produced by a repulsion of the luminous column, the peculiar form being occasioned by the fact that the tinfoil was used in the form of a ring, so that the repulsion must take place symmetrically from all sides, and hence must produce constriction of the column instead of displacing it entirely. We shall, however, see that some caution is necessary before we identify it with the familiar phenomenon of repulsion which has been described as so characteristic of positive relief-effects.

To establish the identity of this effect with the relief-effect when the air-spark is in the positive, the connexion between the negative terminal and the tinfoil was removed, and the tinfoil was connected to earth and the air-spark placed in the positive external circuit. On varying the size of the air-spark as in the case just described, precisely the same series of phenomena were presented except for very minor peculiarities of definition.

Inasmuch as in neither of these effects had there been any appearance of the blue discharge which, as we have seen, is so characteristic of the relief-effect when the air-spark is in the positive circuit, we examined the tube and found that it did not give this effect, or, at all events, it did not give it with the kind of current that was then being used. We therefore took a nitrogen tube in which we had often observed it, and repeated the experiment. No sooner was the air-spark interval of sufficient length than the blue discharge appeared, whether the one or the other of the two effects was being observed. In both cases the effects were at first repulsion, but when the action became more violent it passed into the blue discharge effect. The identity of the effects in the two cases was thus placed beyond all doubt. Indeed, the only difference that can be detected is that the definition is not equally sharp in the two cases.

On testing these tubes by moving the ring of tinfoil up and down the tube it was found that the blue discharge had a greater tendency to appear when the ring was near the positive than when it was near the negative terminal of the tube, thus confirming the law arrived at in the former case: that relief-effects are strongest near to the air-spark terminal, and non-relief effects are strongest at a distance from that terminal.

On looking closely at the blue discharge under the tinfoil, it is seen that it does not lie close to the glass, but is separated from it by a little blank space, precisely as the luminosity on the negative terminal of a tube is separated from the terminal itself by the small non-luminous space known as CROOKES' space. Thus the inside of the glass beneath the tinfoil acts precisely as a negative terminal. In contrast to this we find that in the cases in which the effects are produced by positive pulses, the luminosity starts sharply from the surface of the glass itself, just as the glow on the positive terminal appears to start from the very surface of the terminal itself. Nor is this the only point of resemblance between the effects produced by external negative pulses and the appearances of negative terminals. Any one familiar with the appearances presented by negative terminals shaped like a ring will know that they form round them a blank space, driving off the glow till it at length appears at the centre of the ring only, where it forms a bright line. The resemblance between these appearances and the effects above spoken of, produced by negative pulses, is so close that, so far as the authors of this paper are able to judge, these last-named effects are substantially identical with what would be produced were there a terminal in the tube consisting of a flat metallic ring fitting close to the inner surface of the tube immediately beneath the tinfoil, and this terminal were connected with a source of electricity and played the part of a negative terminal to discharges within the tube.

It is necessary here to make a remark, the purport of which has to some extent been assumed in what has already been stated. Although we have ascribed the sensitive effects directly to pulses of electricity which pass to the tinfoil or the parts of the conducting system nearest to the tube, we do not mean to imply that the free electricity in the tinfoil, or conducting system, affects the discharge by the direct action of the free electricity in the one upon that in the other. According to the theory of the authors of this paper, the effect of the pulses of electricity passing to the outside of the tube is always directly spent in causing a discharge, or its equivalent, to proceed from the interior of the tube, and it is the effect of this upon the luminous discharge already existing in the tube which produces the appearances observed. In the case of positive discharges to the outside of the tube there is no difficulty in accepting this, as the discharge is visible; but in the case of negative pulses the truth of the theory is less obvious. Those tubes which only give a slight repulsion require no special notice; the space formed by such repulsion is in all respects analogous to CROOKES' space, and the interior of the glass beneath the tinfoil is analogous to a ring-shaped negative terminal. But those which commence with very considerable repulsion and, as the air-spark is increased, pass suddenly into what we have called the discharge-relief-effect, require a separate examination.

In such tubes, if the air-spark interval be very great, the appearance is substantially the same as in other tubes. The space separating the blue luminosity from the interior surface of the tube is considerable, and is shaped as we have already described. We may assume then that we have here a case of a negative ring-terminal. When we decrease the air-spark interval, the luminosity draws closer to the tube, until it is separated from it only by a small interval, and the interior of the section of the tube surrounded by this blue luminosity is nearly dark, such darkness not being symmetrically disposed, but extending beyond the tinfoil on the side nearest to the positive terminal of the tube, thus appearing to drive back the positive luminous column. If this be carefully examined, it is impossible to resist the conclusion that we have here the analogue of the familiar negative dark-space, so that the ring-shaped terminal formed on the interior of the tube has its full complement of CROOKES' space, negative halo, and negative dark space. And the correctness of this conclusion is further demonstrated by an examination of the appearances produced by placing a small piece of tinfoil on the tube (not surrounding it, but forming a small patch upon it) and joining it to earth when the air-spark is in the positive. As has been previously described, the blue luminosity will appear between two bright tongues which seem to issue from the luminous column in the tube and stretch towards the blue haze beneath the tinfoil. The space between these two tongues is dark. On closely examining the phenomenon it is seen that the dark space extends from beneath the blue haze to a point considerably on the same side of the tinfoil as the positive terminal of the tube, so that the bright tongue on that side is separated by it from the blue haze for a considerable interval. The other tongue, on the contrary, seems to proceed from a point close to the blue haze. The magnet shows that both these tongues are formed by currents where the positive electricity is passing in the same direction as that in the tube. It is therefore evident that the interior of the tube beneath the tinfoil acts as a negative terminal to the tongue first described, and that this tongue is a positive luminous column, which is separated, as in ordinary cases, from the halo or glow of its negative terminal by a dark space. The other tongue is a positive discharge which proceeds from the interior surface of the tube after the satisfaction of the negative discharge therefrom in the manner just described. This positive discharge is doubtless due to the positive electricity that was fixed on the interior of the tube by the same negative pulse in the tinfoil which caused the negative discharge from the interior of the tube.

This leaves unexplained only the state which precedes the discharge-relief-effect, *i.e.*, that in which the luminous column is repelled. This cannot be taken to be analogous to the repulsion in tubes which do not give both forms of relief-effect (which repulsion we have seen to be due to the formation of a CROOKES' space within the glass), for it is very much greater in amount, and moreover, when the air-spark interval is increased, it disappears, giving place to the discharge-effect which we have seen is the true analogue of the repulsion in tubes which do not give both forms. The interpretation of this repulsion has been felt by the authors of the present paper to be a matter of great difficulty, but they have come to the conclusion that it represents a discharge of the same type as the non-luminous discharges that pass through vacuum tubes under certain conditions of exhaustion and current. This identification must be considered to a great extent hypothetical, but the authors of the present paper believe that it will on examination be found to be free from objections, and to satisfy the conditions of the problem more completely than any other.

Thus we see that the four effects divide themselves into those caused by positive pulses and those caused by negative pulses,[*] and that the effects of these are respectively to form a positive and a negative terminal within the tube beneath the tinfoil.[†]

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[*] Now that the relief-effects can be divided with certainty into those which are produced by positive pulses to the outside of the tube and those which are produced by negative pulses, and these can be recognised by the nature of the appearances within the tube, the full force of the experiment described on pp. 178, 179, becomes felt. If the air-spark be in the positive the effect of joining the tinfoil to earth will be to produce negative pulses to the tube whether the positive or negative terminal be joined to earth, that is to say, whether the tinfoil be maintained at a potential higher or lower than any portion of the circuit in which the tube is situated. And, of course, a similar remark applies to the case where the air-spark is in the negative.

[†] These are the *primary* effects; but in each case there is, as we have seen, a secondary effect of a precisely opposite character, viz.: the formation of a negative and a positive terminal in the respective cases.

IX.—*On the nature of striæ, and the artificial production of striation in the luminous (sensitive) discharge.*

We have seen that positive and negative pulses on the outside of the tube, co-periodic with the discharges that pass through it, cause the interior of the tube within the tinfoil to assume the character of positive and negative terminals respectively. The negative terminals so formed possess all the characteristics usually appertaining to terminals of such shape, but otherwise they do not give us any suggestions specially valuable for our present inquiry. With the positive terminals, however, the case is far different, as we shall proceed to show, taking as an example the special effect when the air-spark is in the positive circuit.

We have seen that the positive discharge due to a ring of tinfoil forms a hollow cone with a sharply-defined luminous outer surface. This cone, if the nearest negative terminal is the negative terminal of the tube, passes into a column of diffused luminosity similar in all respects to the ordinary luminous column which starts from the positive terminal of a tube. But if there is another similar ring of tinfoil also connected with the positive terminal between the former ring and the negative terminal, the luminous column that starts from ring No. 1 is stopped by ring No. 2, and from this latter ring there starts a second hollow luminous cone which stretches away in its turn towards the negative terminal in a diffused luminous column as before described. If these rings be placed at the proper distance from one another, and the size and exhaustion of the tube be suitable, the short luminous column between the rings will dwindle down to a hollow cone with blunt rounded head, this head being greatly superior in brightness to any other part of the cone and stretching to a point close up to or even a little within the next ring, so that it is in the middle of the space enclosed by the hazy blue inside surface of the hollow cone that starts from that ring. And by using additional rings this can be made to repeat itself until the whole luminous column is segmented into these hollow luminous cones or shells with bright rounded heads.

The theory which the authors of this paper desire to put forward is, that each of these luminous cones or shells is a perfect stria both in function and structure. The resemblance in appearance is most striking. In the luminous shells which we have just described there is the same convex bright outline pointing towards the negative terminal, and the same hazy blue ill-defined hollow surface turned towards the shell immediately behind or on the side towards the positive terminal of the tube, and there are the same dark intervals dividing consecutive shells as divide consecutive striæ. In fact, this segmented discharge presents to us all the familiar phenomena of striated discharges in which the striæ have rounded or conical forms. Moreover, the conical is not the only type of striæ which can be successfully produced in this manner. It is equally possible to form flat or annular striæ by proper adjustments of the air-spark interval, as explained in page 192. And as these are precisely the forms which natural striæ tend to assume when they deviate from the conical type, it may be fairly stated that all the forms of natural striæ can be artificially reproduced: a fact which is strongly in favour of the identity of the two phenomena.

But it is not only in structure that these luminous shells resemble striæ. There is also an identity of function. We know that when the positive pulses arrive at the glass they drive off similar positive pulses from the interior of the tube, and thus form the luminous shells. And knowing, therefore, that each luminous shell signifies a positive discharge, and also that no electricity passes through the glass, it is absolutely certain that a like amount of negative electricity must be collected at the surface of the glass within the tube, and must ultimately satisfy an equal amount of the original positive discharge—*i.e.*, of that which comes to it from the positive terminal, or from the shell immediately on the positive side of the one we are considering, as the case may be. We have then a negative discharge from the side of the tube, or from the gas immediately within it, satisfying a positive discharge advancing towards it along the tube; and we find that it causes the luminosity of this discharge to stop short and terminate in a bright, clearly-defined rounded head, which is separated by a dark space from the seat of the negative discharge. This, then, is the function of the shell: the bright part is to serve as the place of departure of the positive electricity that is about to pass across the dark space (or the place of arrival of the negative electricity after so doing), and the hazy interior of the cone is to form the place of its arrival (or the place of departure of the negative electricity); and, so far as we know, this is the sole function of these elements of the shell. Now let us take the case of the striated discharge. Here, also, we know that the positive electricity in the current must leave the bright head of the stria, and, after passing the dark space, arrive at the hazy inside of the next, and the negative electricity must take a reverse course. There is an absolute identity in the functions of the corresponding parts of the two structures, the only difference being that we know, from independent extrinsic evidence, that the electricity in the artificially segmented discharge does not flow continuously, but in intermittent discharges. This independent testimony is absent in the case of the ordinary striated discharge. We shall refer to this point again at the conclusion of this paper, but at present we shall assume what we have already stated to be so highly probable, viz.: that all vacuum discharges are in reality intermittent. Any who do not wish to admit this must take the reasonings of this section as applicable only to those striated discharges which are known to be intermittent.

Returning, then, to the case of the artificially produced conical shells, the *modus operandi* of the discharge is as follows:—When the pulse of positive electricity arrives at the terminal and causes the discharge into the tube, a positive discharge equal[*] to that which passes into the tube moves synchronously from the interior of the tube at each ring of tinfoil, forms the bright shell or stria, and passes on to the next shell or stria; thus supplying the place of the positive pulse that the ring of tinfoil there has just sent on. The last shell passes its discharge to the negative terminal, and the first shell receives a discharge from the positive terminal. In this way a discharge passes through the tube identical in quantity and character to that which passes into it from the positive terminal.

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[*] It may seem an unwarranted assumption to assume that each of these artificially produced discharges is equal to the whole original discharge, but the appearances (with suitable adjustments) seem to warrant it, and as the reasoning is simplified, and the validity of the theory is not affected by this assumption, we shall, through the rest of this section, suppose such to be the case.

If, then, we are right in supposing that the series of artificially produced hollow shells are analogous in their structures and functions to striæ, it is not difficult to deduce, from the explanation above given, the *modus operandi* of an ordinary striated discharge. The passage of each of the intermittent pulses from the bright surface of a stria towards the hollow surface of the next may well be supposed by its inductive action to drive from the next stria a similar pulse, which in its turn drives one from the next stria, and so on. Thus the processes in the naturally and artificially striated columns are precisely similar, save that in the case of the latter the pulses from the several striæ are excited by induction from without the tube, while in the case of the former the induction is that of the discharge itself in its passage from stria to stria. The passage of the discharge is due in both cases to an action consisting of an independent discharge from one stria to the next; and the idea of this action can perhaps be best illustrated by that of a line of boys crossing a brook on stepping stones, each boy stepping on to the stone which the boy in front of him has left.

The truth of the foregoing theory is confirmed by the fact that it is not only by means of the positive special effect that we obtain formations analogous to striæ, but we can also obtain them by means of the other special and relief-effects when the electrical actions are of such a kind as would lead us to expect them according to the theory here put forward. Take, for example, the positive relief-effect. When the pulses passing through the tube are positive and a finger is placed on the tube, the positive relief-effect so caused consists, under certain relations of quantity and tension in the discharge (and notably when the discharge is due to the action of a small coil, as described on page 210, the negative terminal of the tube being put to earth), of a faint haze beneath the finger, and a large dark space extending right across the tube, and stretching for some distance from the spot on which the finger rests along the tube towards the positive end (Plate 18, fig. 16).* The termination of the positive luminous column at the place where it abuts on this dark space, is bright with a sharply-defined rounded outline, like the bright head of a stria, excepting that instead of pointing centrally along the tube it is slightly thrown upwards so as to point to the spot where the finger is placed. Here, again, we have a case in which there is a positive discharge, which gets satisfaction from a source of negative electricity, or, as we may term it, a negative terminal, situated within the tube at the surface of the glass, and we find precisely the same formation present itself. The luminous positive column is separated from the negative terminal by a dark space, and its head becomes bright, rounded, and definite in outline, and assumes all the characteristics of the bright head of a stria. And the experiment may be made still more striking and instructive by placing a second finger a little on the negative side of the first. The luminosity which previously commenced close to the first finger and stretched away towards the negative end of the tube, becomes divided exactly as in the previous case, and the portion intercepted between the fingers resembles in all respects a stria, except that the hazy blue negative surface is imperfectly developed, being really replaced by the haze under the first finger, which is the place at which the interchange of discharges takes place. The length of the body of this stria can be varied at will by increasing or decreasing the distance between the fingers so as to leave a more or less prolonged positive column behind the bright head, but the real structure is in nowise altered thereby.

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* If the positive terminal be put to earth, the striæ will be immediately beneath the fingers as in Plate 18, fig. 16A, as should be the case.

The unit of a striated vacuum discharge is therefore composed of the body of a stria terminating in its bright surface, the dark space in front of it, and the hazy interior surface of the stria on the further side of that dark space. In this unit* we have a positive terminal, with a positive luminous column starting from it, a space across which the discharge passes non-luminously, and a negative terminal; so that in each unit we have represented all the elements of a complete discharge. And in the opinion of the authors of this paper, all striated vacuum discharges are composed of reduplications of this unit, and any phenomena connected with the negative terminal which seem to contradict this view, and to point to a special structure of the discharge near the negative terminal unlike anything that exists in other portions of the discharge, are merely modifications due to the local circumstances of the terminal in a manner now to be explained. We allude, of course, to the phenomena known as the *negative dark space*, the *negative glow*, and CROOKES' *space;* and we now proceed to examine these phenomena, taking first the negative glow, as the one presenting the greatest difficulty on the foregoing theory.

It is well known that the bright surfaces of striæ are usually convex. But this is not always the case. If a striated discharge be produced in a tube which has a bulb in the middle of its length, of diameter considerably greater than that of the tube, it will be found that the striæ in the bulb towards the side nearest the negative are concave on their bright surface, especially in that part of the bulb that rapidly narrows towards the orifice of the tube (Plate 18, fig. 17).* And this phenomenon is not due to the shape of the sides of the glass; for it may be produced in a tube of uniform section. If we join to earth a ring of tinfoil placed round a large tube in which striæ are formed by the use of a coil working with a high-speed break, and thus constrict the striæ in the immediate neighbourhood of the ring, it will be found that those on the positive side of the ring have the same peculiarity as we have above noticed. Their bright surfaces are concave, but as you pass from the constricted portion each is of less curvature than the one before it.

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* It must be borne in mind that the analogue of the positive terminal in this unit is a surface within the body of the striæ, at which the separation of the electricities takes place; and that the body of the striæ is to be regarded as a short positive column terminating in the bright luminous head or conical surface of the striæ. In artificially produced striæ this column can, as we have seen, be made of any length; and even in the natural striæ its length varies within very wide limits.

We see, then, that we can greatly modify the bright outline of a stria by modifying the size and shape of the next stria, *i.e.*, of the negative terminal of this unit. If we make this small, then the bright surface of the next stria will curve round it and be concave; if, on the other hand, it is large, and still more if it be hollow and partially envelop the next stria, the bright surface of the latter will be very convex. Now if we replace this negative terminal (whose hazy, gaseous structure is readily capable of being moulded by and probably owes its form and structure to the discharge itself) by a fixed metallic body, we must expect great modifications in the appearance of the other members of the unit. It is not to be wondered at that the dark space, and the bright surface which forms the other boundary of this dark space, should follow the outlines of the new negative terminal, and that if it consist of a small metallic body projecting into the tube, the general appearance should be as if the bright surface enveloped it, remaining always separated from it by a dark space. This is exactly the appearance of the negative glow, and the above is, we believe, a complete explanation of its formation and function. This negative glow and the haze behind it, which terminates in what is known as the negative dark space, is, according to this theory, a *stria turned inside out* by the influence of the shape and character of the negative terminal (Plate 18, fig. 18).†

We have only to reflect on the very remarkable way in which the negative glow shows the shape of the negative terminal, to see that there is nothing improbable in assuming that such a deformation has taken place. If the negative terminal be spherical, the negative glow forms a spherical envelope round it; if it be merely a

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[*] An exactly similar phenomenon is shown in the familiar experiment of the electric egg, where the striæ which surround the negative terminal are concave on their bright surfaces.

[†] It will doubtless occur to those familiar with striated discharges, that there are considerable differences in the behaviour of striæ and the negative glow when subjected to the action of a magnet. It must, however, be remembered that the positions of the striæ are dependent on one another, since each receives the discharge from the one behind it. But in the case of the negative glow the lines of electric discharge whose termination it marks are fixed to the negative terminal at one end, and thus the glow can only be displaced so far as the deformation of these lines will permit. In fact, the negative glow is like *a stria at anchor*. There is just the same difference between the behaviour of the two as between the effect of waves on floating seaweed and seaweed that is growing on rocks. In this respect the artificial striæ of which we have been speaking resemble the negative glow, with the exception that they have a fixed positive terminal instead of a fixed negative terminal.

straight wire, the negative glow forms itself into a cylindrical envelope around it with a rounded or hemispherical head. If the negative terminal be in the shape of a ring the negative glow envelops it, remaining at a constant distance from the surface of the terminal and thus taking the shape of an anchor ring,* or else (especially if the ring be a small one) no portion of it lies within the circuit of the ring, but it envelops the ring as a whole, the bright surface of the glow having a boss or protuberance opposite to the centre of the ring. If the negative terminal be large and flat, it will be found that the negative glow does not substantially differ in appearance from a flat stria; and by making the negative terminal in the shape of a hollow cone, which is the natural shape of the negative terminal of what we have considered to be the physical unit of a striated discharge, there is little doubt that its resemblance to an ordinary stria would be rendered still more striking.†

If we are right in our conclusion as to the nature of the negative glow, it follows almost necessarily that the *negative dark space* and CROOKES' *space* are merely the representatives of the dark space which occurs as we have seen in every unit of a striated discharge. The former belongs to the unit of which the stria at the head of the positive column is the positive terminal, and the negative haze is the negative terminal, and the latter belongs to the unit of which the negative glow is the positive terminal, and which has for its negative terminal that of the tube itself. No doubt there is one great difficulty in this interpretation, viz.: the contrast in the lengths of the dark spaces in these two units, both when compared with one another and with the dark spaces in the other units throughout the tube. But if we remember the different circumstances of the discharge in the cases of these terminal segments when compared with those that prevail in the other segments of the discharge, we shall see no reason to give great weight to these unexplained difficulties as telling against the truth of the theory. In the first place we are dealing, in the case of CROOKES' space, with the passage of electricity from metal to gas or gas to metal, instead of simply the passage of electricity from one portion of gas to another. And further, we have the far more important circumstance that both in CROOKES' space and the negative dark space we have a case of a discharge in which the negative terminal of the unit is of a shape and disposition wholly unlike that of the negative terminal of any other of the

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* These last remarks will explain a difficulty which would otherwise be felt, as to why it is only in particular cases that we get any analogues of striæ from those special or relief effects which cause negative discharges within the tube. Like all other fixed negative terminals, the glass insists on having its dark space round it in all directions. This dark space is bounded by a glow which, as we have seen, is the analogue of the bright surface of a stria. But its arrangement is such that all we can expect is a stria turned inside out, as at an ordinary negative terminal; or more particularly, at a negative terminal in the form of a flat ring placed close to the interior surface of the tube. And this is precisely what we do obtain.

† Other remarkable instances of the modification of the form of the negative glow, and of the extent to which it may be made to assume the appearance of an ordinary stria under suitable local conditions, will be found in the Postscript to this paper.

segments of the discharge; and hence it is not to be wondered at that we should find that the distance across which the non-luminous discharge takes place in these segments is not identical with that which prevails in the other segments, all of which resemble each other so closely in these respects. And though we would not adduce any considerations as to what is more or less probable, unsupported by experiment, yet we are, we think, justified in saying that it is just in such respects that we should expect to find the details of a unit of striated discharge modified by changes in its terminal arrangements.

It will be seen that the authors of this paper regard each segment as constituting a separate discharge. One phenomenon observed by them, of a different kind to those of which we have been chiefly speaking, appears strongly to confirm this. If a magnet be applied to a striated column, it will be found that the column is not simply thrown up or down as a whole, as would be the case if the discharge passed in direct lines from terminal to terminal, threading the striæ in its passage. On the contrary, each stria is subjected to a rotation or deformation of exactly the same character as would be caused if the stria marked the termination of flexible currents radiating from the bright head of the stria behind it and terminating in the hazy inner surface of the stria in question. An examination of several cases has led the authors of this paper to conclude that the currents do thus radiate from the bright head of a stria to the inner surface of the next, and that there is no direct passage from one terminal of the tube to the other.

It is natural to inquire what, in this theory, is the physical structure of striæ. Are they merely luminous appearances (*i.e.*, loci of maximum luminosity), or are they aggregations of matter having a material structure? This is a question which it is beyond the scope of this paper to discuss, but the most probable view, in the opinion of the authors, is that they should be regarded as septa of complete electric porosity, having a material structure. One of the most important facts favouring this conclusion is that when striæ are formed by a coil working with a high-speed break, the striæ produced by the two currents (the make and the break) adhere persistently together in pairs as though the alternate currents found ready to hand striæ that only needed a little deformation to make them available for their purposes (Plate 17, fig. 15). There are other facts tending to support this conclusion, but a complete examination of the question would carry us beyond the limits of the present paper.

A great difficulty in the way of this, and indeed of all attempted solutions of the stratification of vacuum discharges, arises from the difficulty of imagining any physical cause which could form material aggregations in the form of striæ out of the diffused and mutually repellant particles of the rarefied gas. The following theory is thrown out as a suggestion of a possible mode of accounting for their formation. It is known that the blue haze that surrounds the negative halo or glow is magnetic, and follows the lines of magnetic force just as would be the case were it a mass of magnetized steel filings. Now on the present theory this haze is identical in function, and therefore in structure, with the blue haze at the back of ordinary striæ, and it is a luminosity formed at the places where the non-luminous electric streams arrive at the surface of the stria, or where, as perhaps it should be stated, the positive currents that arrive meet the negative electricity set free by the pulse of positive electricity that has just been expelled from the stria. Now if we suppose that these discharges are effected by or accompanied by convection of free electricity on particles of gas, it does not seem impossible that a positively charged particle might pass so near a negatively charged particle that they would rotate rapidly round one another, under the influence of the mutual attraction of the free electricity with which they are respectively charged, thus forming an analogue to a circular current, which would behave like a magnetic particle just as the electrically laden particles in CROOKES' lines behave like flexible currents. In this way we should have matter capable of mutual attraction, it being assumed that there is on the average some directing force or factor which prevents the directions of the poles of the magnetic matter being completely indifferent. That in some way there must be manufactured out of the gas a substance capable of mutual attraction, which when formed into the glow has a surface tension, is quite evident from Mr. CROOKES' experiments with the radiometer, in which he found that the particles driven off the vanes (when used as a negative terminal) did not produce rotation so long as the glow did not intersect the side of the glass vessel. This is explicable only by supposing that the glow and the vane formed a physically continuous surface kept together by mutual forces.

It may be objected to this theory that the circular currents produced by the two elements of a revolving pair would be in opposite directions, and that they would consequently represent two magnets having opposite polarity, which would neutralize one another. But there is no reason to think that the two currents would be always exactly equal; and any inequality in the currents would produce a preponderance of magnetic polarity, and the preceding remarks would apply.

It is no objection to this theory that the bright portion of the glow does not appear magnetic. Its position is fixed by its being the locus of the extremities of the CROOKES' lines that proceed from the fixed negative terminal, and it can only be affected by the displacement or deformation of these lines, as is found to be the case.

X.—*The passage of the discharge through the tube occupies a time which is sufficiently small in comparison with the interval between the discharges to prevent any interference between successive electrical pulses.*

We shall now revert to the phenomena of the sensitive discharge, and show that all the effects of which we have spoken are due to actions whose durations are comprised within the time that a single pulse takes in passing through the tube, so that these effects are not in any way due to the action of one electrical pulse on a consecutive one. In other words, the whole of the effects take place in each individual pulse, and the repetition in successive pulses only serves to make them appear to the eye as continuous phenomena.

The most valuable direct evidence of the velocity of the discharge through the tube and the independence of each discharge is obtained from the examination of the discharge by a revolving mirror. For this purpose a revolving four-sided prism, whose sides consisted of glass platinised on the upper surface, and which was rotated by hand at a rate of from 400 to 800 revolutions per minute, was made use of. It was found that this instrument was capable of showing the intermittence of the current when in a sensitive state however small the air-spark, showing that the superior limits of the rapidity of the pulsations when produced by the aid of an air-spark is by no means a very high one. The difficulty in increasing the rapidity of the pulsations arose from the tendency of the heated poles of the air-spark to produce an arc between them if they are kept too close to one another.* The revolving mirror was used in the ordinary way by placing a narrow slit upon the tube so that the image of the discharge was a bright line about one-eighth to one-sixteenth of an inch wide. On making the mirror revolve rapidly the appearance in the mirror consisted of parallel bands alternately bright and black; the black bands becoming broader the greater the rapidity with which the mirror revolved and the less the frequency of the discharges.

On trying the above experiments it was found that the discharge could always be resolved into such black and bright bands. There was no broadening out of the bands discernible, showing, so far as such means of observation were capable of demonstrating it, that the discharges were instantaneous. When the air-spark was reversed, *i.e.*, thrown from the positive into the negative, a change occurred in the appearance of the bands; but it was difficult to define the essential characteristics of such change. They seemed to consist in a slight variation of the circumstances of the discharge caused by what may be termed a want of electrical symmetry in some portion of the circuit rather than to any change in the nature of the discharge due to any fundamental contrast in the properties of the two electricities. The definition seemed to be sharper when the air-spark was in the negative than when it was in the positive.

The negative end of the tube was first observed, the slit used being sufficiently long to allow the negative pole, the negative dark space, and the negative end of the positive column to appear in the field at the same time. On the mirror being set to revolve, the black and bright bands appeared generally to extend in straight lines through the whole field right across the negative dark space, although they grew faint and sometimes even invisible in the region of the negative dark space.

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* The air-spark in these experiments passed between two small platinum spheres. Perhaps a variation of the form of the poles might give air-sparks whose periodicity would be much more rapid. This tendency to form an arc was most strongly marked when the air-interval was in the negative part of the circuit.

When the air-spark was lengthened, not only did the black bands get broader, but, associated with the intensely bright bands, there seemed to be one or more other bright bands, much fainter in character, and apparently narrower. Whether they were in fact narrower it is difficult to decide, as their greater faintness would perhaps give them the appearance of being so; but if it is the fact that they were narrower, it is possible that this points to the main bright lines being broader than the image of the slit, which would show that the principal discharge occupies some little time; on the other hand, the apparent width of the bright bands may be due to irradiation. The subsidiary discharges may perhaps arise from the existence of return currents, such as are evidenced by the appearance of negative discharge at the positive pole, when an air-spark of considerable size is introduced, and also by the double effects which tubes so constantly show when we try to produce the relief or non-relief-effects.

A metal ring was now put round the tube about the middle of its length, and the slit was so placed that the ring passed across it about its centre. This ring was then joined successively to the air-spark and non-air-spark terminals, and to earth, and the slit examined in each case. There was not in any instance found to be any crookedness in the bright lines, although when the air-spark was in the positive the slit sometimes included the bright termination of the positive column and a section of the hollow luminous cone which surrounds it. These observations show that there is no want of simultaneity between the discharges caused by the relief or the non-relief-effect and the ordinary discharge in the tube; and as we know that the pulse which causes the former passes along the outside of the tube with the velocity of electrical currents in a conductor, this experiment would seem to demonstrate the existence of a velocity in the discharge within the tube of a similarly high order.

In order further to test the possibility of single pulses giving rise to the effects of which we have been speaking, the following experiment (to which reference has already been made) was tried. The terminals of a tube were connected with the outside and the inside of a small Leyden jar. The poles of the secondary circuit of a coil were placed so that the discharge from the coil charged the jar by leaping over intervals of considerable size (about a quarter of an inch for the negative pole and three-quarters of an inch for the positive pole), so that the make-current was excluded. When the coil was worked there appeared a brilliant discharge caused by the jar periodically discharging itself through the tube. A slit was placed on the tube and the luminous column was examined by the revolving mirror, and it was found that the discharge was quite instantaneous, and that usually it was not followed by the next at any regular interval, but that occasionally it was multiple. The discharge was then tested for both relief and non-relief-effects, and notwithstanding the very large quantity that passed at each discharge, it gave them very markedly. The contact breaker was then worked by hand so as only to give single flashes. These were tested for sensitiveness and were found to be perfectly sensitive. Thus it appears from experiment that the whole of the relief and non-relief-effects are completed within each single pulsation, and that the effect of the rapid repetition of the discharges is merely to give the appearance of permanence to effects which in reality appear and disappear during each separate discharge.

XI.—*The discharge is effected under ordinary circumstances by the passage through the tube from the air-spark terminal of free electricity of the same name as the electricity at that terminal.*

The knowledge derived from our previous experiments of the difference between the effects of positive and negative pulses arriving at the outside of the tube, enables us to ascertain the nature of the electrical disturbances within the tube, for as we have seen there is a complete dissimilarity between the appearances produced within the tube by the two kinds of pulses. Now, if we take a tube containing an ordinary sensitive discharge in which the sensitiveness is produced by an air-spark and connect tinfoil placed upon it to earth, we get the same type of effect wherever we place the tinfoil. Thus, if the air-spark be in the positive portion of the external circuit, the effect of joining to earth a piece of tinfoil placed upon any part of the tube will be what we may term a negative effect, showing that throughout the tube the primary and main effect is a sudden irruption of positive electricity across the section of the tube beneath the tinfoil.* This is the case even though the tinfoil be placed very close to the negative end of the tube. Thus throughout the whole length of the tube the charge of positive electricity passes without awakening a response from the other terminal, or if there be any response it must be a very faint one so as not in any way to make doubtful the very great preponderance of the original charge. If the negative be the air-spark terminal the same result is observed, with the exception that the charge passing through the tube is of negative electricity.

It must be observed that in saying this we do not commit ourselves to the two-fluid theory of electricity, or to any equivalent theory. It may be that instead of a charge of negative electricity being thus sent through the tube the thing that is propagated is a state of deficiency of positive electricity. Nor do we necessarily adopt any special theory of the propagation. It may be that the electricity is carried on material particles of gas; it may be that it is passed from one particle to another without or in addition to the actual motion of the particles charged with it at any moment; or it may be propagated without the assistance of the material particles of the gas. Which of these represents the actual means by which the electricity is propagated through the tube must be decided by other evidence. All that is claimed as following from these experiments is that the charge passes (disruptively) from the air-spark terminal in the shape of what we know as free electricity, and in such shape passes through the tube until it arrives at the other terminal and becomes neutralized by the opposite electricity that is awaiting it there. And this is shown conclusively by the identity of the relief-effect throughout the whole of the tube.*

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* These remarks are strictly only applicable to cases in which the discharge in the tube would, without the interposition of the air-spark, be non-sensitive. Those tubes which of themselves render discharges sensitive must be regarded as tubes, one or both terminals of which act as though there were an air-interval essentially connected with them, so that when using such tubes with an external air-spark we are often in the position of using a tube with an air-spark in both the positive and negative external circuits. The phenomena in such cases are of course somewhat complicated, as special effects (though different in character) are obtained by connecting the tinfoil to either terminal, but otherwise they present no theoretical difficulty, and will not be further noticed in this paper.

The cogency of these arguments will become more evident if we consider certain results obtained with an arrangement which rendered the conditions of the two terminals of the tube electrically symmetrical instead of permitting them to be in so radically different electric conditions as is the case where an air-spark is used.

If we take a coil giving rapid discharges of small quantity, we shall obtain a sensitive discharge. If the tube be symmetrical in figure, and the terminals as nearly alike as possible, then, on examination, we shall find that the sensitiveness is very slight in the centre of the tube and that it increases towards the ends. This is in marked contrast to the ordinary sensitive discharge hitherto described, which shows a sensitiveness gradually decreasing through the whole length of the tube as we pass from the air-spark terminal; and, moreover, the character of the relief-effects differs in the two portions of the tube. In the portion towards the positive terminal the effects are similar to those presented by a tube having an air-spark in the positive, while in the portion towards the negative terminal the effects are similar to those in a tube having an air-spark in the negative. The part of the tube near the centre, which shows little or no sign of sensitiveness, will be called the *neutral zone.*

Nor is this the only peculiarity presented by this form of sensitive discharge. If we place a piece of tinfoil on the tube near to one of the terminals and connect it with the nearer terminal we get a marked effect, wholly different from the to-earth or relief-effect. If, on the other hand, we connect it with the more distant terminal we get an effect which, though stronger and more marked in character, is decidedly of the type of the to-earth-effect. These peculiarities are in no way caused by the double current, as they are well marked when the break-current alone passes.

These phenomena are easily explained by the supposition that owing to the symmetry of the electric condition of the terminals the arrangement is equivalent to an air-spark at both ends, and that the positive and negative pulses are approximately equal in quantity, and rush simultaneously along the tube so as to meet about the centre. The parts near the centre will thus be continually in a state of no electric tension, and hence there can be no relief-effect and no sensitiveness. The pulse at either terminal is of the opposite sign to the pulse at the other terminal, and hence to the pulses that strive to leave the tinfoil upon the tube near that terminal. Hence the connexion of a piece of tinfoil with the more distant terminal will naturally produce the relief-effect, though in a stronger and more decided form. And, moreover, the relief-effects in the two portions of the tube are of opposite characters, showing conclusively that the corresponding electrical pulses within the tube are of opposite signs.

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* These conclusions, coupled with the experiments with the Leyden jar described in the last section, have a curious bearing on the question of the rival theories of electricity. So far as the authors of the present paper are aware, no attempt has ever been made to determine the sign of the electricity in the disruptive discharge. Hitherto if a Leyden jar charged with positive electricity was allowed to discharge itself into a neighbouring conductor, it was uncertain whether positive electricity passed from the jar to the conductor or negative electricity passed from the conductor to the jar, or whether both operations took place. The present experiments indicate a possible method of ascertaining in which of these ways the discharge actually takes place—at all events, when the disruptive discharge takes place in the modified form of a discharge through a vacuum tube. It would, however, be premature to assume that any such considerations as these would decide the vexed questions involved in the present electric theories until the matter has been much more thoroughly examined into than has as yet been the case.

A slight variation of the experiment tends to confirm this view. If we join one of the terminals to earth, so that these discharges cannot produce electrical tension or elevation of potential there, we find at once that the neutral zone disappears, that the luminous discharge presents all the characteristics of an ordinary sensitive discharge with the air-spark at the terminal opposite to that which is to earth, and that the sensitiveness decreases in the usual manner throughout the whole length of the tube as we proceed from the air-spark terminal towards the other end of the tube. And if instead of connecting the terminal to earth we connect it to a condenser of small capacity whose other surface is to earth, so as to diminish, without completely suppressing, the variations of tension, we find an effect of an intermediate character produced. The neutral zone is displaced towards the terminal which is connected with the condenser, and the sensitiveness is found to decrease as we pass from either terminal towards the neutral zone.

A striking confirmation of the correctness of the interpretation here given of these phenomena is obtained by placing a broad piece of tinfoil round the tube midway between its extremities, and putting a telephone in circuit between the tinfoil and the earth. When discharges from a small coil (such as above described) are sent through the tube, the sound heard within the telephone will be faint; but it instantly springs into loudness upon either terminal of the tube being touched with the finger. The reason is obvious. By touching one of the terminals, the discharge becomes one in which the same kind of electricity passes throughout the tube; and the wants of the tinfoil are all of one kind, and are therefore additive. But when both discharges enter the tube, the wants of opposite ends of the tinfoil are of different signs, and it is only the balance which has to come through the telephone circuit.

Still more striking effects in illustration of the subject were obtained by the use of a larger coil with a tube moderately exhausted, and a break working at a somewhat rapid rate. The luminous discharge presented the ordinary characteristics of a coil-discharge as we have just described them. When a ring of tinfoil, somewhat less than an inch in breadth, was placed round the tube not far from one end and connected electrically with the more distant terminal, the portion of the tube between this terminal and the tinfoil became totally dark, and the whole discharge was confined to the portion between the tinfoil and the nearer terminal. And this discharge presented most remarkable features. The hollow luminous cone of the positive non-relief-effect stretched from the interior of the tube within the tinfoil towards the nearer terminal, and also the blue luminosity within the tinfoil which we associate with the positive relief-effect made its appearance. On placing a similar ring of tinfoil in the same position relatively to the opposite terminal, and connecting it with the terminal most distant from it, an exactly similar discharge took place at that end of the tube, and the middle portion of the tube remained dark.

The interpretation of these very striking phenomena follows readily from the theory we have enunciated. Indeed, the action in this case is so plain that it is easy to arrive at the interpretation of the phenomena without any reference to the theory of sensitive discharges; and the importance of these experiments for our purposes is chiefly due to the support they give to the explanations we have put forward of the hollow luminous discharge in the positive non-relief state, and the other characteristic phenomena of sensitive discharges. A positive pulse coming from the secondary circuit rushes at once to the terminal and to the tinfoil ring. At the same instant a negative impulse rushes to the terminal nearest to the tinfoil ring and leaps into the tube. The impulse to the ring has already produced an inductive discharge into the tube of positive electricity, and this meets and satisfies the negative pulse from the terminal, leaving negative electricity predominating near the tinfoil, and this holds for a brief instant the positive charge in the tinfoil. That brief instant suffices for the next pulse from the coil (which will necessarily be of the opposite character) to come up and satisfy the positive charge, and thus set free the negative within the tube, which in its turn satisfies the positive pulse that now comes from the terminal, and then the cycle recommences.

As it has reference to the phenomena of sensitive discharges described in the earlier part of this paper, it is worth remarking that if we do not properly adjust the breadth of the tinfoil, or if the tube be not of a suitable character, we fail to get a complete absence of discharge in the intermediate portion of the tube, but we find that superposed upon the discharges as we have described them there is a somewhat feeble discharge going through the whole tube. This is precisely what we so often obtain in the case of the relief and non-relief-effects in tubes; the phenomena are there with all their characteristics, but blurred or masked by the superposition, as it were, of a further discharge which does not show the usual phenomena. It is reasonable to suppose that the cause is the same, viz.: that the external actions are not of the suitable strength, or the internal conditions are not such as to permit the effect upon the discharge to exist in all its completeness.

We see, then, that observation of the relief-effects enables us to determine the terminal from which the discharge proceeds, and the distance it goes without provoking a response from the other terminal. And this method shows us that in the ordinary case of the air-spark discharge it passes right through the tube before this response comes, if in fact it can be said to come at all. One of the most important consequences that necessarily follows from this is that the discharges at the two terminals of the tube are entirely independent, and are primarily determined each by the conditions at its own terminal and only in a secondary degree, if at all, by the conditions that exist at the opposite terminal. We shall hereafter recur to this point, and shall show still more conclusively the complete independence of the discharge at either terminal of a tube, but it will be well to call attention here to the very important consequences that flow from this principle. Since the discharges are not identically the same at both terminals, the tube must contain different free charges at different times.* It is therefore in no respect like a conductor, but is an independent electrical system, holding much the same position as the air-vessel in a forcing-pump. All the electricity that comes into it goes out again, but the truth of this can only be asserted when we consider the whole discharge from the beginning to the end, and it may not be even approximately true during a small finite time.

This independence of the discharges from the two terminals is a most important fact in the analysis of the mode of passage of electricity through rarefied gases. It dissipates the error of seeking analogies in metallic conduction ; and shows that any appearance of obedience to regular laws as to change of potential as we proceed along the tube, resistance, &c., must arise from the fact that the effects measured are really average effects over a space of time enormously long compared with the duration of individual discharges. These average effects may remain constant notwithstanding great variability during the individual discharges.

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* In order to show this conclusively, a vacuum tube was enclosed in a metal canister (the wires passing to its terminals through tubes of insulating material inserted in small holes in the canister) and a telephone was placed in circuit between the canister and the earth. When a discharge with an air-spark in the external circuit was sent through the tube a sound was heard in the telephone similar to that made by the air-spark. By a fundamental proposition in electricity the free electricity on the surface of the canister (and which escaped through the telephone to the earth) was at any instant equal to the excess of one kind of electricity over the other in the space within the canister. Had the discharge been in the nature of conduction, as in a galvanic current, there would at no instant have been an excess of either kind of electricity, and therefore there would not have been any sound in the telephone. The existence of a sound testified to variations in the algebraical sum of the free electricity in the tube. To show that this was not due to anything depending on the wires leading to the tube, the same experiment was repeated with a tube in which the middle portion was connected with the two end-portions by very narrow passages. The middle portion was placed in the canister and the narrow parts passed through small holes made in its side, so that only a portion of the complete tube was within the canister. The same results were obtained with this arrangement.

By these experiments, together with those previously described, in which a telephone is placed in circuit between the earth and a piece of tinfoil upon the tube, we are able to obtain from the telephone direct evidence of the intermittence of the discharge at the time of its actual passage through the tube. This evidence fully confirms that previously obtained by the use of the revolving mirror. It is, however, of inferior value, for though it shows conclusively that there are rapid periodic alterations of intensity or character in the induction round the tube, it does not enable us to decide whether these are due to intermissions or only to fluctuations in the discharge.

XII.—*On unipolar discharges.*

We have seen that, inasmuch as the discharge from the air-spark terminal produces its special effect without any indistinctness or confusion close up to the opposite terminal, it would appear that the discharges from the two terminals are so far independent that the discharge may take place from one and the free electricity pass right through the tube to the immediate vicinity of the other without evoking a specific response from the latter terminal. And if each such discharge does in any way call forth from the other terminal a specific response, it must be so slight that it does not affect materially the electrical condition of the interior of the tube, or the effect which that condition produces on conducting systems outside the tube. And we have also seen that this independence implies that the electricity leaves the terminal from which it starts in consequence of the electric tension within that terminal, and only in a very subordinate degree in consequence of the correlative action at the opposite terminal. Lest these should seem to be too hastily drawn conclusions we will proceed to describe a class of phenomena which furnish very important evidence of their truth.

If we take two exhausted tubes of the same general type, and connect one terminal of each with one terminal of a large HOLTZ machine, and connect their other terminals with the other terminal of the machine, interposing an air-spark (say in the positive circuit) so that the electricity has two alternative paths, the one through the one tube and the other through the other, the air-spark being common to both paths, a very remarkable phenomenon will be witnessed (Plate 19, fig. 19). If the air-spark be of a suitable magnitude it will be found that one of the tubes is wholly traversed by the discharge, but that the other is occupied only by a luminous column extending from the positive terminal into the tube for a considerable portion of its length, and gradually tapering to a point. If the air-spark do not exceed a certain limit, depending upon the "resistance" of both tubes, there will be no luminosity at the other end of the tube, and no discharge through it. No effect will be produced upon the luminous column, nor on any portion of the discharge, by breaking the connexion with the distant terminal, showing, what the appearance of the column itself sufficiently indicates, that the discharge is unipolar or incomplete. Slight indications of blue haze are sometimes seen at the tip of this tapering column, due probably to some negative electricity gathered from the neighbourhood, but not directly discharged from the opposite terminal. The discharge is, in fact, one which passes into the tube but not with sufficient force to pass through it, and which accordingly returns by the way by which it entered. The cause of this recall we shall examine hereafter; for our present purposes it suffices to point out the fact that here we have a discharge from one pole which is unable to approach near enough to the other pole to get relief there, and actually prefers to return by the way it came rather than to pass through the tube to the other terminal. Such discharges we shall term *unipolar discharges.*

This unipolar discharge is of course intermittent, and therefore sensitive. If we take a glass rod with a piece of tinfoil at the extremity electrically connected by a wire with the positive terminal of the tube, and hold it near to but a little beyond the end of the luminous column, we shall find the luminous column driven back; and by carefully advancing it towards the positive terminal we can often succeed in driving the luminous column wholly back and preventing any visible discharge taking place into the tube (Plate 19, fig. 20). The explanation of this is obvious. At the moment that the charging-up, which causes the discharge, takes place in the positive terminal, there is also a charging up in the tinfoil, and this by its inductive effect tends to prevent the advance of any free positive electricity. Thus, however rapid the pulsation, the force tending to oppose the discharge keeps exact time with it, and causes the heading back of the luminous column. If the tinfoil and wire be connected with earth, or otherwise made a relieving system, we find the usual to-earth-effects produced on this unipolar discharge.

A form of these experiments, which is in some respects even more striking, is obtained by taking a tube with an intermediate terminal and connecting the intermediate and one of the end terminals with the positive terminal of the machine, and the other terminal of the tube with the negative terminal of the machine. Interpose an air-spark in the positive circuit so that it forms part of the path to both of the terminals which were connected with the positive terminal of the machine. With an air-spark of proper dimensions it will be found that while the whole effective current passes from the positive intermediate terminal to the negative terminal, there is seen, besides, first a tongue-shaped luminous column extending from the positive end terminal towards the intermediate terminal; and secondly a similar tongue-shaped luminous column stretching out from the intermediate terminal to meet it (Plate 19, fig. 21). Or again, if we arrange two tubes as first described, and connect both terminals of the second tube with the positive terminal of the machine, we shall have two positive unipolar columns as before. These two do not join; and it is clear that here we have naturally the same effect as that obtained by the use of the tinfoil in the former case (Plate 19, fig. 22). Each of these discharges acts repulsively on the other, and they drive each other back. If we use the tinfoil, as in the former experiment, we can drive each in turn back towards, and sometimes into its terminal, and, within considerable limits, when one column is driven back, the other advances, and *vice versâ.*

This experiment with the intermediate terminal, shows very forcibly how the discharge from an air-spark terminal depends solely on the forces at work at the terminal itself, and has but little reference to the condition of the other terminal of the tube. We see here that the positive electricity from the intermediate terminal actually issues copiously in the direction in which lies not only no negative terminal, but actually a positive terminal, which ultimately succeeds in repelling its advance.

In corroboration of the statement that these tongue-shaped luminous columns are parts of two distinct incomplete discharges, we may add that the magnet shows that they represent discharges going in opposite directions, the positive electricity in each proceeding from the base of each column towards the apex.

Similar phenomena, save in respect of the shape of the luminous columns, are seen when the two terminals are joined to the negative terminal of the machine; but we shall not dwell here on the resemblances or differences between the two phenomena. The two kinds of unipolar discharge, however, are at once distinguishable from one another, and this may be made use of as a final proof of the correctness of the conclusions arrived at as to the nature of the electric displacements which cause the relief-effects. If a piece of tinfoil be placed on a tube through which a discharge passes, and this tinfoil be electrically connected with *one* terminal of a tube of suitable resistance, it will be found that when an air-spark is introduced into the discharge a unipolar discharge will appear in the tube whose terminal is connected with the tinfoil (Plate 20, fig. 24). If the air-spark be in the positive circuit, this will be found to be a positive unipolar discharge; if the air-spark be in the negative circuit it will be a negative unipolar discharge. This test is made still more certain and useful by taking the wire from the tinfoil to both terminals of the auxiliary tube. The result will then be a double unipolar effect, the nature of which is much more readily recognisable. This gives a very convenient test of the sign of the disruptive discharge at any part of the tube, and has been frequently employed with great advantage in cases where the complexity of the circumstances has made it difficult to ascertain it directly.

The following experiment is instructive in relation to unipolar discharges. If, with the arrangement described on page 214, we place upon a tube containing a positive unipolar discharge a ring of tinfoil not very far from the further end of the tube, and connect it with the positive terminal, the discharge is at once completed, and continues to pass throughout the tube (Plate 19, fig. 23). On examining it we find that from the interior of the tube within the ring of tinfoil proceeds the usual hollow conical discharge characteristic of the non-relief state with the air-spark in the positive, and this stretches in the usual way towards the negative terminal of the tube, and the luminosity that starts from the positive column comes up to and stops at the limit of the dark space within this cone in the usual manner. Thus we have by our tinfoil ring cut the tube into two, each of which shows its own separate discharge; and the tension which was insufficient to cause a discharge through the whole tube is equal to the task of supporting the two shorter discharges. And, just as in the corresponding case in the ordinary sensitive discharge, the pulsations in the tinfoil cause positive induction discharges which are strong enough to reach the negative terminal, and the relief of the tension in the tinfoil which ensues on the occurrence of the discharge into the tube from the positive terminal sets free negative electricity from beneath the tinfoil which satisfies the discharge from the positive terminal. Thus the interior of the tube within the ring of tinfoil acts alternately as a positive and negative terminal, forming part first of the discharge that occupies the further portion of the tube, and then of that which occupies the part of the tube nearer to the positive terminal.

We have already spoken of the results of connecting a small condenser with one of the terminals of the tube, when a coil discharge is used, so as to depress the electrical tension produced at that terminal, and shift the position of the neutral zone. It is very instructive to compare these effects with the analogous effects in the case of unipolar discharges. If we join the effective terminal of the tube containing the unipolar discharge to a small condenser composed of, say, two pieces of tinfoil about three square inches in area, with a plate of mica between them, we shall see the luminous tongue slightly shorten. If, now, we connect the other side of this condenser to earth we see a further shortening of the column, which will often almost disappear. If we connect the terminal with a larger condenser or a Leyden jar the discharge wholly disappears. Thus we see that the condenser or Leyden jar has, just as in the case of the coil discharges, the effect of muffling or toning down the intensity of the impulsive changes of electrical tension at the terminals and thereby lessening the violence of the discharges into the tube.

We conclude, then, from these experiments that the independence of the discharge from each terminal is so complete that we can at will cause the discharges from the two terminals to be equal in intensity but opposite in sign (as in the case of the coil) or of any required degree of inequality (as in the case of the coil with a small condenser). Or we can cause the discharge to be from one terminal only, the other terminal acting merely receptively (as in the case of the air-spark discharge); or we can cause the discharge to pass from one terminal only and return to it, the other terminal not taking any part in the discharge; or, finally, we can make the two terminals pour forth independent discharges of the same sign each of which passes back through the terminal from whence it came.

XIII.—*On the state of the tube during the occurrence of the discharge.*

We have already shown that each pulsation may be considered as a separate disturbance the effects of which have passed away before the next pulse comes. It now remains for us to ascertain what is the state of the tube during the continuance of the almost instantaneous pulse which may be considered as the unit of which the visible discharge is built up.

For the purpose of experimentally ascertaining this, it is obvious that if we employ any tests which are not themselves affected with the same periodicity as the discharge that is being examined, we shall get no results that speak with any degree of certainty of the state of the tube at the instant in question. Thus, if we listen with a telephone in circuit between the earth and a piece of tinfoil on the tube, we hear a sound identical in pitch with that of the air-spark, but this only tells us that there is a change of electric tension in the tinfoil at every pulse, and it cannot possibly tell us at what point in the period of the pulse the change occurs. To ascertain this we must use some tests which in themselves partake of the periodicity of the current itself.

The simplest experiment is that of taking two pieces of tinfoil, one near the air-spark end and one near the opposite end of the tube, and connecting them with a thin wire.

In an ordinary sensitive discharge, if we thus place a piece of tinfoil somewhat near to the air-spark terminal and connect it with a larger piece of tinfoil at a considerable distance along the tube, we shall find that we have complete relief-effect at the first-mentioned spot. And if we examine the effect at the larger piece of tinfoil we shall find indications of the non-relief-effect. If now we reverse the arrangement and place the larger piece near the air-spark terminal and the smaller piece at a considerable distance from it, we shall find that while we have feeble relief-effects at the larger piece we have strongly marked non-relief-effects at the other.*

Nor must it be thought that it is only the less definite types of relief and special effects that can be thus produced. If the pieces of tinfoil be made in the form of rings surrounding the tube, all the effects previously described can be produced in this manner. If, for instance, with a positive air-spark a broad ring of tinfoil near the positive end of the tube be connected to a narrower ring near the opposite terminal, we shall, with a proper adjustment of the air-spark, obtain the segmentation of the luminous column and the formation of the hollow cone, just as definitely as if we had joined the narrow ring to the positive terminal of the tube. A similar effect can be produced with a negative air-spark, provided the narrower ring is the one that is nearer to the air-spark terminal.

Now these effects can scarcely be reconciled with any theory which makes the discharge simultaneous throughout the tube. Suppose that it is so, then if we take A to be the smaller piece of tinfoil (which we shall suppose is the nearer to the air-spark terminal) and B to be the larger piece, it is obvious that the sudden change of tension at A is greater than that at B, since it compels B to give it relief. But however large B might be, such relief could never be greater than would amount to making the potential at A sink to that at B, which would by no means be tantamount to connecting A to earth, since the impulsive change at B will ordinarily be very considerable. Hence, as we get complete relief at A by joining it to B, it must be that B has not suffered any increase of potential at the moment that A seeks relief, and that thus it is not prevented from giving to A that relief which its size and distance from the spot where the free electricity is at the instant in question enable it to give—a relief which, as we have seen, is to all intents and purposes complete. But it does so at the expense of receiving a pulse of electricity from A of the same type as it would receive were it joined to the air-spark terminal, and we therefore get indications of non-relief-effect. We see then that the discharge is progressive, and that it proceeds from the air-spark terminal. For the electric disturbance at A makes its demand for relief on B, and finds B quite as capable of giving it as though no discharge had taken place, so that the discharge has not yet reached B.

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* We have seen that the negative special effect is identical with the positive relief-effect; and it might, therefore, be thought that, after all, the effects seen are nothing but the relief-effects belonging to a discharge from the terminal nearest to the tinfoil, somewhat as was the case when a coil is used. But it is easy to show that this is not the right view to take of the matter; for if, instead of connecting the tinfoil that is further from the air-spark with that which is nearer, we connect it with earth, so as to give to that part the full relief it desires, we find it to be an effect of an opposite character, showing that the former effect was truly a special effect.

Another conclusion of some importance may be drawn from these experiments. We see the relief-effect in A causes B to be so raised in potential as to give marked non-relief-effects. Hence the magnitude of the electric displacements which accompany relief-effects, though of course not so great as those accompanying non-relief-effects, which are the same as those of the discharges themselves, are yet comparable with them, and are quite capable of causing inductive discharges in the tube.

Another very instructive form of these experiments is obtained by placing a long strip of tinfoil along the tube from end to end, taking care, of course, that it is not connected with the terminals (Plate 20, fig. 25). Suppose the air-spark is in the positive, then, if the tube be sufficiently sensitive, it will be found that near the positive terminal there is the discharge relief-effect. A little further on there is repulsion, which gradually diminishes and is finally replaced by attraction of the luminous column which, as we have seen, is an ordinary form of non-relief-effect.* The portion of the tinfoil near the air-spark terminal has derived relief from the remainder by reason of its being the first to experience the sudden change of tension, and the more distant parts receive in consequence pulses of electricity sufficient to cause the non-relief-effects. If the tinfoil be divided a little beyond the part of the tube occupied by the discharge relief-effect the positive end of the further portion will now present the discharge relief-effect; because instead of being a part of the relieving system to the portion of the strip of tinfoil between it and the positive terminal, it is now the part of the further piece of tinfoil which is the first to feel the call for relief, and consequently is the part that derives the full benefit of the relieving system formed by the piece of tinfoil of which it forms a part. And at the same time the negative end of the nearer piece of tinfoil (which previously showed relief-effects) will show non-relief-effects, inasmuch as it has to afford relief to the part nearest to the positive terminal and has no longer any more distant piece of tinfoil from which to draw supplies of positive electricity.

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* When the air-spark is in the negative the relief-effect usually assumes the appearance of an attraction of the luminosity to the inner surface of the tube. In reality it is due to the luminous positive discharges from the side of the tube, such discharges always appearing (as we have seen) to start from the actual surface of the glass. The same remarks apply to the non-relief-effect when the air-spark is in the positive.

This experiment, then, proves that the instantaneous increase of tension arrives at the tinfoil nearer the air-spark end, which we will call A, sooner than at the other, which we will call B. Now what is the state of the tube at B at the moment when the rise of tension occurs at A? We have shown that it is not affected by the rise of tension at A, but we have not otherwise ascertained its electrical state. This is ascertained in the following way. Fasten the piece of tinfoil B to a glass rod and let the wire that connects it be a fine wire running near to the tube during the whole of its course, so that the capacity of the system relieving A may depend almost exclusively on the capacity of B (Plate 20, fig. 26). Now if we move B as far as possible from the tube, keeping the wire straight, it will be found that *the capacity of B to relieve A is diminished.* In other words, by placing B on the tube we place it in a neighbourhood where there is a rising demand for the very electricity which is driven off from A. Suppose, for example, that the air-spark is in the positive. The electricity that leaves A is positive; and hence it follows that at the part of the tube upon which B rests there is a demand for positive, which amounts to saying that there is a rising negative potential there; or in other words the quantity of negative electricity within the tube at B must be increasing.

There would seem to be but one interpretation of this phenomenon. The negative terminal (in this case) is connected with its source of electricity during the whole of the interval between the pulses, and it must be pouring its electricity into the tube during all that time, thus raising gradually the quantity of negative electricity in the tube. When the pulse of positive comes it sweeps up all the negative that it encounters and rushes out at the opposite terminal probably leaving the tube in a positive state. Indeed, this is so marked that we often see indications of a reverse current so soon as we introduce an air-spark, showing that the positive pulse is so violent that it causes over-relief. When this pulse has passed, the non-air-spark terminal, or (as we may better term it) the connected terminal, commences to pour out its electricity, undoing the effect of the pulse on the tube and gradually filling it with negative electricity till the next pulse comes.

So much then for the condition of the tube in front of the electric pulse. We must next inquire what is the condition of the tube behind the pulse during the discharge. This is much more difficult, because we can no longer use the test above described, as we cannot tell whether the reaction which undoubtedly follows the relief-effect occurs during or after the passage of the pulse.

If, however, we put several pieces of tinfoil on the tube and connect the one nearest to the air-spark terminal to earth, placing a telephone in circuit, we shall find that the loudness of the tone is increased by connecting the other pieces of tinfoil with the first piece. This shows that their relief demands are additive, and as we know that the sign of the electricity in the relief-pulse is the same for all, and consequently that the pulse which constitutes the re-action must also necessarily be of the same sign for all, we see that the last piece must be in a condition demanding relief before the first has ceased to need it. In other words, the discharge is like a pair of lazy tongs and not like a bullet; and if we remember that we saw reasons for coming to the conclusion that the discharge was produced by action at the air-spark terminal and was almost independent of the condition of the other terminal, we see a sufficient reason why the discharge should be as we have found it to be. As each kind of electricity repels its like, there is every reason why the pulse should extend itself throughout the tube, and it is not easy to conceive any action that should make it pass through the tube in a compact mass like a bullet. This theory would also account for the observed fact that the relief-action is more violent near to the air-spark terminal than at a distance from it. With the special effect it is not so—indeed, the effect is feebler the nearer we are to the air-spark terminal—and this also is, as we have previously remarked, in full accordance with the above theory, inasmuch as it is natural that the effect of the external pulses should be less marked in proportion as the conflicting action within the tube is more powerful and commences more immediately after the special action commences.

To check the correctness of the conclusions arrived at in this section, it may be well to contrast the results above described with the analogous results obtained from a coil discharge where there is a neutral zone. According to theory, the nature of the relief required in such a case is different upon opposite sides of the neutral zone. Hence it should be possible to give complete relief to a part of the tube on one side of the neutral zone by connecting it with tinfoil placed on the tube at a suitable spot on the other side of the neutral zone. This is found to be the case, and further, it is found that the two corresponding spots approach and recede from the neutral zone together. Moreover, if a long slip of tinfoil be placed on the tube extending from end to end it is found that there are relief effects all along it, though of course they are of opposite types on opposite sides of the neutral zone, and that a very slight change is caused by connecting the strip of tinfoil to earth, showing that the demands of the two portions of the tube are well nigh complementary. And further, although the experiment is difficult to try in such a way as to leave no doubt as to the result, it has been several times observed that the effect of connecting a piece of tinfoil (which is connected to earth through a telephone) near one terminal of a tube containing such a discharge with a piece near the other is to lessen the sound instead of increasing it, thus showing that the electrical wants of the two pieces are opposite in character; and in further confirmation of this we may refer to the experiment described on page 212, which in a more marked and obvious way exemplifies the fact that the electrical wants of pieces of tinfoil on opposite sides of the neutral zone are of opposite signs.

Concluding remarks.

Having thus examined the sensitive state of vacuum discharges, and ascertained the special cause to which their sensitiveness is due, and also the way in which that cause directly operates in producing the effects observed, we are brought face to face with the question, How far are our conclusions relative to these discharges applicable to the case of ordinary or uninterrupted vacuum discharges? or in other words, How far they are independent of the peculiarity of the interruptedness of the discharge? In considering this most important question, it must be conceded that all due weight must be given to the consideration that our investigations have been carried on by means of experiments which would, one and all, have failed had they been applied to ordinary vacuum discharges; so that any conclusions sought to be drawn from them relative to such ordinary discharges must be received with great caution. But this would not justify us in neglecting the remarkable resemblances that exist between the two classes of discharges, or in refusing to examine the information which our knowledge of the constitution of the one gives us of the probable constitution of the other.

As we have previously remarked, it may be stated generally that there is no phenomenon presented by ordinary vacuum discharges which may not be shown by sensitive or interrupted discharges; and conversely there is no phenomenon presented by the latter (except such as directly depend on and are due to its sensitive and interrupted character) which is not also shown by the former. In illustration of this we will select two important features, viz.: striation and the terminal peculiarities. And first as to striation: Every one of the manifold shapes presented by the striæ of the one can be found among striæ produced by the other; and, moreover, the very same element which predisposes to the presence or absence of striation (viz.: the greater or less quantity of electricity discharged) in the one case predisposes also in the other. No doubt the sensitive discharge, in the conditions under which it has been here studied, is much less often striated than is the non-sensitive, but as the character of such a discharge is essentially variable and unsteady, it is not at all surprising that effects which are known to depend on complete steadiness of the discharge should be those least frequently met with. And that we are right in assigning these as reasons for the tendency in interrupted discharges to be non-striated is rendered more probable by the fact that, when we compel the discharges to operate with great regularity and within the prescribed limits of quantity and of time, as for example in the case of the high-speed break or of artificially produced striation, we at once find it easy to produce the steady effects commonly observable in ordinary discharges.

Next as to the terminal peculiarities.* These are, it will be admitted, common to both classes of discharges. But inasmuch as we have seen that the relief and non-relief-effects are substantially identical with terminal peculiarities, we can now go further, and say that they are shown wherever there is an instantaneous electric discharge from the space close to the surface of any fixed body. The consideration, therefore, of either of these important features alike leads us to the conclusion that the essentials of the discharge of electricity through rarefied gases are the same whether the discharge be interrupted, uninterrupted, or wholly discontinuous, and perhaps alternate, as in the case last mentioned.

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* If the theory of the authors of this paper be correct, these terminal peculiarities are in reality cases of striation where the circumstances of the discharge are such as to maintain perforce the requisite steadiness and uniformity of conditions. But for the present purposes it is better to consider them separately.

Now the simplest, and indeed the only obvious explanation of this result is, that the character which was found to be fundamental in sensitive discharges, viz.: disruptiveness, is common to both kinds of discharge; and that the difference between the two kinds is to be sought in the scale on which that character is displayed.* In both discharges each terminal pours forth its electricity to satisfy its own needs, and only in a very secondary degree to satisfy the needs of the opposite terminal. The one terminal does not feel the electric state of the other directly, as would be the case were they metallically connected, but pours forth its electricity in the shape of free electricity, and leaves it to wander at its own will in that shape. If these matters could be demonstrated conclusively—in other words, if it could be shown that all disruptive discharges are discontinuous, and that the appearance of continuity is, in the case of the ordinary discharge, solely due to the extreme rapidity with which consecutive pulses follow each other, and the correlative fact that each individual pulse is of extreme minuteness, a great step would be gained in our knowledge of the nature of electric discharges; and though this is not to be hoped for at present, we trust that the results recorded in this paper add considerably to the evidence in favour of them, and we shall now proceed to indicate the considerations which have led us to this conclusion.

In the first place it is undeniable that all the phenomena of vacuum discharges, and especially the terminal phenomena, *may* be presented by discontinuous discharges, each individual member of which is wholly independent of those before or after it, so that we may say that they may be produced by single instantaneous discharges. And if the circumstances be such as to produce considerable regularity in the successive discharges, we shall have all the characteristic phenomena of the steady uninterrupted discharge presented by them. Hence the idea of continuity of discharge is not necessary to account for these effects. Nay, further, we can show that a continuous displacement of electricity from the surface of a body will fail to produce these effects when a more rapid displacement (of an equal amount) will produce them. In the experiment given on page 170 it is found that, if the wire to the outside of the tube comes from the *positive* terminal of the machine, the appearance within the tube is of a *negative* terminal and *vice versâ*. Now, as the action must necessarily be alternate within the tube, there must be as great a discharge of positive electricity during the one part of the process as there is of negative during the other. The only difference is that the one occupies only the time required for a spark to pass between the terminals, and the other occupies the vastly longer period that elapses between two consecutive sparks.*

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* To this we must add the restriction that, in the case of the ordinary discharge, the discontinuous pulses in which the electricity leaves the terminals must be very minute, a condition which is almost a corollary from the greater rapidity of intermittence in that discharge.

The probability that both kinds of discharge are really pulsatory is increased by the consideration that the striæ are *formed* by the discharge. This increases the difficulty of supposing that a strictly continuous current could imitate effects which we have seen to be caused by discharges known to be instantaneous and disconnected. If the evidence given in the paper as to the form of the discharge from one stria to another (see page 205) be considered sufficient, these remarks have still greater weight, for it is scarcely conceivable that a strictly continuous current should take so strange a course. The passage from stria to stria must then be taken as disruptive and discontinuous, and if this be granted, then, as striæ are only particular cases of terminals, it follows almost as a matter of course that all discharges in rarefied air are equally so. And this consideration becomes of greater weight when we consider the intimate relation of striæ and their component parts to what we have termed the *physical unit* of a striated discharge. Such a discharge is, as we have seen, made up of repetitions of a unit consisting of the bright surface of a stria, the dark space between it and the next stria and the hazy interior of the latter. Such a structure then must either be necessary for the passage of the discharge or its natural consequence. Take the hollow cone discharge produced by the positive non-relief-effect. Here we compel a positive discharge to leave the interior surface of the glass under such circumstances that an equal amount of negative electricity is left behind to satisfy the original positive discharge. This it effects by causing the positive column to stop suddenly at a considerable interval from the place of the discharge, and to terminate in a sharply-defined bright head, which is in all respects identical with the outer surface of a stria, while on its part the negative discharge forms the hazy interior of the hollow cone. Thus the discharge requires and compels the formation of a unit of the structure above described. It seems in the highest degree improbable that structures formed and maintained to satisfy the needs of disruptive discharges (which have to leap from the one part and alight on the other of the differently constituted terminal portions of the structure) should also be formed and maintained by discharges which, in virtue of their continuous character, must be supposed to flow with the same evenness as in a metallic current, and to remain obedient to the law that the same amount of electricity flows across every section of the circuit in the same infinitesimal portion of time.

But if we admit that the ordinary discharge is discontinuous, the rapidity of the intermittence must be very great. For if this were not the case the individual pulsations would be of considerable magnitude, and would cause sufficient rise and fall of electric tension outside the tube to give to us the phenomena of the sensitive state. But if they are thus rapid there is no difficulty in the non-sensitiveness of the discharge. There may be many discharges from each terminal into the tube in the space of time that it takes the discharge to pass through the tube, and the alterations of tension at any point outside the glass may be infinitesimal, so as to be wholly unequal to producing any relief-effect within.

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* This is selected as the most striking instance. But most of the phenomena of the relief and non-relief-effects would probably be found on examination to afford instances of it.

It may be objected that this theory supposes something like a discontinuity in the range of rapidity of pulsations in electrical discharge. Either they are of the comparatively slow period that gives the sensitive state, or of the indefinitely more rapid period that gives the non-sensitive state. This difficulty is, however, only parallel to the difficulty with which we are met when we see an air-spark interval enlarged to beyond the striking distance, when the discharge across it at once becomes silent, and the current that passes is found, when conducted through the tube, to have all the characteristics of a continuous discharge. No one can believe that the discharge across the interval has suddenly become continuous—the real nature of the change must be that the intermittent pulses (whether carried by convection on particles of air or not) have become so numerous that the average of the discharge, when taken over an extremely short space of time, is constant. And the appositeness of the comparison is shown by the fact that the discharge through the tube is non-sensitive.*

We have seen that the discharges in the sensitive discharge are sudden and of sufficient quantity to affect to a considerable extent the electric tension in the space around. This suddenness is an essential. And it is to this that we must attribute the indifference of the discharge to the potential of the body affecting it and the dependence of the effects on the change of electric state alone. If an electrified body be surrounded by air, it must of course form an electric field around it, which will or will not alter with time according as the air can or can not get at the body and convey away its free electricity. But when this electric field has been formed, it is affected just as easily by a new electric force brought suddenly into the field as though the original body were not there. The change is to all intents and purposes the same. We do not know what material changes accompany the formation of such an electric field, but for the sake of argument assume that there is a certain driving off of gaseous particles until equilibrium is attained. Then the new force suddenly brought in will send away a like stream of particles whether or not the field on which it operates has already been subjected to a similar operation.

The true analogue to these effects is to be found in the consideration of impulsive and continuous forces in dynamics. It is a well-known principle that in calculating the effect of impulsive forces all continuous forces, whether they be of the nature of pressures, strains, external attractions, or otherwise, may be neglected. And the same is true of the changes of electric tension with which we are dealing in the sensitive discharges. They are so sudden and yet of such considerable magnitude that in considering the change they effect we may proceed as though no other electrical forces were at work. Thus, if we take the case of the positive unipolar discharge, a conductor connected with the acting terminal will, if approached towards the head of the column, drive it back, because when the sudden rush of electricity comes to the terminal, and from it into the tube, there is an equally sudden rush of the same kind to the conductor. Each causes an impulsive change in the electric tension around it; and as these are of a character to nullify one another within the tube the column is shortened. But a Leyden jar charged with the same kind of electricity to a much higher potential than the source of the discharge will not drive back the column. All that it does is to set up a permanent electric field which offers no resistance whatever to the sudden impulsive electrical changes that occur in the terminal and in the space within the tube.

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* In this case, as in the other, there is probably no discontinuous change in the character of the discharge. No doubt much of the discharge is carried in the silent form as we approach striking distance, and the apparent discontinuity is due to the fact that the increase in the proportion so carried is very sudden as we get near that point.

The authors of this paper have abstained purposely from dwelling upon any questions involving theories of the nature of electricity or the difference between positive and negative electricity, or upon other differences which have come to light during the present investigation. But it is important to observe that the difference in behaviour of the two kinds of electricity is not confined to cases in which these electricities leave the terminals proper of the tube, but is equally present when they come from quasi-terminals (as in the case of the relief and non-relief-effects), or even when they form their own terminals out of gaseous materials, as in the case of striæ.

Postscript.

(June 28, 1879.)

A.—*On the variations of form of the negative glow.*

It is obviously a matter of great importance to the theory of striæ put forward in this paper, that the negative glow should be shown to be merely a stria modified by the local conditions that exist at the negative terminal of the tube. We have therefore thought it desirable to record one or two cases of special forms assumed by the negative glow under special circumstances, which have been met with by us since the date at which this paper was originally presented to the Royal Society, and which appear to us to show very clearly that the negative glow is only a modified stria.

The first case was observed by one of the authors of this paper in a tube belonging to Mr. Warren De La Rue, which was being experimented upon in his laboratory with a current of several thousand cells of his well-known battery. The negative terminal was in the form of a rather large ring, but the exhaust of the tube was so good, and consequently the breadth of the Crookes' space was so considerable, that the negative glow did not assume the form of an anchor ring, but of a surface of revolution, the section of which through its axis was bean-shaped, as represented in Plate 18, fig. 27. The glow as a whole had the ordinary whitish appearance, but the blunt protuberance or boss, opposite to the centre of the ring (marked R in the figure), where we should naturally assume any action that might be going on in the glow to be most intense, was deeper in tone and was precisely like a stria in appearance, both as regarded colour and shape, save that it was at its edges continuous with the negative glow. And when a certain change was made in the current by an alteration of some of the circumstances of the external circuit, a small conical stria appeared *within* the negative glow, just in front of the boss above spoken of, and quite separated from it, the two resembling in all respects (save in their position relative to the negative glow) two small striæ separated by the usual dark space, the dimensions of the latter being, like those of the two striæ, very small.

Again, Plate 18, fig. 28, represents the variations of the negative glow, and its transition into a complete striated form, under the influence of a perforated disk terminal. In this it will be seen that the so-called negative glow not only follows the general contour of the disk, but that it projects through the aperture. This part of the discharge assumes the character of a positive column in a constricted tube, and exhibits the well-known bead-like striæ.

It has been already suggested in the text of this paper, that the long dark space intervening between the negative glow (or stria, as it is here regarded) and the head of the striated column is due to the local action of the terminal itself. An experiment, shown in Plate 20, fig. 29, corroborates this view. Beside the end terminals which were used for the discharge, the tube was furnished with an intermediate ring terminal, which remained disconnected, except so far as leakage through it from the air. The column of striæ did not usually extend from the positive end so far as the ring. But when the column was drawn out by the influence of a magnet, and made fully to reach the ring, the column of striæ was immediately thrown back, so as tc form a comparatively long dark space between the stria so anchored on the ring and the column behind. This sudden throwing back was repeated every time a stria was brought on to the ring by the magnet, and was doubtless due to the fact that the ring was acting as a (weak) negative pole through leakage from the air.

B.—*On the double character of the coil discharge.*

We have given experimental proof in Section XI. of this paper, that when we take a discharge of small quantity from a coil of symmetrical make, the electricity passes into the tube simultaneously at both terminals, and that the two discharges meet and form a neutral zone near the middle of the tube. Since the paper was originally written, the authors have met with cases in which the distinction between the nature of the discharges from the two terminals and the position of their place of meeting are of themselves visible to the eye without needing to be demonstrated by the tests we have described in the paper.

The tube in which this was first observed was one of considerable length, but only of moderate diameter and moderately good exhaust. The appearance of the discharge is shown in Plate 20, fig. 30. The break current alone passed. The luminous column starting from the positive terminal was much smaller in diameter than the interior of the tube, and consequently left a dark space between it and the sides of the tube. The portion in the half of the tube nearer to the negative terminal, on the contrary, appeared wholly to fill up the tube. On touching the positive terminal the discharge filled the tube throughout; on touching the negative terminal the discharge of smaller section extended throughout the whole length of the tube, and when a small condenser was used, as described on page 211, the portion of the length occupied by each could be varied at will. It should be added that the narrow and broad discharges showed, by the nature of their relief-effects, that they were respectively caused by positive and negative electricity, and the part of the tube where they were seen to join was found in all cases to be the neutral zone. It also should be added that either column could be shortened by producing relief-effects near its root as though such a proceeding delayed the discharge, just as the small condenser would do.

The cause of the difference in the two discharges is not difficult to explain. Both the discharges get some portion of relief from the inner surface of the glass even when the finger is not placed thereon, and this in the case of the positive discharge naturally results in the formation of a dark space at the surface of the glass, while in the case of the negative discharge it leads to the formation of positive luminosity close up to the surface of the glass.

A similar difference in appearance in the two discharges has since been observed in other tubes, and it is probably not an uncommon phenomenon. It is doubtless analogous to what is very frequently seen in the case of a discharge with a positive air-spark, in which it is found that the luminous column is very slender and does not nearly fill the tube while the air-spark is small.

C.—*On the interference of intermittent discharges.*

In making certain experiments with intermittent discharges, it happened that a discharge was sent through two tubes placed in series, but lying parallel to one another, and at no great distance apart. It was observed that the discharge in the one tube strongly affected that in the other, and from this the authors were led to examine more closely the question of the interference of discharges which do not take place in the same tube.

For this purpose intermittent discharges were sent through two parallel tubes in series, the current passing sometimes in the same, and sometimes in contrary directions through the two tubes. The two discharges were found to be greatly affected by one another, as could be clearly shown by putting the one tube at a greater or less distance from the other. On examining the nature of these effects, they seemed to follow the same law as would have prevailed had the tubes formed one long one which could be doubled up or bent so as to make the one portion of it parallel to the other portion. This law is that the discharge in the portion of the tube nearer the air-spark tends to produce special effects on the further portion, and itself to manifest relief-effects. One remarkable result of this law deserves mention. When the air-spark is in the positive the special effect is of course accompanied by positive luminous discharge from the inside of the glass. Accordingly, the tube through which the discharge passed last had positive luminous discharge throughout its whole length, on the side where the other tube lay, and thus the whole luminosity in that tube appeared to be attracted towards the other discharge. This was of course the case whether the discharge passed in the same or contrary directions through the tubes, so that the remarkable phenomenon was presented of a luminous discharge (which we know behaves like a current) appearing to attract violently a discharge going in the opposite direction. After the explanation given above, it will be understood that the phenomenon was not really a case of two currents going in opposite directions attracting one another, but was a case of the luminous effects of the interference of the two co-periodic discharges through the medium of the impulsive static induction of the pulses of which they were constituted.

When a discharge is intermittent it can be made to pass through several tubes which are so connected as to offer alternative paths to the discharge. When this was done it was found that these discharges affected each other either by apparent attraction or repulsion of the luminosity which accompanied them. The authors have not had time to go further into the subject so as to ascertain precisely the law in such cases, but they have no doubt that on examination the effects could be accounted for by similar considerations to those which were found to hold in the previous case.

An extremely interesting experiment showing how intermittent discharges can in this way be made to interfere when they are synchronous, but not otherwise, was made by using two equal small coils to produce the discharges in two parallel tubes. When the contact breakers were made to work in unison any effect which the discharges produced upon one another was steady, as would have been the case had their co-periodicity been due to their forming part of one discharge as in the previous cases. But when they were a little out of unison, so that audible beats were given in addition to the two notes produced by the contact breakers, it was found that the luminosities in the tubes flickered (*i.e.*, were attracted or repelled) synchronously with each such beat. The reason is obvious. It was only just at the time that the contact breakers were moving approximately synchronously that the two intermittent discharges were sufficiently synchronous to interfere with one another.

VII. *On Stresses in Rarified Gases arising from Inequalities of Temperature.*

By J. Clerk Maxwell, *F.R.S., Professor of Experimental Physics in the University of Cambridge.*

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1. In this paper I have followed the method given in my paper "On the Dynamical Theory of Gases" (Phil. Trans., 1867, p. 49). I have shown that when inequalities of temperature exist in a gas, the pressure at a given point is not the same in all directions, and that the difference between the maximum and the minimum pressure at a point may be of considerable magnitude when the density of the gas is small enough, and when the inequalities of temperature are produced by small* solid bodies at a higher or lower temperature than the vessel containing the gas.

2. The nature of this stress may be thus defined:—Let the distance from a given point, measured in a given direction, be denoted by h; then the space-variation of the temperature for a point moving along this line will be denoted by $\frac{d\theta}{dh}$, and the space-variation of this quantity along the same line by $\frac{d^2\theta}{dh^2}$.

There will, in general, be a particular direction of the line h for which $\frac{d^2\theta}{dh^2}$ is a maximum, another for which it is a minimum, and a third for which it is a maximum-minimum. These three directions are at right angles to each other, and are the

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* The dimensions of the bodies must be of the same order of magnitude as a certain length λ, which may be defined as the distance travelled by a molecule with its mean velocity during the time of relaxation of the medium.

The time of relaxation is the time in which inequalities of stress would disappear if the rate at which they diminish were to continue constant. Hence

$$\lambda=2\left(\frac{2p}{\pi\rho}\right)^{\frac{1}{2}}\cdot\frac{\mu}{p}=2\mu\left(\frac{2}{\pi\rho p}\right)^{\frac{1}{2}}.$$

On the hypothesis that the encounters between the molecules resemble those between "rigid elastic" spheres, the free path of a molecule between two successive encounters has a definite meaning, and if l is its mean value,

$$l=\frac{3}{2}\mu\left(\frac{\pi}{2p\rho}\right)^{\frac{1}{2}}=\frac{3\pi}{8}\lambda=1\cdot178\lambda.$$

So that the mean path of a molecule may be taken as representing what we mean by "small."

If the force between the molecules is supposed to be a continuous function of the distance, the free path of a molecule has no longer a definite meaning, and we must fall back on the quantity λ, as defined above.

axes of principal stress at the given point; and the part of the stress arising from inequalities of temperature is, in each of these principal axes,

$$3\frac{\mu^2}{\rho\theta}\frac{d^2\theta}{dh^2},$$

where μ is the coefficient of viscosity, ρ the density, and θ the absolute temperature.

3. Now for dry air at 15° C, $\mu=1\cdot 9\times 10^{-4}$ in centimetre-gramme-second measure, and $\frac{3\mu^2}{\rho\theta}=\frac{1}{p}0\cdot 315$, where p is the pressure, the unit of pressure being one dyne per square centimetre, or nearly one millionth part of an atmosphere.

If a sphere of $2a$ centimetres in diameter is T degrees centigrade hotter than the air at large distances from it, then, when there is a steady flow of heat, the temperature at a distance of r centimetres from the centre will be

$$\theta=\theta_0+\frac{\mathrm{T}a}{r}, \text{ and } \frac{d^2\theta}{dr^2}=\frac{2\mathrm{T}a}{r^3}.$$

Hence, at a distance of r centimetres from the centre of the sphere, the pressure in the direction of the radius arising from inequality of temperature will be $\frac{\mathrm{T}a}{pr^3}0\cdot 63$ dynes per square centimetre.

4. In Mr. CROOKES' experiments the pressure, p, was often so small that this stress would be capable, if it existed alone, of producing rapid motion in a radiometer.

Indeed, if we were to consider only the normal part of the stress exerted on solid bodies immersed in the gas, most of the phenomena observed by Mr. CROOKES could be readily explained.

5. Let us take the case of two small bodies symmetrical with respect to the axis joining their centres of figure. If both bodies are warmer than the air at a distance from them, then, in any section perpendicular to the axis joining their centres, the point where it cuts this line will have the highest temperature, and there will be a flow of heat outwards from this axis in all directions.

Hence $\frac{d^2\theta}{dh^2}$ will be positive for the axis, and it will be a line of maximum pressure, so that the bodies will repel each other.

If both bodies are colder than the air at a distance, everything will be reversed; the axis will be a line of minimum pressure, and the bodies will attract each other.

If one body is hotter and the other colder than the air at a distance, the effect will be smaller, and it will depend on the relative sizes of the bodies, and on their exact temperatures, whether the action is attractive or repulsive.

6. If the bodies are two parallel disks very near to each other, the central parts will produce very little effect, because between the disks the temperature varies uniformly, and $\frac{d^2\theta}{dh^2}=0$. Only near the edges will there be any stress arising from inequality of temperature in the gas.

7. If the bodies are encircled by a ring having its axis in the line joining the bodies, then the repulsion between the two bodies, when they are warmer than the air in general, may be converted into attraction by heating the ring so as to produce a flow of heat inwards towards the axis.

8. If a body in the form of a cup or bowl is warmer than the air, the distribution of temperature in the surrounding gas is similar to the distribution of electric potential near a body of the same form, which has been investigated by Sir W. THOMSON. Near the convex surface the value of $\frac{d^2\theta}{dh^2}$ is nearly the same as if the body had been a complete sphere, namely $2T\frac{1}{a^2}$, where T is the excess of temperature, and a is the radius of the sphere. Near the concave surface the variation of temperature is exceedingly small.

Hence the normal pressure will be greater on the convex surface than on the concave surface, and if we were to neglect the tangential pressures we might think this an explanation of the motion of Mr. CROOKES' cups.

Since the expressions for the stress are linear as regards the temperature, everything will be reversed when the cup is colder than the surrounding air.

9. In a spherical vessel, if the two polar regions are made hotter than the equatorial zone, the pressure in the direction of the axis will be greater than that parallel to the equatorial plane, and the reverse will be the case if the polar regions are made colder than the equatorial zone.

10. All such explanations of the observed phenomena must be subjected to careful criticism. They have been obtained by considering the normal stresses alone, to the exclusion of the tangential stresses, and it is much easier to give an elementary exposition of the former than of the latter. If, however, we go on to calculate the forces acting on any portion of the gas in virtue of the stresses on its surface, we find that when the flow of heat is steady, these forces are in equilibrium. Mr. CROOKES tells us that there is no molar current or wind in his radiometer vessels. It is not easy to prove this by experiment, but it is satisfactory to find that the system of stresses here described as arising from inequalities of temperature will not, when the flow of heat is steady, generate currents.

11. Consider, then, the case in which there are no currents of gas but a steady flow of heat, the condition of which is

$$\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}+\frac{d^2\theta}{dz^2}=0.$$

(In the absence of external forces such as gravity, and if the gas in contact with solid bodies does not slide over them, this is always a solution of the equations, and it is the only permanent solution.) In this case the equations of motion show that every particle of the gas is in equilibrium under the stresses acting on it. Hence, any finite portion of the gas is also in equilibrium; also, since the stresses are linear functions of the temperature, if we superpose one system of temperatures on another, we also superpose the corresponding systems of forces.

Now the system of temperatures due to a solid sphere of uniform temperature immersed in the gas, cannot of itself give rise to any force tending to move the sphere in one direction rather than in another. Let the sphere be placed within the finite portion of gas which, as we have said, is already in equilibrium. The equilibrium will not be disturbed. We may introduce any number of spheres at different temperatures into the portion of gas, so as to form a body of any shape, heated in any manner, and when the flow of heat has become steady the whole system will be in equilibrium.

12. How, then, are we to account for the observed fact that forces act between solid bodies immersed in rarified gases, and this, apparently, as long as inequalities of temperature are maintained?

I think we must look for an explanation in the phenomenon discovered in the case of liquids by HELMHOLTZ and PIOTROWSKI,[*] and for gases by KUNDT and WARBURG,[†] that the fluid in contact with the surface of a solid must slide over it with a finite velocity in order to produce a finite tangential stress.

The theoretical treatment of the boundary conditions between a gas and a solid is difficult, and it becomes more difficult if we consider that the gas close to the surface is probably in an unknown state of condensation. We shall therefore accept the results obtained by KUNDT and WARBURG on their experimental evidence.

They have found that the velocity of sliding of the gas over the surface due to a given tangential stress varies inversely as the pressure.

The coefficient of sliding for air on glass was found to be $G=\frac{8}{p}$ centimetres, where p is the pressure in millionths of an atmosphere. Hence at ordinary pressures G is insensible, but in the vessels exhausted by Mr. CROOKES it may be considerable.

Hence, if close to the surface of a solid there is a tangential stress S acting on a surface parallel to that of the body in a direction h parallel to that surface, there will also be a sliding of the gas in contact with the solid over its surface in the direction h with a finite velocity $=\frac{SG}{\mu}$.

13. I have not attempted to enter on the calculation of the effect of this sliding motion, but it is easy to see that if we begin with the case in which there is no sliding, the instantaneous effect of permission being given to the gas to slide must be to diminish the action of all tangential stresses on the surface, without affecting the normal stresses, and in course of time to set up currents sweeping over the surfaces of solid bodies, thus completely destroying the simplicity of our first solution of the problem.

14. When external forces, such as gravity, act on the gas, and when the thermal phenomena produce differences of density in different parts of the vessel, then the well-

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[*] Wiener Sitzb., xl., 1860, p. 607.

[†] POGG. Ann., clv., 1875, p. 337.

known convection currents are set up. These also interfere with the simplicity of the problem and introduce very complicated effects. All that we know is that the rarer the gas and the smaller the vessel the less is the effect of the convection currents, so that in Mr. CROOKES' experiments they play a very small part.

We now proceed to the calculations :—

(1.) *Encounter between two Molecules.*

The motion of the two molecules after an encounter depends on their motion before the encounter, and is capable of being determined by purely dynamical methods. If the encounter of the molecules does not cause rotation or vibration in the individual molecules, then the kinetic energy of the centres of mass of the two molecules must be the same after the encounter as it was before.

This will be true on the average, even if the molecules are complex systems capable of rotation and internal vibration, provided the temperature is constant. If, however, the temperature is rising, the internal energy of the molecules is, on the whole, increasing, and therefore the energy of translation of their centres of mass must be, on an average, diminishing at every encounter. The reverse will be the case if the temperature is falling.

But however important this consideration may be in the theory of specific heat and that of the conduction of heat, it has only a secondary bearing on the question of the stresses in the medium; and as it would introduce great complexity and much guess-work into our calculations, I shall suppose that the gas here considered is one the molecules of which do not take up any sensible amount of energy in the form of internal motion. KUNDT and WARBURG[*] have shown that this is the case with mercury gas.

Let the masses of the molecules be M_1 and M_2, and their velocity-components ξ_1, η_1, ζ_1, and ξ_2, η_2, ζ_2 respectively. Let V be the velocity of M_1 relative to M_2.

Before the encounter let a straight line be drawn through M_1 parallel to V, and let a perpendicular b be drawn from M_2 to this line. The magnitude and direction of b and V will be constant as long as the motion is undisturbed.

During the encounter the two molecules act on each other. If the force acts in the line joining their centres of mass, the product bV will remain constant, and if the force is a function of the distance, V and therefore b will be of the same magnitude after the encounter as before it, but their directions will be turned in the plane of V and b through an angle 2θ, this angle being a function of b and V, which vanishes for values of b greater than the limit of molecular action. Let the plane through V and b make an angle ϕ with the plane through V parallel to x, then all values of ϕ are equally probable.

If ξ_1' be the value of ξ_1 after the encounter,

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[*] POGG. Ann., clvii., 1876, p. 353.

$$\xi_1'=\xi_1+\frac{M_2}{M_1+M_2}((\xi_2-\xi_1)2\sin^2\theta+[(\eta_2-\eta_1)^2+(\zeta_2-\zeta_1)^2]^{\frac{1}{2}}\sin 2\theta\cos\phi)\quad . \quad . \quad (1)$$

When the two molecules are of the same kind, $\frac{M_2}{M_1+M_2}=\frac{1}{2}$, and in the present investigation of a single gas we shall assume this to be the case.

If we use the symbol δ to indicate the increment of any quantity due to an encounter, and if we remember that all values of ϕ are equally probable, so that the average value of $\cos\phi$ and of $\cos^3\phi$ is zero, and that of $\cos^2\phi$ is $\frac{1}{2}$, we find

$$\delta(\xi_1+\xi_2)\ =0\quad \ldots \quad (2)$$

$$\delta(\xi_1^2+\xi_2^2)=-[3(\xi_2-\xi_1)^2-V^2]\sin^2\theta\cos^2\theta.\quad \ldots \quad (3)$$

$$\delta(\xi_1^3+\xi_2^3)=-\tfrac{3}{2}(\xi_1+\xi_2)[3(\xi_2-\xi_1)^2-V^2]\sin^2\theta\cos^2\theta.\quad \ldots \quad (4)$$

From these by transformation of coordinates we find

$$\delta(\xi_1\eta_1+\xi_2\eta_2)=-3(\xi_2-\xi_1)(\eta_2-\eta_1)\sin^2\theta\cos^2\theta\quad \ldots \quad (5)$$

$$\begin{aligned}\delta(\xi_1\eta_1^2+\xi_2\eta_2^2)=-\tfrac{1}{2}[&9(\xi_1\eta_1^2+\xi_2\eta_2^2)-3(\xi_1\eta_2^2+\xi_2\eta_1^2)\\ &-(\xi_1+\xi_2)(6\eta_1\eta_2+V^2)]\sin^2\theta\cos^2\theta\quad \ldots \quad (6)\end{aligned}$$

$$\begin{aligned}\delta(\xi_1\eta_1\zeta_1+\xi_2\eta_2\zeta_2)=-\tfrac{1}{2}[&9(\xi_1\eta_1\zeta_1+\xi_2\eta_2\zeta_2)-3(\xi_1\eta_1\zeta_2+\xi_1\eta_2\zeta_1+\xi_1\eta_2\zeta_2+\xi_2\eta_1\zeta_1\\ &+\xi_2\eta_1\zeta_2+\xi_2\eta_2\zeta_1)]\sin^2\theta\cos^2\theta\quad \ldots \quad (7)\end{aligned}$$

[*Application of Spherical Harmonics to the Theory of Gases.*

If we suppose the direction of the velocity of M_1 relative to M_2 to be indicated by the position of a point P on a sphere, which we may call the sphere of reference, then the direction of the relative velocity after the encounter will be indicated by a point P′, the angular distance PP′ being 2θ, so that the point P′ lies in a small circle, every position in which is equally probable.

We have to calculate the effect of an encounter upon certain functions of the six velocity-components of the two molecules. These six quantities may be expressed in terms of the three velocity-components of the centre of mass of the two molecules (say u, v, w), the relative velocity of M_1 with respect to M_2 which we call V, and the two angular coordinates which indicate the direction of V. During the encounter, the quantities u, v, w, and V remain the same, but the angular coordinates are altered from those of P to those of P′ on the sphere of reference.

Whatever be the form of the function of ξ_1, η_1, ζ_1, ξ_2, η_2, ζ_2, we may consider it expressed in the form of a series of spherical harmonics of the angular coordinates, their coefficients being functions of u, v, w, V, and we have only to determine the effect of the encounter upon the value of the spherical harmonics, for their coefficients are not changed.

Let $Y^{(n)}$ be the value at P of the surface harmonic of order n in the series considered.

After the encounter, the corresponding term becomes what $Y^{(n)}$ becomes at the point P′, and since all positions of P′ in a circle whose centre is P are equally probable, the mean value of the function after the encounter must depend on the mean value of the spherical harmonic in this circle.

Now the mean value of a spherical harmonic of order n in a circle, the cosine of whose radius is μ, is equal to the value of the harmonic at the pole of the circle multiplied by $P^{(n)}(\mu)$, the zonal harmonic of order n, and amplitude μ.

Hence, after the encounter, $Y^{(n)}$ becomes $Y^{(n)}P^{(n)}(\mu)$, and if F_n is the corresponding part of the function to be considered, and δF_n the increment of F_n arising from the encounter, $\delta F_n = F_n(P^{(n)}(\mu) - 1)$.

This is the mean increment of F_n arising from an encounter in which $\cos 2\theta = \mu$. The rate of increment is to be found from this by multiplying it by the number of encounters of each molecule per second in which μ lies between μ and $\mu + d\mu$, and integrating for all values of μ from -1 to $+1$.

This operation requires, in general, a knowledge of the law of force between the molecules, and also a knowledge of the distribution of velocity among the molecules.

When, as in the present investigation, we suppose both the molecules to be of the same kind, and take both molecules into account in the final summation, the spherical harmonics of odd orders will disappear, so that if we restrict our calculations to functions of not more than three dimensions, the effect of the encounters will depend on harmonics of the second order only, in which case $P^{(2)}(\mu) - 1 = \frac{3}{2}(\mu^2 - 1) = \frac{3}{2}\sin^2 2\theta$. —Note added May, 1879.]

(2.) *Number of Encounters in unit of Time.*

We now abandon the dynamical method and adopt the statistical method. Instead of tracing the path of a single molecule and determining the effects of each encounter on its velocity-components and their combinations, we fix our attention on a particular element of volume, and trace the changes in the average values of such combinations of components for all the molecules which at a given instant happen to be within it. The problem which now presents itself may be stated thus: to determine the distribution of velocities among the molecules of any element of the medium, the current-velocity and the temperature of the medium being given in terms of the coordinates and the time. The only case in which this problem has been actually solved is that in which the medium has attained to its ultimate state, in which the temperature is uniform and there are no currents.

Denoting by

$$dN = f_1(\xi, \eta, \zeta, x, y, z, t)\,d\xi d\eta d\zeta dx dy dz$$

the number of molecules of the kind M_1 which at a given instant are within the element of volume $dxdydz$, and whose velocity-components lie between the limits $\xi\pm\frac{1}{2}d\xi$, $\eta\pm\frac{1}{2}d\eta$ $\zeta\pm\frac{1}{2}d\zeta$, BOLTZMANN has shown that the function f_1 must satisfy the equation

$$\frac{df_1}{dt}+\xi_1\frac{df_1}{dx}+\eta_1\frac{df_1}{dy}+\zeta_1\frac{df_1}{dz}+X\frac{df_1}{d\xi_1}+Y\frac{df_1}{d\eta}+Z\frac{df_1}{d\zeta}+$$

$$+\iiint d\xi_2 d\eta_2 d\zeta_2\int b\,db\int d\phi\,V(f_1f_2-f_1'f_2')=0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where f_2, f_1', f_2' denote what f becomes when in place of the velocity-components of M_1 before the encounter we put those of M_2 before the encounter, and those of M_1 and M_2 after the encounter, respectively, and the integration is extended to all values of ϕ and b and of ξ_2, η_2, ζ_2, the velocity-components of the second molecule M_2.

It is impossible, in general, to perform this integration without a knowledge, not only of the law of force between the molecules, but of the form of the functions f_1, f_2, f_1', f_2', which have themselves to be found by means of the equation.

It is only for particular cases, therefore, that the equation has hitherto been solved.

If the medium is surrounded by a surface through which no communication of energy can take place, then one solution of the equation is given by the conditions

$$f_1f_2-f_1'f_2'=0,$$

and

$$\xi\frac{df}{dx}+\eta\frac{df}{dy}+\zeta\frac{df}{dz}+X\frac{df}{d\xi}+Y\frac{df}{d\eta}+Z\frac{df}{d\zeta}=0,$$

which give

$$f_1=A_1e^{-h(2\psi_1+\xi^2+\eta^2+\zeta^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where ψ_1 is the potential of the force whose components are X_1, Y_1, Z_1, and A_1 is a constant which may be different for each kind of molecules in the medium, but h is the same for all kinds of molecules.

This is the complete solution of this problem, and is independent of any hypothesis as to the manner in which the molecules act on each other during an encounter. The quantity h which occurs in this expression may be determined by finding the mean value of ξ^2, which is $\frac{1}{2h}$. Now in the kinetic theory of gases,

$$\rho\xi^2=p=R\rho\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where p is the pressure, ρ the density, θ the absolute temperature, and R a constant for a given gas. Hence

$$\frac{1}{2h}=R\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

We shall suppose, however, with BOLTZMANN, that in a medium in which there are inequalities of temperature and of velocity

$$dN=N(1+F(\xi,\eta,\zeta))f_0(\xi,\eta,\zeta)d\xi d\eta d\zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

where F is a rational function of ξ, η, ζ, which we shall suppose not to contain terms of more than three dimensions, and f_0 is the same function as in equation (9).

Now consider two groups of molecules, each defined by the velocity-components, and let the two groups be distinguished by the suffixes $(_1)$ and $(_2)$. We have to estimate the number of encounters of a given kind between these two groups in a unit of volume in the time δt, those encounters only being considered for which the limits of b and ϕ are $b\pm\frac{1}{2}db$ and $\phi\pm\frac{1}{2}d\phi$.

Let us first suppose that both groups consist of mere geometrical points which do not interfere with each other's motion. The group dN_1 is moving through the group dN_2 with the relative velocity V, and we have to find how many molecules of the first group approach a molecule of the second group in a manner which would, if the molecules acted on each other, produce an encounter of the given kind. This will be the case for every molecule of the first group which passes through the area $bdbd\phi$ in the time δt. The number of such molecules is $dN_1 Vbdbd\phi\delta t$ for every molecule of the second group, so that the whole number of pairs which pass each other within the given limits is

$$Vbdbd\phi dN_1 dN_2 \delta t,$$

and if we take the time δt small enough, this will be the number of encounters of the real molecules in the time δt.

(3.) *Effect of the Encounters.*

We have next to estimate the effect of these encounters on the average values of different functions of the velocity-components. The effect of an individual encounter on these functions for the pair of molecules concerned is given in equations (3), (4), (5), (6), (7), each of which is of the form

$$\delta P=Q\sin^2\theta\cos^2\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

where P and Q are functions of the velocity-components of the two molecules, and if we write $\overline{P}$ for the average value of P for the N molecules in unit of volume, then taking the sum of the effects of the encounters—

$$\Sigma\delta P=N\delta\overline{P} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

We thus find

$$\frac{\delta P}{\delta t}=N\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int Q\sin^2\theta\cos^2\theta Vbdbd\phi f_1 f_2 d\xi_1 d\eta_1 d\zeta_1 d\xi_2 d\eta_2 d\zeta_2 \quad . \quad . \quad . \qquad (15)$$

Now, since θ is a function of b and V, the definite integral

$$V\int_0^{2\pi}\int_0^{\infty} b\sin^2\theta\cos^2\theta dbd\phi=B \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

will be a function of V only.

If the molecules are "rigid-elastic" spheres of diameter s,

$$B=\tfrac{1}{6}\pi s^2 V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If they repel each other with a force inversely as the fifth power of the distance, so that at a distance r the force is κr^{-5}, then

$$B=\left(\frac{2\kappa}{M}\right)^{\frac{1}{2}} A_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where A_2 is the numerical quantity 1·3682. In this case B is independent of V.

The experiments of O. E. Meyer,[*] Kundt and Warburg,[†] Puluj,[‡] Von Obermayer,[§] Eilhard Wiedemann,[||] and Holman,[¶] show that the viscosity of air varies according to a lower power of the absolute temperature than the first, probably the 0·77 power. If the viscosity had varied as the first power of the absolute temperature, B would have been independent of V. Though this is not the case, we shall assume, for the sake of being able to effect the integrations, that B is independent of V.

We shall find it convenient to write for B,

$$B=\frac{p}{3N\mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where p is the hydrostatic pressure, N the number of molecules in unit of volume, and μ a new coefficient which we shall afterwards find to be the coefficient of viscosity.

Equation (15) may now be written

$$\frac{\delta P}{\delta t}=\frac{p}{3\mu}\iiiint\!\!\iint Q f_1 f_2 d\xi_1 d\eta_1 d\zeta_1 d\xi_2 d\eta_2 d\zeta_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where the integrations are all between the limits $-\infty$ and $+\infty$, and f_1 and f_2 are of the form

$$f=(1+F(\xi,\eta,\zeta))h^{\frac{3}{2}}\pi^{-\frac{3}{2}}e^{-h(\xi^2+\eta^2+\zeta^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

F (ξ, η, ζ) being small compared with unity.

We may write F in the form

$$\begin{aligned}F=&(2h)^{\frac{1}{2}}(\alpha\xi+\beta\eta+\gamma\zeta)+2h(\tfrac{1}{2}\alpha^2\xi^2+\tfrac{1}{2}\beta^2\eta^2+\tfrac{1}{2}\gamma^2\zeta^2+\beta\gamma\eta\zeta+\gamma\alpha\zeta\xi+\alpha\beta\xi\eta)\\&+(2h)^{\frac{3}{2}}(\tfrac{1}{6}\alpha^3\xi^3+\tfrac{1}{6}\beta^3\eta^3+\tfrac{1}{6}\gamma^3\zeta^3+\tfrac{1}{2}\alpha^2\beta\xi^2\eta+\tfrac{1}{2}\alpha^2\gamma\xi^2\zeta+\tfrac{1}{2}\beta^2\gamma\eta^2\zeta\\&+\tfrac{1}{2}\beta^2\alpha\eta^2\zeta+\tfrac{1}{2}\gamma^2\alpha\zeta^2\xi+\tfrac{1}{2}\gamma^2\beta\zeta^2\eta+\alpha\beta\gamma\xi\eta\zeta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)\end{aligned}$$

where each combination of the symbols $\alpha\beta\gamma$ is to be taken as a single independent symbol, and not as a product of the component symbols.

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[*] Pogg. Ann., 1873, Bd. 148, p. 222.

[†] Pogg. Ann., 1876, Bd. 159, p. 403.

[‡] Wiener Sitz., 1874 and 1876.

[§] Wiener Sitz., 1875.

[||] Arch. des Sci. Phys. et Nat., 1876, t. 56, p. 273.

[¶] American Academy of Arts and Sciences, June 14, 1876. Phil. Mag., s. 5, vol. 3, No. 16, Feb., 1877.

(4.) *Mean Values of Combinations of* ξ, η, ζ.

To find the mean value of any function of ξ, η, ζ for all the molecules in the element, we must multiply this function by f, and integrate with respect to ξ, η, and ζ.

If the non-exponential factor of any term contains an odd power of any of the variables, the corresponding part of the integral will vanish, but if it contains only even powers, each even power, such as $2n$, will introduce a factor

$$\mathrm{R}^n\theta^n(2n-1)(2n-3)\;.\;.\;3\cdot1$$

into the corresponding part of the integral.

First, let the function be 1, then

$$1=\iiint f\,d\,\xi\,d\,\eta\,d\,\zeta \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (23)$$

or

$$1=1+\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (24)$$

which gives the condition

$$\alpha^2+\beta^2+\gamma^2=0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (25)$$

Let us next find the mean value of ξ in the same way, denoting the result by the symbol $\bar{\xi}$,

$$\bar{\xi}=(\mathrm{R}\theta)^{\frac{1}{2}}[\alpha+\tfrac{1}{2}(\alpha^3+\alpha\beta^2+\alpha\gamma^2)] \; . \; . \; . \; . \; . \; . \; . \; . \quad (26)$$

Since in what follows we shall denote the velocity-components of each molecule by $u+\xi$, $v+\eta$, $w+\zeta$, where u, v, w are the velocity-components of the centre of mass of all the molecules within the element, it follows that the mean values of ξ, η, ζ are each of them zero. We thus obtain the equations

$$\left.\begin{aligned}\alpha+\tfrac{1}{2}(\alpha^3+\alpha\beta^2+\alpha\gamma^2)=0\\ \beta+\tfrac{1}{2}(\alpha^2\beta+\beta^3+\beta\gamma)=0\\ \gamma+\tfrac{1}{2}(\alpha^2\gamma+\beta^2\gamma+\gamma^3)=0\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (27)$$

Remembering these conditions, we find that the mean values of combinations of two, three, and four dimensions are of the forms

$$\left.\begin{aligned}\overline{\xi^2}\;&=\mathrm{R}\theta(1+\alpha^2)\\ \overline{\xi\eta}\;&=\mathrm{R}\theta\alpha\beta\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (28)$$

$$\left.\begin{aligned}\overline{\xi^3}\;&=(\mathrm{R}\theta)^{\frac{3}{2}}\alpha^3\\ \overline{\xi\eta^2}&=(\mathrm{R}\theta)^{\frac{3}{2}}\alpha\beta^2\\ \overline{\xi\eta\zeta}&=(\mathrm{R}\theta)^{\frac{3}{2}}\alpha\beta\gamma\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (29)$$

$$\left.\begin{aligned}\overline{\xi^4} &= 3\mathrm{R}^2\theta^2(1+2\alpha^2)\\ \overline{\xi^3\eta} &= 3\mathrm{R}^2\theta^2\alpha\beta\\ \overline{\xi^2\eta^2} &= \mathrm{R}^2\theta^2(1+\alpha^2+\beta^2)\\ \overline{\xi^2\eta\zeta} &= \mathrm{R}^2\theta^2\beta\gamma\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

(5.) *Rates of Decay of these Mean Values.*

If any term of Q in equation (20) contains symbols belonging to one group alone of the molecules, the corresponding term of the integral may be found from the above table, but if it contains symbols belonging to both groups we must consider the sextuple integral (20). But we shall not find it necessary to do this for terms of not more than three dimensions, for in these, if both groups of symbols occur, the index of one of them must be odd, and the integral vanishes.

We thus find from equations (3), (4), (5), (6), and (7)

$$\frac{\delta}{\delta t}\alpha^2 = -\frac{p}{\mu}\alpha^2 \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$\frac{\delta}{\delta t}\alpha\beta = -\frac{p}{\mu}\alpha\beta . \quad . \quad . \quad . \quad . \quad (32)$$

$$\frac{\delta}{\delta t}\alpha^3 = \frac{1}{2}\frac{p}{\mu}(-2\alpha^3+\alpha\beta^2+\alpha\gamma^2) \quad . \quad . \quad . \quad . \quad (33)$$

$$\frac{\delta}{\delta t}\alpha\beta^2 = \frac{1}{6}\frac{p}{\mu}(\alpha^3-8\alpha\beta^2+\alpha\gamma^2) \quad . \quad . \quad . \quad . \quad (34)$$

$$\frac{\delta}{\delta t}\alpha\beta\gamma = -\frac{3}{2}\frac{p}{\mu}\alpha\beta\gamma \quad . \quad . \quad . \quad . \quad . \quad (35)$$

[Any rational homogeneous function of $\xi\ \eta\ \zeta$ is either a solid harmonic, or a solid harmonic multiplied by a positive integral power of $(\xi^2+\eta^2+\zeta^2)$, or may be expressed as the sum of a number of terms of these forms.

If we express any one of these terms as a function of u, v, w, V and the angular coordinates of V, we can determine the rate of change of each of the spherical harmonics of the angular coordinates.

If we then transform the expression back to its original form as a function of ξ_1, η_1, ζ_1, ξ_2, η_2, ζ_2, and if we add the corresponding functions for both molecules, we shall obtain an expression for the rate of change of the original function.

Thus among the terms of two dimensions we have the five conjugate solid harmonics

$$\frac{1}{3}(2\xi^2-\eta^2-\zeta^2),$$

$$\xi\eta, \qquad \xi\zeta,$$

$$\eta^2-\zeta^2, \quad \eta\zeta.$$

The rate of increase of each of these arising from the encounters of the molecules is found by multiplying it by $-\frac{p}{\mu}$. We may therefore call $\frac{p}{\mu}$ the "modulus of the time of relaxation" of this class of functions.

The function $\xi^2+\eta^2+\zeta^2$ is not changed by the encounters.

Homogeneous functions of three dimensions are either solid harmonics of the third order or solid harmonics of the first order multiplied by $\xi^2+\eta^2+\zeta^2$, or combinations of these.

The time modulus for solid harmonics of the third order is $\frac{3}{2}\frac{p}{\mu}$.—Note added May, 1879.]

That of ξ, η, or ζ, multiplied by $\xi^2+\eta^2+\zeta^2$ is $\frac{2}{3}\frac{p}{\mu}$.

(6.) *Effect of External Forces.*

The only effect of external forces is expressed by equations of the form

$$\frac{\delta u}{\delta t}=X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

The average values of ξ, η, ζ and their combinations are not affected by external forces.

(7.) *Variation of Mean Values within an Element of Volume.*

We have employed the symbol δ to denote the variation of any quantity within an element, arising either from encounters between molecules or from the action of external forces.

There is a third way, however, in which a variation may occur, namely, by molecules entering the element or leaving it, carrying their properties with them.

We shall use the symbol ∂ to denote the actual variation within a specified element.

If MQ is the average value of any quantity for each molecule within the element, then the quantity in unit of volume is ρQ. We have to trace the variation of ρQ.

We begin with an element of volume moving with the velocity-components U, V, W, then by the ordinary investigation of the "equation of continuity"

$$\frac{\partial}{\delta t}[Q\rho]+\frac{d}{dx}[Q(u+\xi-U)]+\frac{d}{dy}[Q(v+\eta-V)]+\frac{d}{dz}[Q(w+\zeta-W)]=\rho\frac{d}{\delta t}Q \quad . \quad . \quad (37)$$

If after performing the differentiations we make $U=u$, $V=v$, $W=w$, the equation becomes for an element moving with the velocity (u, v, w)

$$\frac{\delta}{\delta t}(Q\rho)+\rho Q\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+\frac{d}{dx}(\rho Q\xi)+\frac{d}{dy}(\rho Q\eta)+\frac{d}{dz}(\rho Q\zeta)=\rho\frac{\delta}{\delta t}Q \quad . \quad . \quad . \quad (38)$$

(8.) *Equation of Density.*

Let us first make Q=1, then, since the mass of a molecule is invariable, the equation becomes

$$\frac{\delta\rho}{\delta t}+\rho\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

which is the ordinary "equation of continuity."

Eliminating by means of this equation the second term of the general equation (38) we obtain the more convenient form—

$$\rho\frac{\delta Q}{\delta t}+\frac{d}{dx}(\rho Q\xi)+\frac{d}{dy}(\rho Q\eta)+\frac{d}{dz}(\rho Q\zeta)=\rho\frac{\delta Q}{\delta t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

(9.) *Equations of Motion.*

Putting $Q=u+\xi$, this equation becomes

$$\rho\frac{\delta u}{\delta t}+\frac{d}{dx}(\rho\xi^2)+\frac{d}{dy}(\rho\xi\eta)+\frac{d}{dz}(\rho\xi\zeta)=\rho X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

where any combination of the symbols ξ, η, ζ is to be taken as the average value of that combination.

Substituting their values as given in (28)

$$\rho\frac{\delta u}{\delta t}+R\frac{d}{dx}(\rho\theta)+R\left[\frac{d}{dx}(\rho\theta\alpha^2)+\frac{d}{dy}(\rho\theta\alpha\beta)+\frac{d}{dz}(\rho\theta\alpha\gamma)\right]=\rho X \quad . \quad . \quad . \quad (42)$$

which is one of the three ordinary equations of motion of a medium in which stresses exist.

(10.) *Terms of Two Dimensions.*

Put $Q=(u+\xi)^2$. Since the resulting equation is true whatever be the values of u, v, w, we may, after differentiation, put each of these quantities equal to zero. We shall thus obtain the same result which we might have obtained by elimination between this and the former equations. We find

$$\rho\frac{\delta}{\delta t}\xi^2+2\rho\xi^2\frac{du}{dx}+2\rho\xi\eta\frac{du}{dy}+2\rho\xi\zeta\frac{du}{dz}+\frac{d}{dx}(\rho\xi^3)+\frac{d}{dy}(\rho\xi^2\eta)+\frac{d}{dz}(\rho\xi^2\zeta)=\rho\frac{\delta}{\delta t}\xi^2 \quad . \quad . \quad (43)$$

or by substituting the mean values of these quantities from (29)

$$\rho\frac{\delta\theta}{\delta t}+\rho\frac{\delta}{\delta t}(\theta\alpha^2)+2\rho\theta\frac{du}{dx}+2\rho\theta\left(\alpha^2\frac{du}{dx}+\alpha\beta\frac{du}{dy}+\alpha\gamma\frac{du}{dz}\right)$$
$$+\mathrm{R}^{\frac{1}{2}}\left[\frac{d}{dx}(\rho\theta^{\frac{3}{2}}\alpha^3)+\frac{d}{dy}(\rho\theta^{\frac{3}{2}}\alpha^2\beta)+\frac{d}{dz}(\rho\theta^{\frac{3}{2}}\alpha^2\gamma)\right]=-\frac{\mathrm{R}\rho^2\theta^2}{\mu}\alpha^2 \quad . \quad . \quad . \quad (44)$$

with two other equations of similar form.

Similarly we obtain by putting $\mathrm{Q}=(u+\xi)(v+\eta)$

$$\rho\frac{\delta}{\delta t}(\theta\alpha\beta)+\rho\theta\left(\frac{dv}{dx}+\frac{du}{dy}\right)$$
$$+\rho\theta\left(\alpha^2\frac{dv}{dx}+\alpha\beta\frac{dv}{dy}+\alpha\gamma\frac{dv}{dz}+\alpha\beta\frac{du}{dx}+\beta^2\frac{du}{dy}+\beta\gamma\frac{du}{dz}\right)$$
$$+\mathrm{R}^{\frac{1}{2}}\left[\frac{d}{dx}(\rho\theta^{\frac{3}{2}}\alpha^2\beta)+\frac{d}{dy}(\rho\theta^{\frac{3}{2}}\alpha\beta^2)+\frac{d}{dz}(\rho\theta^{\frac{3}{2}}\alpha\beta\gamma)\right]=-\frac{\mathrm{R}\rho^2\theta^2}{\mu}\alpha\beta \quad . \quad . \quad . \quad . \quad (45)$$

with two other equations of like form for $\beta\gamma$ and $\gamma\alpha$.

(11.) *Terms of Three Dimensions.*

Putting $\mathrm{Q}=(u+\xi)^3$ and in the final equation making $u=v=w=0$ and eliminating $\frac{\delta u}{\delta t}$ by (41) we find

$$\rho\frac{\delta}{\delta t}\xi^3+3\rho\xi^3\frac{du}{dx}+3\rho\xi^2\eta\frac{du}{dy}+3\rho\xi^2\zeta\frac{du}{dz}$$
$$+\frac{d}{dx}(\rho\xi^4)+\frac{d}{dy}(\rho\xi^3\eta)+\frac{d}{dz}(\rho\xi^3\zeta)$$
$$-3\xi^2\left[\frac{d}{dx}(\rho\xi^2)+\frac{d}{dy}(\rho\xi\eta)+\frac{d}{dz}(\rho\xi\zeta)\right]=\rho\frac{\delta}{\delta t}\xi^3 \quad . \quad . \quad . \quad . \quad . \quad (46)$$

which gives

$$\mathrm{R}^{\frac{3}{2}}\rho\frac{\delta}{\delta t}(\theta^{\frac{3}{2}}\alpha^2)+3\rho(\mathrm{R}\theta)^{\frac{3}{2}}\left(\alpha^3\frac{du}{dx}+\alpha^2\beta\frac{du}{dy}+\alpha^2\gamma\frac{du}{dz}\right)$$
$$+3\mathrm{R}^2\rho\theta\frac{d\theta}{dx}+3\mathrm{R}^2\rho\theta\left(\alpha^2\frac{d\theta}{dx}+\alpha\beta\frac{d\theta}{dy}+\alpha\gamma\frac{d\theta}{dz}\right)+3\mathrm{R}^2\rho\theta\frac{d}{dx}(\alpha^2\theta)$$
$$-3\mathrm{R}^2\theta\alpha^2\left[\frac{d}{dx}(\rho\theta\alpha^2)+\frac{d}{dy}(\rho\theta\alpha\beta)+\frac{d}{dz}(\rho\theta\alpha\gamma)\right]=\rho(\mathrm{R}\theta)^{\frac{3}{2}}\frac{1}{2}\frac{p}{\mu}(-2\alpha^3+\alpha\beta^2+\alpha\gamma^2) \quad . \quad (47)$$

Since the combinations of $\alpha\beta\gamma$ represent small numerical quantities, we may at this stage of the calculation, when we are dealing with terms of the third order, neglect terms involving them, except when they are multiplied by the large coefficient p/μ. The equation may then be written approximately :—

$$3\mathrm{R}^2\rho\theta\frac{d\theta}{dx}=\rho(\mathrm{R}\theta)^{\frac{3}{2}}\frac{1}{2}\frac{p}{\mu}(-2\alpha^3+\alpha\beta^2+\alpha\gamma^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

Similarly, by putting $\mathrm{Q}=(u+\xi)\ (v+\eta)^2$, we obtain the approximate equation

$$\mathrm{R}^2\rho\theta\frac{d\theta}{dx}=\rho(\mathrm{R}\theta)^{\frac{3}{2}}\frac{1}{6}\frac{p}{\mu}(\alpha^3-8\alpha\beta^2+\alpha\gamma^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

and in the same way we find

$$\mathrm{R}^3\rho\theta\frac{d\theta}{dx}=\rho(\mathrm{R}\theta)^{\frac{3}{2}}\frac{1}{6}\frac{p}{\mu}(\alpha^3+\alpha\beta^2-8\alpha\gamma^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

(12.) *Approximate Values of Terms of Three Dimensions.*

From equations (48), (49), and (50), we find

$$\left.\begin{aligned}
&\alpha^3=-\frac{9}{2}\frac{\mu}{p}\left(\frac{\mathrm{R}}{\theta}\right)^{\frac{1}{2}}\frac{d\theta}{dx}, \quad \alpha\beta^2=\alpha\gamma^2=-\frac{3}{2}\frac{\mu}{p}\left(\frac{\mathrm{R}}{\theta}\right)^{\frac{1}{2}}\frac{d\theta}{dx}\\
&\text{From which by substitution we obtain}\\
&\beta^3=-\frac{9}{2}\frac{\mu}{p}\left(\frac{\mathrm{R}}{\theta}\right)^{\frac{1}{2}}\frac{d\theta}{dy}, \quad \alpha^2\beta=\beta\gamma^2=-\frac{3}{2}\frac{\mu}{p}\left(\frac{\mathrm{R}}{\theta}\right)^{\frac{1}{2}}\frac{d\theta}{dy}\\
&\gamma^3=-\frac{9}{2}\frac{\mu}{p}\left(\frac{\mathrm{R}}{\theta}\right)^{\frac{1}{2}}\frac{d\theta}{dz}, \quad \alpha^2\gamma=\beta^2\gamma=-\frac{3}{2}\frac{\mu}{p}\left(\frac{\mathrm{R}}{\theta}\right)^{\frac{1}{2}}\frac{d\theta}{dz}
\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

The value of $\alpha\beta\gamma$ is of a smaller order of magnitude, and we do not require it in this investigation.

(13.) *Equation of Temperature.*

Adding the three equations of the form (44), and omitting terms containing small quantities of two dimensions, and also products of differential coefficients such as $\frac{d\mu}{dx}\frac{d\theta}{dx}$, we find

$$\frac{\delta\theta}{\delta t}=\frac{5}{2}\frac{\mu}{\rho}\left(\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}+\frac{d^2\theta}{dz^2}\right)+\frac{2}{3}\frac{\theta}{\rho}\frac{\delta\rho}{\delta t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

The first term of the second member represents the rate of increase of temperature due to conduction of heat, as in FOURIER's Theory, and the second term represents the increase of temperature due to increase of density. We must remember that the gas here considered is one for which the ratio of the specific heats is 1·66.

(14.) *Stresses in the Gas.*

Subtracting one-third of the sum of the three equations from (44), we obtain

$$p\alpha^2=-2\mu\frac{du}{dx}+\frac{2}{3}\mu\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+3\frac{\mu^2}{\rho\theta}\frac{d^2\theta}{dx^2}+\frac{3}{2}\frac{\mu^2}{\rho\theta}\left(\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}+\frac{d^2\theta}{dz^2}\right) \quad . \quad . \quad (53)$$

This equation gives the excess of the normal pressure in x above the mean hydrostatic pressure p. The first two terms of the second member represent the effect of viscosity in a moving fluid, and are identical with those given by Professor STOKES (Cambridge Transactions, vol. viii., 1845, p. 297). The last two terms represent the part of the stress which arises from inequality of temperature, which is the special subject of this paper.

There are two other equations of similar form for the normal stresses in y and z.

The tangential stress in the plane xy is given by the equation

$$p\alpha\beta=-\mu\left(\frac{du}{dy}+\frac{dv}{dx}\right)+3\frac{\mu^2}{\rho\theta}\frac{d^2\theta}{dxdy} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

There are two other equations of similar form for the tangential stresses in the planes of yz and zx.

(15.) *Final Equations of Motion.*

We are now prepared to complete the equations of motion by inserting in (42) the values of the quantities α^2, $\alpha\beta$, $\alpha\gamma$, and we find for the equation in x

$$\rho\frac{\delta u}{\delta t}+\frac{dp}{dx}-\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)+\frac{1}{3}\mu\frac{d}{dx}\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)$$

$$+\frac{9}{2}\frac{\mu^2}{\rho\theta}\frac{d}{dx}\left(\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}+\frac{d^2\theta}{dz^2}\right)=\rho X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

If we write

$$p'=p+\frac{1}{3}\mu\left(\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\right)+\frac{9}{2}\frac{\mu^2}{\rho\theta}\left(\frac{d^2\theta}{dx^2}+\frac{d^2\theta}{dy^2}+\frac{d^2\theta}{dz^2}\right) \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$=p+\frac{9}{5}\frac{\mu}{\theta}\frac{\delta\theta}{\delta t}-\frac{23}{15}\frac{\mu}{\rho}\frac{\delta\rho}{\delta t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

or, if the pressure p is constant, so that $\rho\delta\theta+\theta\delta\rho=0$

$$p'=p+\frac{10}{3}\frac{\mu}{\theta}\frac{\delta\theta}{\delta t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

then the equation (55) may be written

$$\rho\frac{\delta u}{\delta t}+\frac{dp'}{dx}-\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)=\rho X \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

If there are no external forces such as gravity, then one solution of the equations is

$$u=v=w=0,\ p'=\text{constant},$$

and if the boundary conditions are such that this solution is consistent with them, it will become the actual solution as soon as the initial motions, if any exist, have subsided. This will be the case if no slipping is possible between the gas and solid bodies in contact with it.

But if such slipping is possible, then wherever in the above solution there is a tangential stress in the gas at the surface of a solid or liquid, there cannot be equilibrium, but the gas will begin to slide over the surface till the velocity of sliding has produced a frictional resistance equal and opposite to the tangential stress. When this is the case the motion may become steady. I have not, however, attempted to enter into the calculation of the state of steady motion.

[I have recently applied the method of spherical harmonics, as described in the notes to sections (1) and (5), to carrying the approximations two orders higher. I expected that this would have involved the calculation of two new quantities, namely, the rates of decay of spherical harmonics of the fourth and sixth orders, but I found that, to the order of approximation required, all harmonics of the fourth and sixth orders may be neglected, so that the rate of decay of harmonics of the second order, the time-modulus of which is $\mu\div p$, determines the rate of decay of all functions of less than 6 dimensions.

The equations of motion, as here given (equation 55) contain the second derivatives of u, v, w, with respect to the coordinates, with the coefficient μ. I find that in the more approximate expression there is a term containing the fourth derivatives of u, v, w, with the coefficient $\mu^3\div\rho p$.

The equations of motion also contain the third derivatives of θ with the coefficient $\mu^2\div\rho\theta$. Besides these terms, there is another set consisting of the fifth derivatives of θ with the coefficient $\mu^4\div\rho^2p\theta$.

It appears from the investigation that the condition of the successful use of this method of approximation is that $l\frac{d}{dh}$ should be small, where $\frac{d}{dh}$ denotes differentiation with respect to a line drawn in any direction. In other words, the properties of the medium must not be sensibly different at points within a distance of each other, comparable with the "mean free path" of a molecule.—Note added June, 1879.]

Appendix.

(Added May, 1879.)

In the paper as sent in to the Royal Society, I made no attempt to express the conditions which must be satisfied by a gas in contact with a solid body, for I thought it very unlikely that any equations I could write down would be a satisfactory representation of the actual conditions, especially as it is almost certain that the stratum of gas nearest to a solid body is in a very different condition from the rest of the gas.

One of the referees, however, pointed out that it was desirable to make the attempt, and indicated several hypothetical forms of surfaces which might be tried. I have therefore added the following calculations, which are carried to the same degree of approximation as those for the interior of the gas.

It will be seen that the equations I have arrived at express both the fact that the gas may slide over the surface with a finite velocity, the previous investigations of which have been already mentioned;* and the fact that this velocity and the corresponding tangential stress are affected by inequalities of temperature at the surface of the solid, which give rise to a force tending to make the gas slide along the surface from colder to hotter places.

This phenomenon, to which Professor OSBORNE REYNOLDS has given the name of Thermal Transpiration, was discovered entirely by him. He was the first to point out that a phenomenon of this kind was a necessary consequence of the Kinetic Theory of Gases, and he also subjected certain actual phenomena, of a somewhat different kind, indeed, to measurement, and reduced his measurements by a method admirably adapted to throw light on the relations between gases and solids.

It was not till after I had read Professor REYNOLDS' paper that I began to reconsider the surface conditions of a gas, so that what I have done is simply to extend to the surface phenomena the method which I think most suitable for treating the interior of the gas. I think that this method is, in some respects, better than that adopted by Professor REYNOLDS, while I admit that his method is sufficient to establish the existence of the phenomena, though not to afford an estimate of their amount.

The method which I have adopted throughout is a purely statistical one. It considers the mean values of certain functions of the velocities within a given element of the medium, but it never attempts to trace the motion of a molecule, not even so far as to estimate the length of its mean path. Hence all the equations are expressed in the forms of the differential calculus, in which the phenomena at a given place are connected with the space variations of certain quantities at that place, but in which no quantity appears which explicitly involves the condition of things at a finite distance from that place.

The particular functions of the velocities which are here considered are those of one, two, and three dimensions. These are sufficient to determine approximately the principal phenomena in a gas which is not very highly rarified, and in which the space-variations within distances comparable to λ are not very great.

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* Sect. 12 of introduction.

The same method, however, can be extended to functions of higher degrees, and by a sufficient number of such functions any distribution of velocities, however abnormal, may be expressed. The labour of such an approximation is considerably diminished by the use of the method of spherical harmonics as indicated in the note to Section I. of the paper.

On the Conditions to be Satisfied by a Gas at the Surface of a Solid Body.

As a first hypothesis, let us suppose the surface of the body to be a perfectly elastic smooth fixed surface, having the apparent shape of the solid, without any minute asperities.

In this case, every molecule which strikes the surface will have the normal component of its velocity reversed, while the other components will not be altered by impact.

The rebounding molecules will therefore move as if they had come from an imaginary portion of gas occupying the space really filled by the solid, and such that the motion of every molecule close to the surface is the optical reflection in that surface of the motion of a molecule of the real gas.

In this case we may speak of the rebounding molecules close to the surface as constituting the *reflected* gas. All directed properties of the incident gas are reflected, or, as Professor LISTING might say, *perverted* in the reflected gas; that is to say, the properties of the incident and the reflected gas are symmetrical with respect to the tangent plane of the surface.

The incident and reflected gas together constitute the actual gas close to the surface. The actual gas, therefore, cannot exert any stress on the surface, except in the direction of the normal, for the oblique components of stress in the incident and reflected gas will destroy one another.

Since gases can actually exert oblique stress against real surfaces, such surfaces cannot be represented as perfectly reflecting surfaces.

If a molecule, whose velocity is given in direction and magnitude, but whose line of motion is not given in position, strikes a fixed elastic sphere, its velocity after rebound may with equal probability be in any direction.

Consider, therefore, a stratum in which fixed elastic spheres are placed so far apart from one another that any one sphere is not to any sensible extent protected by any other sphere from the impact of molecules, and let the stratum be so deep that no molecule can pass through it without striking one or more of the spheres, and let this stratum of fixed spheres be spread over the surface of the solid we have been considering, then every molecule which comes from the gas towards the surface must strike one or more of the spheres, after which all directions of its velocity become equally probable.

When, at last, it leaves the stratum of spheres and returns into the gas, its velocity must of course be *from* the surface, but the probability of any particular magnitude and direction of the velocity will be the same as in a gas at rest with respect to the surface.

The distribution of velocity among the molecules which are leaving the surface will therefore be the same as if, instead of the solid, there were a portion of gas at rest, having the temperature of the solid, and a density such that the number of molecules which pass from it through the surface in a given time is equal to the number of molecules of the real gas outside which strike the surface.

To distinguish the molecules, which, after being entangled in the stratum of spheres, afterwards return into the surrounding gas, we shall call them, collectively, the *absorbed and evaporated* gas.

If the spheres are so near together that a considerable part of the surface of each sphere of the outer layer is shielded from the direct impact of the incident molecules by the spheres which lie next to it, then if we call that point of each sphere which lies furthest from the solid the *pole* of the sphere, a greater proportion of molecules will strike any one of the outer layer of spheres near its pole than near its equator, and the greater the obliquity of incidence of the molecule, the greater will be the probability that it will strike a sphere near its pole.

The direction of the rebounding molecule will no longer be with equal probability in all directions, but there will be a greater probability of the tangential part of its velocity being in the direction of the motion before impact, and of its normal part being opposite to the normal part before impact.

The condition of the molecules which leave the surface will therefore be intermediate between that of evaporated gas and that of reflected gas, approaching most nearly to evaporated gas at normal incidence and most nearly to reflected gas at grazing incidence.

If the spheres, instead of being hard elastic bodies, are supposed to act on the molecules at finite, though small distances, and if they are so close together that their spheres of action intersect, then the gas which leaves the surface will be still more like reflected gas, and less like evaporated gas.

We might also consider a surface on which there are a great number of minute asperities of any given form, but since in this case there is considerable difficulty in calculating the effect when the direction of rebound from the first impact is such as to lead to a second or third impact, I have preferred to treat the surface as something intermediate between a perfectly reflecting and a perfectly absorbing surface, and, in particular, to suppose that of every unit of area a portion f absorbs all the incident molecules, and afterwards allows them to evaporate with velocities corresponding to those in still gas at the temperature of the solid, while a portion $1-f$ perfectly reflects all the molecules incident upon it.

We shall begin by supposing that the surface is the plane $y\,z$, and that the gas is on that side of it for which x is positive.

The incident molecules are those which, close to the surface, have their normal component of velocity negative. We shall distinguish these molecules by the suffix $(_1)$. For these, and these only, ξ_1 is negative.

The rebounding molecules are those which have ξ positive. We shall distinguish them by the suffix $(_2)$. Those which are evaporated will be further distinguished by an accent.

Symbols without any mark refer to the whole gas, incident, reflected, and evaporated, close to the surface.

The quantity of gas which is incident on unit of surface in unit of time, is $-\rho_1\xi_1$.

Of this quantity the fraction $1-f$ is reflected, so that the sign of ξ is reversed, and the fraction f is evaporated, the mean value of ξ in evaporated gas being ξ', where the accent distinguishes symbols belonging to unpolarized gas at rest relative to the surface, and having the temperature, θ', of the solid.

Equating the quantity of gas which is incident on the absorbing part of the surface to that which is evaporated from it, we have

$$f\rho_1\xi_1+f\rho_2'\xi_2'=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Equating the whole quantity of gas which leaves the surface to the reflected and evaporated portions

$$\rho_2\xi_2=(f-1)\,\rho_1\xi_1+f\rho_2'\xi_2' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

If we next consider the momentum of the molecules in the direction of y, that of the incident molecules is $\rho_1\xi_1\eta_1$. A fraction $(1-f)$ of this is reflected and becomes $(1-f)\rho_1\xi_1\eta_1$, and a fraction f of it is absorbed and then evaporated, the mean value of η being now $-v$, namely, the velocity of the surface relatively to the gas in contact with it.

The momentum of the evaporated portion in the direction of y is therefore $-f\rho_2'\xi_2'v$, and this, together with the reflected portion, makes up the whole momentum which is leaving the surface, or

$$\rho_2\xi_2\eta_2=(f-1)\,\rho_1\xi_1\eta_1-f\rho_2'\xi_2'v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Eliminating $f\rho_2'\xi_2'$ between equations (61) and (62)

$$(1-f)\rho_1\xi_1\eta_1+\rho_2\xi_2\eta_2+v\,[(1-f)\rho_1\xi_1+\rho_2\xi_2]=0 \quad . \quad . \quad . \quad . \quad (63)$$

The values of functions of ξ, η and ζ for the incident molecules are to be found by multiplying the expression in equation (22) by the given function, and integrating with respect to ξ between the limits $-\infty$ and 0, and with respect to η and ζ between the limits $\pm\infty$.

The values of the same functions for the molecules which are leaving the surface are to be found by integrating with respect to ξ from 0 to ∞.

We must remember, however, that since there is an essential discontinuity in the conditions of the gas at the surface, the expression in equation (22) is a much less accurate approximation to the actual distribution of velocities in the gas close to the surface than it is in the interior of the gas. We must, therefore, consider the surface conditions at which we arrive in this way as liable to important corrections when we shall have discovered more powerful methods of attacking the problem.

For the present, however, we consider only terms of three dimensions or less, and we find

$$\left.\begin{aligned}\rho_1\xi_1&=-\rho(2\pi)^{-\frac{1}{2}}(\mathrm{R}\theta)^{\frac{1}{2}}(1+\tfrac{1}{2}\alpha^2)\\ \rho_2\xi_2&=\rho(2\pi)^{-\frac{1}{2}}(\mathrm{R}\theta)^{\frac{1}{2}}(1+\tfrac{1}{2}\alpha^2)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

$$\left.\begin{aligned}\rho_1\xi_1\eta_1&=\tfrac{1}{2}\rho\mathrm{R}\theta\alpha\beta-\tfrac{1}{2}\rho(2\pi)^{-\frac{1}{2}}\mathrm{R}\theta\alpha^2\beta\\ \rho_2\xi_2\eta_2&=\tfrac{1}{2}\rho\mathrm{R}\theta\alpha\beta+\tfrac{1}{2}\rho(2\pi)^{-\frac{1}{2}}\mathrm{R}\theta\alpha^2\beta\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Substituting these expressions in equation (63), and neglecting α^2 in comparison with unity, we find

$$(2-f)\rho\mathrm{R}\theta\alpha\beta+f(2\pi)^{-\frac{1}{2}}\rho\mathrm{R}\theta\alpha^2\beta+2f(2\pi)^{-\frac{1}{2}}(1+\tfrac{1}{2}\alpha^2)(\mathrm{R}\theta)^{\frac{1}{2}}\rho v=0 \quad . \quad . \quad . \quad (66)$$

If we write

$$\mathrm{G}=\tfrac{1}{2}\mu(2\pi)^{\frac{1}{2}}(p\rho)^{-\frac{1}{2}}\left(\frac{2}{f}-1\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

and substitute for $\alpha\beta$ and $\alpha^2\beta$ their values as given in equations (54) and (51), and divide by $2(p\rho)^{\frac{1}{2}}$, equation (66) becomes

$$v-\mathrm{G}\left(\frac{dv}{dx}-\frac{3}{2}\frac{\mu}{\rho\theta}\frac{d^2\theta}{dxdy}\right)-\frac{3}{4}\frac{\mu}{\rho\theta}\frac{d\theta}{dy}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

If there is no inequality of temperature, this equation is reduced to

$$v=\mathrm{G}\frac{dv}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

If, therefore, the gas at a finite distance from the surface is moving parallel to the surface, the gas in contact with the surface will be sliding over it with the finite velocity v, and the motion of the gas will be very nearly the same as if the stratum of depth G had been removed from the solid and filled with the gas, there being now no slipping between the new surface of the solid and the gas in contact with it.

The coefficient G was introduced by HELMHOLTZ and PIOTROWSKI under the name of *Gleitungs-coefficient*, or coefficient of slipping. The dimensions of G are those of a line, and its ratio to l, the mean free path of a molecule, is given by the equation

$$\mathrm{G}=\frac{2}{3}\left(\frac{2}{f}-1\right)l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

KUNDT and WARBURG found that for air in contact with glass, $\mathrm{G}=2l$, whence we find $f=\frac{1}{2}$, or the surface acts as if it were half perfectly reflecting and half perfectly absorbent. If it were wholly absorbent, $\mathrm{G}=\frac{2}{3}l$.

It is easy to write down the surface conditions for a surface of any form. Let the direction-cosines of the normal ν be l, m, n, and let us write

$$\frac{d}{d\nu} \text{ for } l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz}.$$

We then find as the surface conditions

$$\left.\begin{aligned} u-\mathrm{G}\frac{d}{d\nu}[(1-l^2)\,u-\;lmv-lnw]+\frac{3}{4}\frac{\mu}{\rho\theta}\left(\frac{d}{dx}-\;l\frac{d}{d\nu}\right)\left(\theta+4\mathrm{G}\frac{d\theta}{d\nu}\right)=0 \\ v-\mathrm{G}\frac{d}{d\nu}[(1-m^2)v-mnw-mln]+\frac{3}{4}\frac{\mu}{\rho\theta}\left(\frac{d}{dy}-m\frac{d}{d\nu}\right)\left(\theta+4\mathrm{G}\frac{d\theta}{d\nu}\right)=0 \\ w-\mathrm{G}\frac{d}{d\nu}[(1-n^2)\,w-\;nlu-nmv]+\frac{3}{4}\frac{\mu}{\rho\theta}\left(\frac{d}{dz}-n\frac{d}{d\nu}\right)\left(\theta+4\mathrm{G}\frac{d\theta}{d\nu}\right)=0 \end{aligned}\right\} \quad . \quad (71)$$

In each of these equations the first term is one of the velocity-components of the gas in contact with the surface, which is supposed fixed; the second term depends on the slipping of the gas over the surface, and the third term indicates the effect of inequalities of temperature of the gas close to the surface, and shows that in general there will be a force urging the gas from colder to hotter parts of the surface.

Let us take as an illustration the case of a capillary tube of circular section, and for the sake of easy calculation we shall suppose that the motion is so slow, and the temperature varies so gradually along the tube that we may suppose the temperature uniform throughout any one section of the tube.

Taking the axis of the tube for that of z, we have for the condition of steady motion parallel to the axis

$$\frac{dp}{dz}-\mu\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (72)$$

Since everything is symmetrical about the axis, if we write r^2 for x^2+y^2 we find as the solution of this equation

$$w=\mathrm{A}+\frac{1}{4\mu}\frac{dp}{dz}r^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (73)$$

If Q denotes the quantity of gas which passes through a section of the tube in unit of time

$$\begin{aligned} \mathrm{Q}&=2\pi\int\rho wrdr \\ &=\pi\rho a^2\left(\mathrm{A}+\frac{1}{8\mu}\frac{dp}{dz}a^2\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (74) \end{aligned}$$

At the inner surface of the tube we have $r=a$, and

$$\begin{aligned} w&=\mathrm{A}+\frac{1}{4\mu}\frac{dp}{dz}a^2 \\ &=\frac{\mathrm{Q}}{\pi\rho a^2}+\frac{1}{8\mu}\frac{dp}{dz}a^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (75) \end{aligned}$$

also

$$\frac{dw}{d\nu}=-\frac{1}{2\mu}\frac{dp}{dz}a \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (76)$$

The last of equations (71) may therefore be written

$$\frac{\mathrm{Q}}{\pi\rho a^2}+\frac{1}{8\mu}(a^2+4\mathrm{G}a)\frac{dp}{dz}-\frac{3}{4}\frac{\mu}{\rho\theta}\frac{d\theta}{dz}=0 \quad . \; . \; . \; . \; . \; . \; . \quad (77)$$

Equation (77) gives the relation between the quantity of gas which passes through any section of the tube, the rate of variation of pressure, and the rate of variation of temperature in passing along the axis of the tube.

If the pressure is uniform there will be a flow of gas from the colder to the hotter end of the tube, and if there is no flow of gas the pressure will increase from the colder to the hotter end of the tube.

These effects of the variation of temperature in a tube have been pointed out by Professor OSBORNE REYNOLDS as a result of the Kinetic Theory of Gases, and have received from him the name of Thermal Transpiration : a name in strict analogy with the use of the word Transpiration by GRAHAM.

But the phenomenon actually observed by Professor REYNOLDS in his experiments was the passage of gas through a porous plate, not through a capillary tube ; and the passage of gases through porous plates, as was shown by GRAHAM, is of an entirely different kind from the passage of gases through capillary tubes, and is more nearly analogous to the flow of a gas through a small hole in a thin plate.

When the diameter of the hole and the thickness of the plate are both small compared with the length of the free path of a molecule, then, as Sir WILLIAM THOMSON has shown, any molecule which comes up to the hole on either side will be in very little danger of encountering another molecule before it has got fairly through to the other side.

Hence the flow of gas in either direction through the hole will take place very nearly in the same manner as if there had been a vacuum on the other side of the hole, and this whether the gas on the other side of the hole is of the same or of a different kind.

If the gas on the two sides of the plate is of the same kind but at different temperatures, a phenomenon will take place which we may call *thermal effusion.*

The velocity of the molecules is proportional to the square root of the absolute temperature, and the quantity which passes out through the hole is proportional to this velocity and to the density. Hence, on whichever side the product of the density into the square root of the temperature is greatest, more molecules will pass from that side than from the other through the hole, and this will go on till this product is equal on both sides of the hole. Hence the condition of equilibrium is that the density must be inversely as the square root of the temperature, and since the pressure is as the product of the density into the temperature, the pressure will be directly proportional to the square root of the absolute temperature.

The theory of thermal effusion through a small hole in a thin plate is therefore a very simple one. It does not involve the theory of viscosity at all.

The finer the pores of a porous plate, and the rarer the gas which effuses through it, the more nearly does the passage of gas through the plate correspond to what we have called effusion, and the less does it depend on the viscosity of the gas.

The coarser the pores of the plate and the denser the gas, the further does the phenomenon depart from simple effusion, and the more nearly does it approach to transpiration through a capillary tube, which depends altogether on viscosity.

To return to the case of transpiration through a capillary tube. When the temperature is uniform

$$Q=-\frac{\pi\rho a^4}{8\mu}\frac{dp}{dz}\left(1+4\frac{G}{a}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

By experiments on capillary tubes of glass, MM. KUNDT and WARBURG found* for the value of G for air at different pressures and at from 17° C. to 27° C.,

$$G=\frac{8}{p} \text{ centimetres} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

where p is the pressure in dynes per square centimetre, which is nearly the same as in millionths of an atmosphere. For hydrogen on glass

$$G=\frac{15}{p} \text{ centimetres} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

When there is no flow of gas in a tube in which the temperature varies from end to end, the pressure is greater at the hot end than at the cold end. Putting $Q=0$ we have

$$\frac{dp}{d\theta}=6\frac{\mu^2}{\rho\theta}\frac{1}{a^2+4Ga} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

The quantity $6\frac{\mu^2}{\rho\theta}$ is just double of that calculated in section (3) of the introduction, and is therefore in C.G.S measure $0{\cdot}63\div p$ for dry air at 15° C. Let us suppose $a=0{\cdot}01$ centimetre, and the pressure 40 millimetres of mercury, then $G={\cdot}00016$ centimetre.

If one end of the tube is kept at 0° C. and the other at 100° C., the pressure at the hot end will exceed that at the cold end by about 1·2 millionths of an atmosphere.

The difference of pressure might be increased by using a tube of smaller bore and air of smaller density, but the effect is so small that though the theoretical proof of its existence seems satisfactory, an experimental verification of it would be difficult.

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[*] POGG. Ann., July, 1876.

VIII. *Researches on the Action of Organic Substances on the Ultra-violet Rays of the Spectrum.*

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[Plates 21–34.]

PART I.

Method of Experimenting.—Examination of Alcohols and Fatty Acids.

In the year 1863 the late Dr. W. A. Miller described his method of examining the diactinic properties of various saline solutions and organic substances. (Miller, "On the Photographic Transparency of Various Bodies, &c.," Phil. Trans., 1863, I.)

He found that the spectrum transmitted by lenses and prisms of rock-crystal was far superior in length as determined by sensitive photographic plates, to that obtained when lenses and prisms of glass and other materials were employed. He found particularly that the spectrum produced by passing electric sparks between silver points was very rich in photographic rays, and he ascertained how far various substances were transparent to these rays. Professor Stokes at the same time communicated results he had obtained when studying the long spectrum of the electric light by receiving the rays on a fluorescent screen. (Stokes, "On the Long Spectrum of Electric Light," Phil. Trans., 1863, I.) He discovered the fact that certain solutions showed light and dark bands which were otherwise invisible. More recently, M. Soret has studied the absorption of the ultra-violet rays of the spectrum transmitted by various substances, using for the purpose a spectroscope of his invention, which has a fluorescent eye-piece. (Soret, "Recherches sur l'Absorption des Rayons Ultra-violets par diverses substances," 'Archives des Sciences Physiques et Naturelles;' Geneva, January, 1878.) In 1872 the apparatus of Dr. Miller was reconstructed by one of us, and certain improvements were introduced. The intention of repeating his experiments and carrying them out in a more complete manner was prompted by a desire to know what relation, if any, exists between the molecular constitution and the actinic absorption of organic substances. No encouragement to pursue this line of research was offered by the results of Dr. Miller, since he states, "I have not been able to trace any special connexion between the chemical complexity of a substance and its diactinic power." ('Journal of the Chemical Society,' vol. ii., p. 68.) Thus water was found to be perfectly diactinic, and ethylic alcohol more than methylic alcohol, though less so than water. Since all the physical properties of organic substances are dependent on their molecular constitution, it is very probable that the photographic transparency of a substance is a delicate test of its purity, and that notwithstanding the care taken in the purification of specimens, yet some of the experiments of Dr. MILLER were made with impure substances. In the face of this difficulty it was thought that by observing a large number of bodies of similar constitution, many of which would be metameric substances, such as the ethereal salts of the organic acids, and homologous series of the normal alcohols and acids, evidence might be forthcoming of the influence of impurities, and the variations in the absorption of the invisible rays caused by each increment of CH_2 in the molecule. As according to the researches of MILLER the physical state of a substance does not generally affect its absorptive power, we may avoid the difficulty of making observations on equal volumes of organic substances in a state of vapour, and easily arrive at the maximum absorption due to a molecule of a substance, by taking into account its specific volume in the liquid state, and by making the thickness of the layer of liquid experimented on proportional to its specific volume, or by dissolving molecular weights of substances in solvents of known transparency.

The Method of Experimenting.

After considerable time spent in giving a trial to the various methods of studying the ultra-violet rays, the photographic method appeared to be decidedly the most satisfactory.

Rays which are very indistinct, or even quite invisible on any fluorescent screen, may be brought out distinctly on a properly prepared photographic plate. It is useful, however, to have a focussing screen of uranium glass in the camera, or what answers the purpose equally well, a glass plate coated with gelatine in the solution of which some æsculin has been dissolved. To observe the spectrum by reflected light, and view the visible and invisible rays simultaneously, a piece of paper steeped in a solution of æsculin, to which a little ammonia has been added, may be employed, and it answers the purpose even when dry.

The instruments employed were the following :—1st. An induction coil and Leyden jar for producing an unbroken stream of sparks between metal points. 2nd. A collimator tube 3 feet long, carrying a pair of clips at one end for holding the points close to the slit, and a quartz lens at the other. 3rd. A quartz prism placed at the angle of minimum deviation for the sodium line D. 4th. Another quartz lens and the body of a photographic camera which could be extended to 36 inches. The separate portions of this lengthy apparatus were firmly screwed down to two heavy tables to prevent the shifting of any part, since the proper adjustment of the whole takes some time. The coil, which could give a 6 or 7-inch spark in air, was excited by a battery of five GROVE's cells. Other batteries were tried, but none gave such satisfactory results when working for hours together. The Leyden jar was of such a size that each surface of tinfoil measured 72 square inches.

The carriers for the metallic points are so fixed to the collimator tube that by the motion of two screws they may be moved from right to left across the slit, and at the same time as close as desirable under different circumstances. The proper position of the carrier is, of course, a vertical one, but it could be turned over in a horizontal direction so as to afford convenience for fixing the points. The electrodes are more conveniently held by screw forceps than by spring clips. Broad electrodes, as former experimenters have pointed out, are usually the best to work with (MILLER and STOKES, *loc. cit.*), but we have in certain cases obtained better results with points, as for instance, with nickel wire. The spark apparatus was enclosed in a wooden box along with the end of the collimator tube to prevent the light emitted by the electrodes from escaping into the dark room, and to muffle the irritating noise of the electric discharge. We have had no difficulty in obtaining an uninterrupted stream of sparks yielding a perfectly steady light for three quarters of an hour. Professor STOKES used a condensing lens of 2½ inches focal length to concentrate the rays of the spark upon the slit, but since we sometimes employed an amalgam containing several metals as the lower electrode, we could not use the same, for the reason that mercury vapour would condense upon the lens. Instead of condensing the rays with a lens, we diminished the loss of light very greatly by fixing the electrodes as near as possible to the slit, and placing the cell holding the liquid to be examined in a little wooden box behind the slit, into which the collimator tube passed. This box was about 4 inches long, with a hinged lid, and it was possible to place a train of three or four cells full of liquid within it, through which the rays were transmitted.

The cells themselves were cut from wide glass tube by slicing it longitudinally in two pieces and cutting off lengths of three-quarters of an inch. These were fitted in metal frames, and pieces of quartz were affixed to each end by means of screws pressing upon a rim of metal. The situation of the liquid behind the slit, and therefore at a distance from the spark, obviated a difficulty caused by the ignition of the vapour of volatile liquids. The tube of the collimator, however, became filled with vapour, which of itself would exert a strong absorption in many cases, and in order to remove this vapour a nozzle was fixed at right angles to the tube at a point as close to the lens as possible, and this was placed in communication with a concertina aspirator, two or three strokes of which will draw five or six litres of air through the tube.

The lens behind the prism was capable of being raised and lowered by means of a sliding front as on an ordinary landscape camera. This enabled us, when occasion required, to take three photographs on a plate 2 inches in width. According as was found most convenient we have made use of electrodes of silver, cadmium, zinc, aluminium, nickel (containing as we find by its spectrum a trace of copper), indium, and iron. In some cases an amalgam containing silver, zinc, cadmium, sodium, aluminium, and magnesium has been used together with a point of iron. The spectrum most generally useful is that yielded by nickel points. In certain cases, particularly when several absorption bands occur, it is advisable to photograph with the slit open, so that the nickel lines may not be too distinct.

The Photographic Process.

The original method of photographing employed by Dr. MILLER we have found to be defective: first, because the more refrangible end of the ultra-violet spectrum is extremely weak, if not entirely wanting, when photographed on plates containing a plain iodised collodion; secondly, a wet collodion process is disadvantageous when long exposure is sometimes necessary; and thirdly, when working in a small room the ozone generated by the electric discharge acts upon wet collodion plates in such a manner that they become coated with a thick deposit of silver directly the developing solution is applied, the deposit being densest where the bath solution has most accumulated. Much larger and better spectra are obtained by using ordinary bromo-iodised collodion and an iron developer, but of course this process is rendered useless by the action of the ozone on the silver solution.

The extraordinary improvements made of late years in the preparation and development of dry plates, together with the foregoing facts, combined to recommend a dry process.

Successive trials have been made with plates coated with washed collodio-bromide emulsion, with the BEECHY standard dry plates, uranium plates, Mr. KENNETT's and WRATTEN and WAINWRIGHT's gelatine pellicle plates. The gelatine plates are to be preferred for two reasons; if they are wanted for the production of negatives to print from, the film is exceedingly fine and even in texture, and on the other hand, if transparencies showing absorption bands are desired they need not be varnished. The BEECHY plates are more sensitive to very feeble rays in a certain part of the spectrum than those coated with gelatine pellicle; this may be seen by comparing the spectra of iron photographed on these two varieties of plates. One advantage of the gelatine plates is that they photograph more of the less refrangible rays than any others we have tried, and though the lengthening of the spectrum in consequence is comparatively slight, yet it is of importance in the examination of certain coloured substances, which while they transmit rays of higher refrangibility absorb the blue and violet. Such bodies are solutions of the nitrophenols and nitranilines. The exposure of the sensitive plates has been varied from 10 seconds to an hour and a-half, according to circumstances, depending partly on the plates employed and the object to be attained; but it is seldom that a longer period than a minute is necessary even for the production of negatives to print from, and when photographing absorption spectra with an open slit 10 seconds will suffice.

The Measurement of Bands and their Position.

The lines of zinc, cadmium, and aluminium offer a convenient means of determining the position of absorption bands, &c., and M. SORET has employed them. The lines of cadmium in the ultra-violet as well as in the visible portion of the spectrum have been measured by M. MASCART and their wave-lengths calculated. M. SARASIN has made use of these measurements in his determination of the refraction indices of quartz for the ordinary and extraordinary rays in the ultra-violet spectrum. ('Archives des Sciences Physiques et Naturelles;' Geneva, vol. lxi, p. 109; January, 1878.) The wave-lengths for the cadmium lines in the invisible portions of the spectrum are shown in Diagram No. 1, and it will be seen that several of the following diagrams have been made after this scale, it being an advantage to have the metallic lines placed at intervals corresponding to their wave-lengths. When nickel points instead of cadmium have been used, two strongly marked air-lines below the solar line L (Plate 21, fig. 1) serve as a starting point for measurements; another air-line between the lines 11 and 12 also answers the same purpose. Inasmuch, however, as the appearance of spectra as seen when photographed is different from that when the lines have been mapped out in accordance with their wave-lengths, it has been considered desirable to follow the plan of M. SORET and lay down the spectra in some cases just as they appear in the photographs.

Some of the advantages derived from the employment of photography in studying the ultra-violet rays are the following: The measurements are free from any personal error, the photographs are permanent and unmistakable records easily referred to, and the observations are made more rapidly and with much less fatigue to the eye than by any other means. Furthermore, liquids which are slightly turbid and unfit for observation with a fluorescent screen or eye-piece can be examined photographically by lengthening the exposure of the plate.

Absorption Caused by Films of Canada Balsam.

It was considered very desirable to know whether Canada balsam, as used in the construction of optical apparatus, was capable of cutting off the ultra-violet rays to any considerable extent. We are aware, from the researches of M. SORET, that Iceland spar is extremely transparent, the 26th line of cadmium being distinguishable through a thickness of 65 m.m. of the substance. It was only necessary, then, to take a small NICOL's prism and interpose it between the slit and the first lens. The result showed that lenses and prisms, cemented with Canada balsam, are utterly useless for such experiments as these, the more refrangible rays being cut off sharply at the 12th line. Several kinds of beautifully clear and colourless optical glass, made by FEIL, of Paris, and lent us by Mr. HILGER, have the same effect upon the spectrum, whether the glasses be made of lead or aluminium; they are all utterly valueless for observations on rays more refrangible than line No. 12 of cadmium. Fluor spar is quite transparent.

Examination of Organic Substances.

After making a preliminary series of observations on the alcohols, ethereal salts, and fatty acids, we abandoned the use of silver electrodes used by Dr. MILLER, and at once, by the aid of other metallic spectra (referred to on diagram No. 1), investigated the comparative transparency of quartz in use for constructing cells, the cells being filled with water and with different alcohols.

It will be seen that the sides of three quartz cells are almost quite transparent, and that, when filled with water, they are practically unchanged in this respect.

It is evident that we need not fear to employ the three cells in a train, or to use pure water as a solvent for such substances as we may desire to examine in solution. The thickness of liquid examined was never less than three-quarters of an inch.

Many of the following organic substances were obtained from Mr. KAHLBAUM'S agents; they were prepared in his factory, and were afterwards carefully rectified and purified by us.

Methylic Alcohol. Specimen No. 1.—This was apparently perfectly pure; it had no odour or colour, and it boiled at 66° C. It transmitted very few rays beyond 12 Cd, and it is worthy of remark that, both before and after distillation from quicklime prepared from pure marble, it yielded the same absorption spectrum. The identical specimen of methylic alcohol examined by Dr. MILLER was compared with this, and was found to give the same spectrum. The bottle containing it was labelled "Methylic alcohol from methylic oxalate."

Specimen No. 2.—This sample was kindly furnished, in a raw state, by Mr. JOHN WILLIAMS, President of the Pharmaceutical Society. It had been prepared from oil of winter-green. After redistillation with caustic potash, and dehydration with caustic lime, it was rectified, and found to boil between 66° and 66°·5. As it possessed a faint, sweet smell, it was manifestly impure, the impurity doubtless being a small quantity of some essential oil not capable of saponification. The somewhat longer spectrum of this substance favoured the belief that a specimen of absolute purity would be nearly photographically transparent. The specimen examined by M. SORET transmitted very little beyond the line 18 Cd, and was therefore not of such purity as this, since only 10 m.m. or about half the thickness of liquid was examined.

Specimen No. 3.—About a kilogramme of oxalate of methyl, beautifully crystallised and perfectly white, was decomposed by strong soda solution, which was made from pure soda prepared from sodium. Repeated rectifications from lumps of soda yielded more than 200 grammes of methylic alcohol, which, as it gives a spectrum showing almost as great a transparency as water, must be regarded as absolutely pure. It boils at 66° C., and has neither colour, odour, nor distinctive taste. Two photographs of this specimen are represented in the diagram; the longer spectrum resulted from the use of indium and zinc points. It shows how very nearly the alcohol approaches water in transparency.

Ethylic Alcohol. Specimen No. 1.—This specimen was obtained from Mr. Kahlbaum, of Berlin. After rectification it boiled at 78°·5. It exhibits slightly less transparency than methylic alcohol.

Specimen No. 2.—A sample of ordinary commercial absolute alcohol obtained from Messrs. Burgoyne and Burbidges, BP. 79°. We tested two or three Winchester quarts of this liquid and made use of it for dissolving such substances as were insoluble in water. They yielded the same spectrum as Nos. 1 and 2. Several other samples we have examined are not of such great purity, although from the same firm.

Propylic Alcohol.—This was carefully purified and redistilled, BP. 97°·5. It is a substance very difficult to dehydrate completely, but, as a trace of water does not affect its diactinic quality, this may be disregarded. It will be seen by a glance at the diagram that one cell-full of the liquid is less diactinic than ethylic alcohol, and that the absorption due to three times the thickness of one cell is not much greater than that caused by one cell only. As the specific volumes of the three alcohols are as the following numbers :—

Methyl	40·8
Ethyl	62·8
Propyl	84·8

we may examine the liquids in what are nearly molecular proportions by taking layers of liquid differing in thickness in the proportions of 2, 3, and 4 according to the number of carbon atoms in the molecule. It will be seen that these three substances show a gradual increase in the absorption of the more refrangible rays corresponding to the size of their molecules, or, in other words, to the number of carbon atoms they contain. The increased absorption is perfectly evident whether we take equal volumes of the liquids or thicknesses proportional to their specific volumes, from which we may conclude that one cell-full of liquid exerts nearly the maximum amount of absorption of the substance.

Normal Octylic Alcohol.—It has been shown that this substance is obtainable from the oil of *Heracleum Spondylium.* This specimen was prepared from heracleum oil by Mr. Kahlbaum. After purification it boiled at 192°·5 instead of at 198°, the correct boiling point. One cell-full of the liquid exerts a marked degree of absorption, though, judging from the alcohols previously examined, the spectrum is very much what one might expect from a substance containing as many as eight carbon atoms.

It is worth remarking that the impure specimens of methylic and ethylic alcohols, when diluted with an equal or double volume of water, were perfectly diactinic.

The Fatty Acids.

Formic Acid. Specimen No. 1.—This acid was evidently very carefully prepared and of great purity ; it was said to be crystallisable at 48° F., though we did not succeed in obtaining it in the solid form. Obtained from Mr. Kahlbaum.

Specimen No. 2.—Another sample similar to No. 1, obtained from Messrs. HOPKIN and WILLIAMS.

Specimen No. 3.—Prepared by us from a beautifully crystallised specimen of formiate of lead, by decomposing the salt, after drying it at 130° C., in a current of pure dry sulphuretted hydrogen, and subsequent distillation from a further quantity of formiate of lead crystals.

Specimen No. 4.—Consisted of two or three portions of the acid fractionally distilled during the preparation of the substance from oxalic acid and glycerin.

All these specimens were manifestly impure, for they yielded spectra in which the more refrangible rays were absorbed to a varying extent in every case, the absorption being considerable. The acid prepared from the lead salt by means of sulphuretted hydrogen gives the shortest spectrum and probably contains some sulphur compound. We have been much troubled by repeated attempts to prepare pure formic acid. It was decided, if possible, to oxidise pure methylic alcohol to the acid. When bichromate of potash and sulphuric acid are used for the purpose there is a danger of oxidising the acid to carbonic acid; and when we consider, too, that in order to oxidise 16 grammes of alcohol a litre of a solution of the bichromate salt is required, it is easily seen that the formic acid will be diluted so largely that in the process of separation by distillation the product would be destroyed—at any rate this method was unsuccessful. The action of sulphuric acid and manganic oxide or tartaric acid does not yield a pure substance. The action of oxalic acid on formiate of barium yields a distillate which at first is different from the product of other processes in being free from a slight but peculiar odour, and which it is highly probable is not proper to formic acid, but belongs to some ever present impurity. On continuing the distillation, the usual smell was developed, which fact seems to show that heat causes a change in the formic acid. It is said by LORIN (*vide* WATTS' 'Dictionary of Chemistry,' vol. vi.) that formic acid of 87 per cent. can be obtained by the dry distillation of cupric formate. It is quite true that a small quantity of very weak formic acid can be obtained by this process, but as soon as all the crystalline water has been distilled off no yield of formic acid is obtainable from the salt. The products of the action of heat on the dried salt when maintained at a temperature of 200° in a paraffin bath, are metallic copper, carbonic acid and water, and, in all probability, carbon monoxide is formed.

The specimens of formic acid examined by Dr. MILLER and by M. SORET give much shorter spectra than ours. The formic acid examined by the latter observer absorbed all the rays beyond the 8th cadmium line, while the two better of our specimens transmit rays to a distance midway between the 12th and 17th lines (see Diagram 2), and that too through double the thickness of liquid.

Acetic Acid.—Two or three specimens of the glacial acetic acid of commerce were examined, as well as specimens carefully purified by distillation and crystallisation. The spectra were the same in each case, transmitting rays just beyond 17.

Propionic Acid. Specimen No. 1.—This was a very good sample of the acid, boiling at 140°–141° after rectification.

Specimen No. 2.—This was prepared from the nitrile. Boiling point 141°.

Specimen No. 3.—This was a fraction of the distillate from No. 2. It boiled at 139°–141°. These three specimens do not greatly differ in the length of transmitted spectrum; the two acids prepared from the nitrile are more diactinic than the other. The acid examined by M. Soret corresponds well with these.

Butyric Acid.—No. 1 boils at 161°·5–162°, No. 2 at 160°–161°·5. Both portions are very nearly the same in diactinic property, and equal thicknesses of the liquid being taken they do not differ much from the propionic acids.

The specific volumes of these four fatty acids are the following:—

Formic acid	42
Acetic	64
Propionic	86
Butyric	108

Consequently we must examine thicknesses of liquid proportional to these numbers in order to see the change in diactinic property corresponding to molecular differences. Since, however, we have not been able to obtain pure formic acid we have very carefully crystallised formate of sodium and compared it with sodium salts of those acids, molecular weights of the salts being dissolved and made up to equal volumes of liquid. One of the purest specimens of the sodium formate was prepared from a beautifully crystallised copper salt which had been completely dried and heated to near the temperature at which it decomposes. This was dissolved in water and mixed with a solution of carbonate of soda in just sufficient quantity to precipitate the copper. A little cupric hydrate remained dissolved, and was separated by electrolysis in a platinum dish with the aid of one of Grove's cells. The aqueous solution of the sodium salt was slightly yellowish, but a beautifully white product was obtained by precipitation from a highly concentrated liquid by the addition of pure absolute alcohol, in which the formate is much less soluble than in water. The crystalline precipitate was recrystallised from alcohol. Notwithstanding, however, the excessive care with which sodium formate was prepared, it was found practically impossible to obtain it in a state of sufficient purity to make it available for trustworthy observation. Crystals of snow-like whiteness yielded yellowish solutions soon becoming yellow; various crops of such crystals gave spectra of different lengths; sometimes the portion giving the longest spectrum, after the most careful recrystallisations would transmit only half the rays, and by no artifice could a fraction be made to transmit the original spectrum of the salt. A very careful examination showed us that formate of soda is changed with extreme ease into oxalate, and there is no doubt that from this fact arises the unsuccessful issue of our experiments on this salt. The solution of the four sodium salts contained the following weights of the substances:—Formate, 4·8 grammes; acetate, 6·1; propionate, 7·4, and butyrate, 8·7 grammes in each 12 c.c. of liquid, or the solutions of acetate, propionate, and butyrate contained 50·8, 61·6, and 72·5 per cent. of the respective salts. The acetate and propionate formed supersaturated solutions.

Single and double thicknesses of liquid were examined. In Diagram No. 2 is shown the effect on the ultra-violet rays of the spectrum due to three successive members of a homologous series of acids. Each solution contained a molecule of the sodium salt of the acid. The two series of observations 1 and 2 represent the action of single and double cells or thicknesses of $\frac{3}{4}$ and $1\frac{1}{2}$ inch respectively. It is particularly striking how the second series of observations shows a regular decrease in transmitted rays caused by each increment of CH_2 in the molecule.

The same fact is again noticeable in Diagram No. 3, Plate 23. Here, in order to get rid of the inconvenience caused by the change of alkaline formates into oxalates on evaporation, we made use of very beautiful specimens of barium salts. In this case the formate is the most transparent of salts, and it may be seen that the acetate bears the same relation to it in transparency that in the case of the sodium salt the propionate bore to the acetate.

A glance at Diagram No. 3 will show that a cell-full of the acids corresponding to the ethylic and propylic alcohols, which we have every reason to believe are perfectly pure substances, are less diactinic than the alcohols as shown in Diagram No. 1, which fact points to the carboxyl group in the molecule possessing a special absorptive power greater than that of hydroxyl. We have further evidence of this in the short spectrum of oxalic as compared with acetic acid. The specimen examined was prepared with great care and repeatedly recrystallised. There is every reason to believe in its absolute purity.

The Ethereal Salts of Fatty Acids.

In examining regular series of formates, acetates, propionates, &c., of the radicles, methyl, ethyl, and propyl, it was expected that probably the majority of substances would be of such a degree of purity that some relation between their constitution and their optical properties would be traceable. About twenty specimens of these compound ethers were obtained from Mr. KAHLBAUM; they varied in quantity according to the nature of the substances, but the majority of the specimens weighed 50 grammes. They were purified by washing with distilled water and with carbonate of potash; they were dried by standing over fused carbonate of potash, and were subsequently distilled. The boiling points of the various specimens examined are given on Diagram No. 4.

It will be seen that it was hopeless to get any result from this examination, many of the substances manifestly containing impurities which injured their photographic transparency. The only three bodies which appear to be pure are the ethylic acetate, propionate, and butyrate. Even in this case two specimens of the same substance, with precisely the same boiling point, vary slightly in diactinic quality. Unless very large quantities of liquid are operated upon, substances cannot be purified by fractional distillation, and even under most favourable circumstances distillation must be several times repeated. The remarkably short spectrum transmitted by amyl-butyrate was due evidently to some impurity, since the behaviour of the substance on dilution with alcohol was quite in accordance with this view. The generally transparent character of the alcohols and ethereal salts is remarkable.

Normal Hydrocarbons.

Only three of these—hexane, heptane, and octane—have been examined; they were specimens kindly lent us by Prof. Schorlemmer, and believed to be perfectly pure. Three series of photographs were taken of them, but they yielded spectra in no way corresponding to their difference in composition, from which we are inclined to believe they were impure (see Diagram No. 2).

Compound Ammonias.

A series of five of these bodies was examined, the solutions containing 33 per cent. of the substances in the case of ethylamine and methylamine. It is believed that, with the exception of ethylamine, they contained some impurity, since the methylamine and dimethylamine yielded shorter spectra than the ethylamine. These were commercial samples, and most probably were prepared from ammonia obtained from gas-liquor. M. Soret has shown that commercial ammonia, even after many recrystallisations as sulphate, still shows an absorption band, due to some impurity which is constantly present; it is therefore scarcely likely that pure products would be obtained unless specially prepared ammonia be used for the purpose. We considered that the volcanic ammonia, which is free from some of the impurities present in that made from gas-liquor, might be perfectly diactinic. Three separate samples, each measuring half a gallon, were examined, with the result that all the rays beyond 17 Cd were cut off. Even when so dilute that only one volume of ammonia was contained in 64 of water, all traces of absorption had not been destroyed (see Diagram No. 5, Plate 25). This absorption is not due to nitric acid, because M. Soret has shown that the absorption band due to that substance occurs between 12 Cd and 17 Cd.

Conclusions.

(1.) The normal alcohols of the series $C_nH_{2n-1}OH$, are remarkable for transparency to the ultra-violet rays of the spectrum, pure methylic alcohol being nearly as much so as water.

(2.) The normal fatty acids exhibit a greater absorption of the more refrangible rays of the ultra-violet spectrum than the normal alcohols containing the same number of carbon atoms.

(3.) There is an increased absorption of the more refrangible rays corresponding to each increment of CH_2 in the molecule of the alcohols and acids.

(4.) Like the alcohols and acids the ethereal salts derived from them are highly transparent to the ultra-violet rays, and do not exhibit absorption bands.

PART II.

Examination of Substances containing the Benzene Nucleus.

In the examination of such substances as contain a closed chain of carbon atoms doubly linked together, one fact presented itself in a striking manner. All these bodies are strongly adiactinic, those which are the least so being the hydrocarbons; Diagram No. 2, will at once make this apparent. Professor STOKES has pointed out that one of these substances, salicine, in an aqueous solution, the strength of which is not stated, shows a characteristic absorption band. Now salicine is a substance the constitution of which is perfectly well known: it is a glucoside of saligenin,

$$\left.\begin{matrix} C_6H_{11}O_5 \\ C_6H_4(OH)CH_2 \end{matrix}\right\} O$$

and consequently the absorption band may be due to the saligenin. At any rate it seemed worth while to dilute solutions of known strength, containing such allied substances as phenol, salicylic acid and salicylate of methyl, and ascertain whether any absorption bands are thus made visible. The result has disclosed the fact that almost if not entirely all benzene derivatives, including the hydrocarbon itself, are characterised by one or more absorption bands, which resist modification or extinction by dilution to an enormous extent.

For this part of the research it has been necessary to make solutions of known strength, to dilute them with definite quantities of water, and photograph the various liquids thus obtained. These photographs have been placed upon sectional paper, upon which the lines of cadmium as seen in a photograph of the metallic spectrum have been drawn, and the proportions of substance in solution have been co-ordinated with the position of the absorption bands relative to the cadmium lines. A series of curves has thus been obtained, and each of these proves to be a highly characteristic feature of the substance observed. The number of photographs necessary for completing the curves has varied very much with different substances, sometimes four or five, and occasionally eighteen or twenty have been taken.

When the substance to be examined was a liquid, two cubic centimetres were measured and diluted with alcohol if necessary, or water if possible, so as to form a total volume of 100 c.c. From this solution others more dilute were obtained. If the substance were a solid 0·2 gr. or 0·02 gr. in weight of the body were dissolved in alcohol or water so as to form a total volume of 100 c.c., and when it was desirable to compare a liquid substance with a solid, both were weighed and dissolved in the same manner. In this way salicylate of methyl was compared with salicylic acid and salicine.

Benzene (Diagram No. 6).—A very pure specimen, kindly lent us by Professor GUTHRIE. About ten ounces of the substance had been separated from more than a gallon and a-half of the commercial benzene, by successive crystallisations in a freezing mixture. The first feature noticeable here is the small alteration in the absorption of the rays caused by a dilution with 99 volumes of alcohol. It is impossible to describe the changes noticed on repeated dilutions without reference to the diagram, but it may be well to record the proportions of alcohol to benzene in the liquids photographed.

One part of benzene was successively diluted so as to measure 100, 150, 190, 220, 235, 250, 375, 500, 750, 850, 1000, 1275, 1500, 2000, and 2500 times its original volume. With a dilution of 750, six absorption bands are seen. In other words, as the cell is $\frac{3}{4}$ inch in thickness, a layer of benzene $\frac{1}{1000}$th of an inch in thickness would show the same spectrum. As the bands are perfectly well seen after a dilution of 1500 times, it follows that a film of benzene $\frac{1}{2000}$th of an inch in thickness would show them.

Toluene.—Methylbenzene, BP. 111° C.—The various solutions examined contained $\frac{1}{100}$, $\frac{1}{500}$, $\frac{1}{1000}$, $\frac{1}{2000}$, $\frac{1}{2500}$, and $\frac{1}{3000}$ of their volume of the hydrocarbon. It is evident, from Diagram No. 7, that a similarity exists between this substance and benzene, though fewer rays are transmitted by toluene, which greatly modifies the appearance of the two diagrams. The two absorption bands between 17 Cd and 18 Cd are highly characteristic.

Ethylbenzene, BP. 135°–6 (Diagram 8).—The solutions examined contained the following volumes of the liquid, $\frac{1}{150}$, $\frac{1}{300}$, $\frac{1}{500}$, $\frac{1}{1000}$, $\frac{1}{2000}$, $\frac{1}{3000}$, $\frac{1}{3500}$, $\frac{1}{4000}$, and $\frac{1}{6000}$. At $\frac{1}{500}$ a strong band of transmitted rays appears from about half way between 18 Cd and 23 Cd, and extends to 24 Cd; this widens out, until at $\frac{1}{3000}$ it meets the narrow absorption band noticed in the previous spectrum as situated between 17 Cd and 18 Cd.

Trimethylbenzene.—Mesitylene.—The methyl molecules occupy the position indicated by 1–3–5, BP. 163–4° C. (Diagram No. 9). Observations were made on solutions containing $\frac{1}{240}$, $\frac{1}{500}$, $\frac{1}{1000}$, $\frac{1}{2000}$, $\frac{1}{3000}$, $\frac{1}{6000}$ and $\frac{1}{10000}$ of their volume of the liquid. A transmitted band occurs between 18 Cd and 23 Cd, commencing at $\frac{1}{500}$ dilution. At $\frac{1}{2000}$ two narrow bands of transmitted rays occur between 17 Cd and 18 Cd.

These hydrocarbons are the least opaque to the photographic rays of the many substances containing the benzene nucleus which we have examined, although compared with the substances previously treated of, namely, the alcohols, fatty acids, and ethereal salts, they are greatly wanting in transparency.

We now pass to substances containing the hydroxyl group replacing hydrogen.

Phenol, $C_6H_5.OH$ (Diagram No. 10).—Solutions in water containing the following quantities of the body were examined: $\frac{1}{400}$, $\frac{1}{800}$, $\frac{1}{1700}$, $\frac{1}{3200}$, $\frac{1}{6400}$, $\frac{1}{12800}$, and $\frac{1}{25600}$.

In this case, as in all the following, the intensity of all the absorption bands renders it necessary to reduce the dimensions of the ordinates to the curve, which represent the quantities of water used in dilution, otherwise the diagram would stretch to five times the length of those representing the hydrocarbons. The absorption of the rays more refrangible that 24 Cd was not extinguished by the dilution to $\frac{1}{12800}$, nor is it entirely destroyed by doubling such dilution.

Thymol (Diagram No. 11).—This body is a methyl-iso-propyl-oxybenzene,

$$C_6H_3\left\{\begin{array}{l}OH\\CH_3\\CH(CH_3)_2\end{array}\right.$$

Solutions of the following strength were examined: 1 gramme was contained in 100 c.c., 1000 c.c., 2000 c.c., 5000 c.c., 7500 c.c., 10,000 c.c., 20,000 c.c., 40,000 c.c., 80,000 c.c., 100,000 c.c.

The smaller effect of dilution on the absorption rendered it necessary again to reduce the dimensions of the ordinates to half those of the phenol curves.

It seems as if complexity of structure increases the intensity of the absorption bands.

Pyrogallol, $C_6H_3\ (OH)_3$ (Diagram No. 12). — Solutions containing the following proportions of the substance were examined: $\frac{1}{500}$, $\frac{1}{1000}$, $\frac{1}{5000}$, $\frac{1}{10000}$, $\frac{1}{50000}$: $\frac{1}{75000}$, and $\frac{1}{100000}$. After dilution has reduced the proportion of substance in solution to $\frac{1}{5000}$, rays beyond 18 Cd are faintly transmitted, with a proportion of $\frac{1}{50000}$ there is nearly complete transmission as far as 23 Cd, but there is still some absorption with $\frac{1}{100000}$.

Benzoic, hippuric, and phthalic acids being substances in which the carboxyl group occurs, are of much interest.

Benzoic Acid, $C_6H_5.COOH$ (Diagram No. 13).—Solutions containing $\frac{1}{200}$, $\frac{1}{1000}$, $\frac{1}{3000}$, $\frac{1}{5000}$, $\frac{1}{6000}$, $\frac{1}{8000}$, $\frac{1}{10000}$, $\frac{1}{15000}$, $\frac{1}{20000}$, $\frac{1}{40000}$, $\frac{1}{80000}$, and $\frac{1}{120000}$ their weight of substance were examined.

The most diluted of these liquids still partially absorbed the rays beyond 23 Cd—that is to say, they were enfeebled. There is a feeble transmission of the rays between 17 Cd and 18 Cd with a dilution of $\frac{1}{5000}$. This extends to the main band of transmitted rays at $\frac{1}{6000}$, and a continuous spectrum is transmitted as far as 18 Cd with $\frac{1}{15000}$.

Phthalic Acid, $C_6H_4.(COOH)_2$ (Diagram No. 14).—The general appearance of this and the preceding spectrum is remarkable, though there is a distinct difference between them. Solutions containing $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{5000}$, $\frac{1}{6000}$, $\frac{1}{7500}$, $\frac{1}{10000}$, $\frac{1}{15000}$, $\frac{1}{20000}$, $\frac{1}{40000}$, $\frac{1}{60000}$, $\frac{1}{80000}$, $\frac{1}{100000}$, $\frac{1}{120000}$, $\frac{1}{150000}$, $\frac{1}{200000}$, $\frac{1}{250000}$, $\frac{1}{300000}$, and $\frac{1}{400000}$ths. their weight were examined. The last solution on comparison with distilled water still showed a slight absorption of the extreme rays. It is the increased opacity of the latter substance which constituted the chief difference between the spectra of benzoic and phthalic acids. This strengthens the view already expressed, that the introduction of the carboxyl group into a molecule increases the actinic absorption, phthalic acid standing to benzoic as oxalic to acetic acid, while benzoic acid and phenol are related in the same manner as acetic acid and alcohol.

Hippuric Acid, $C_9H_9NO_3$, (Diagram No. 15).—This being a benzamido-acetic acid it was interesting to ascertain whether the benzoic acid residue was traceable; its peculiarities, however, do not appear. Solutions examined contained $\frac{1}{500}$, $\frac{1}{2000}$, $\frac{1}{5000}$, $\frac{1}{10000}$, and $\frac{1}{40000}$ of their weight of the substance.

Aniline, $C_6H_5.NH_2$, BP. 182° (Diagram No. 16).—Solutions of aniline in alcohol were made which were afterwards diluted with water. Solutions containing the following proportions by volume of aniline were examined: $\frac{1}{1000}$, $\frac{1}{10000}$, $\frac{1}{12000}$, $\frac{1}{15000}$, $\frac{1}{20000}$, $\frac{1}{30000}$, $\frac{1}{100000}$, $\frac{1}{110000}$, $\frac{1}{120000}$, $\frac{1}{180000}$, and $\frac{1}{220000}$. A first absorption band lies immediately below 17 Cd; it first makes a definite appearance when the rays just beyond have been transmitted after a dilution with 12,000 parts of water. The absorption is not destroyed by carrying the dilution $\frac{1}{30000}$. No rays are transmitted further than midway between 18 Cd and 23 Cd until a dilution of $\frac{1}{110000}$ is attained, and all absorption of the more refrangible rays is not effected until after dilution with 220,000 parts of water.

One of the chief objects in view in commencing a study of the benzene derivatives was to ascertain the difference in the absorption spectra of strictly isomeric substances. We have accordingly examined the three oxybenzoic acids, two nitro-phenols, and two nitranilines.

Salicylic Acid, $C_6H_4.OH.COOH$, *or* 1·2 *Oxybenzoic Acid* (Diagram No. 17).—This substance, prepared some years ago by one of us from the oil of winter-green, was in the form of large crystals. We recrystallised it from alcohol, and took the choicest crystals for preparing our solutions. Solutions were made containing $\frac{1}{200}$, $\frac{1}{300}$, $\frac{1}{400}$, $\frac{1}{800}$, $\frac{1}{1000}$, $\frac{1}{1700}$, $\frac{1}{2000}$, $\frac{1}{3000}$, $\frac{1}{3200}$, $\frac{1}{4000}$, $\frac{1}{4400}$, $\frac{1}{4500}$, $\frac{1}{4700}$, $\frac{1}{5000}$, $\frac{1}{6000}$, $\frac{1}{8000}$, $\frac{1}{10000}$, $\frac{1}{15000}$, and $\frac{1}{20000}$ of their weight of the substance. There is nothing remarkable in the spectrum until after a dilution of $\frac{1}{1000}$; from this point, however, until $\frac{1}{5000}$ is reached; the curves obtained are very curious.

Oxybenzoic Acid, 1·3 (Diagram No. 18).—This compound is remarkable for its adiactinic quality; it transmits no rays beyond 12 Cd until after a dilution of $\frac{1}{5000}$, and there is still a distinct absorption between 12 Cd and 17 Cd until $\frac{1}{20000}$, when even the rays beyond 18 Cd are only feebly transmitted.

Paroxybenzoic Acid, 1·4 (Diagram No. 19).—This acid shows an increase in adiactinic quality exceeding that of the preceding substances. Solutions containing $\frac{1}{13000}$, $\frac{1}{26000}$, $\frac{1}{39000}$, $\frac{1}{52000}$, $\frac{1}{77000}$, $\frac{1}{103000}$, $\frac{1}{154000}$, $\frac{1}{205000}$, and $\frac{1}{140000}$ their weight of the substance were examined, and the diagrams constructed from these two other sets of observations were, however, taken with various proportions of solid to solution; they gave, however, the same result. No rays beyond 17 Cd are transmitted until after a dilution of $\frac{1}{30000}$ has been reached. There is a broad absorption band between

18 Cd and 23 Cd, with one part of the substance in 100,000 of water, and this band continues, though diminished in power, until 205,000 parts of water have been added. Absorption of the more refrangible rays beyond 26 Cd continue till 410,000 parts of water have been used in dilution. Had this diagram been drawn on the same scale as that representing benzene and the allied hydrocarbons, it would have extended beyond a length of 170 feet, each inch only of which would have represented a dilution with 200 parts of water.

The two following substances were examined on account of their close relationship to salicylic acid and its isomers :—

Salicylate of Methyl, $C_6H_4.OH.COOCH_3$, BP. 220°·7 (Diagram No. 20).—This was rectified from about a kilogramme of oil of winter-green. Two sets of observations were made, and solutions containing the following proportions by weight of the substances were used : $\frac{1}{1000}$, $\frac{1}{1500}$, $\frac{1}{2500}$, $\frac{1}{5000}$, $\frac{1}{10000}$, $\frac{1}{15000}$, $\frac{1}{30000}$, $\frac{1}{45000}$, and $\frac{1}{60000}$. At first, all rays beyond 12 Cd are absorbed, and this condition continues until beyond a dilution of $\frac{1}{1500}$, when the rays between 17 Cd and 18 Cd are transmitted. The absorption band lying between 12 Cd and 17 Cd becomes nearly extinguished by an addition of 45,000 parts of the diluent ; but at the same time a second band of rays, lying between 25 Cd and 26 Cd, is transmitted, which gradually extends but does not overcome the absorption band lying between 18 Cd and 23 Cd until after a dilution with 60,000 parts of water.

Salicine, $\left.\begin{matrix}C_6H_{11}O_5\\ C_6H_4(OH)CH_2\end{matrix}\right\}O$ (Diagram No. 21).—Solutions examined contained the following proportions of substance : $\frac{1}{400}$, $\frac{1}{800}$, $\frac{1}{1600}$, $\frac{1}{4000}$, $\frac{1}{8000}$, $\frac{1}{20000}$, and $\frac{1}{50000}$. The absorption band is highly characteristic. Transmitted rays, a little less refrangible than 23 Cd, make their appearance in solutions containing $\frac{1}{400}$ of substance, in addition to the continuous transmission of rays less refrangible than 17 Cd. In solutions containing $\frac{1}{8000}$ the absorption band has practically disappeared.

Ortho-nitro-phenol, $C_6H_4.NO_2.OH$, 1·2 (Diagram No. 22).—Aqueous solutions only of the phenols were employed notwithstanding their very slight solubility in water. Solutions containing $\frac{1}{4000}$, $\frac{1}{8000}$, $\frac{1}{12000}$, $\frac{1}{16000}$, $\frac{1}{20000}$, $\frac{1}{25000}$, $\frac{1}{28000}$, $\frac{1}{30000}$, $\frac{1}{32000}$, $\frac{1}{35000}$, $\frac{1}{36000}$, $\frac{1}{40000}$, $\frac{1}{45000}$, $\frac{1}{50000}$, $\frac{1}{55000}$, and $\frac{1}{60000}$ of their weight of the crystals were examined.

Although this is a colourless substance it is remarkable how it cuts off nearly all but the visible rays, even when dissolved in 20,000 times its weight of water. When the dilution has reached 25,000, the rays a little beyond 11 Cd and 12 Cd are transmitted ; at 35,000 a second band of rays lying between 18 Cd and 23 Cd makes its appearance ; and at 60,000 there is almost a continuous spectrum nearly as far as 26 Cd, though traces of an absorption band are seen even in presence of 120,000 parts of water.

Para-nitro-phenol, 1·4 (Diagram No. 23).—This substance is strongly coloured yellow, and more than 16,000 parts of water do not yield a solution which is colourless when an inch of liquid is examined.

Solutions were made containing the following proportions of the salt by weight:— $\frac{1}{6000}$, $\frac{1}{8000}$, $\frac{1}{16000}$, $\frac{1}{20000}$, $\frac{1}{32000}$, $\frac{1}{40000}$, $\frac{1}{50000}$, $\frac{1}{55000}$, and $\frac{1}{60000}$.

The curve representative of this substance is very remarkable. The chief features are a narrow band of transmitted rays which must be almost contained within the visible spectrum, which scarcely increases in width even in presence of 16,000 parts of water, together with a second more refrangible group of rays between 17 Cd and 23 Cd, which widens out very considerably.

Meta-nitraniline, $C_6H_4.NO_2.NH_2$, 1·3 (Diagram No. 24).—This body was observed in aqueous solution, containing the following proportions of substance: $\frac{1}{4000}$, $\frac{1}{5000}$, $\frac{1}{6000}$, $\frac{1}{7000}$, $\frac{1}{8000}$, $\frac{1}{20000}$, $\frac{1}{28000}$, $\frac{1}{40000}$, $\frac{1}{60000}$, and $\frac{1}{100000}$.

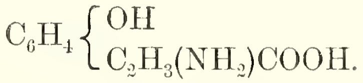
The chief feature in the curve derived from the spectra of this subtance is the long, straight, almost unaltered absorption band at about 9 Cd, which stretches from 8000 to 40,000.

There is still absorption near 23, 24, 25, and 26 Cd, in presence of 120,000 parts of water.

Para-nitraniline, 1·4 (Diagram No. 25).—Solutions containing $\frac{1}{4000}$, $\frac{1}{8000}$, $\frac{1}{16000}$, $\frac{1}{20000}$, $\frac{1}{40000}$, $\frac{1}{60000}$, and $\frac{1}{100000}$ parts by weight of the substance were examined. The general appearance of the curve representing this body resembles in a striking degree that belonging to the corresponding nitrophenol. The absorption band which in presence of 50,000 parts of water lies between the air-line and 10 Cd continues, though narrowed and much enfeebled, until double this dilution has been effected. Throughout the various dilutions the blue and violet rays were absorbed, although more refrangible rays were transmitted.

The following examples afford an instance of one of the applications of this kind of work to the purposes of research.

Tyrosine was for some time considered by BARTH and other chemists to be a derivative of paroxy-benzoic acid, but more recently it has been regarded as an oxyphenyl-amido-propionic acid,



$$C_6H_4\begin{cases} OH \\ C_2H_3(NH_2)COOH. \end{cases}$$

But these views have not been confirmed by the synthesis of the substance.

If it be a derivative of paroxy-benzoic acid it may be expected to partake of the peculiarities of this substance in its action on the ultra-violet rays of the spectrum.

We have accordingly examined solutions of tyrosine (see Diagram No. 26), and find that it more nearly resembles a phenol in actinic quality than an acid in which the carboxyl group replaces hydrogen in the benzene nucleus. Hence the latter view of its constitution, which represents the group oxyphenyl as substituting hydrogen in amido-propionic acid, is more correct than the former, since it is in accordance with its optical character.

The solutions of tyrosine examined contained $\frac{1}{500}$, $\frac{1}{1000}$, $\frac{1}{2000}$, and $\frac{1}{6000}$ of their weight of the substance.

Phlorizine (Diagram No. 27).—It was considered of interest to examine this substance, since it is a glucoside containing the residue of phloroglucin, a body isomeric with pyrogallol, and phloretic acid an oxyphenyl-propionic acid. Solutions containing the following proportions of the substance were taken : $\frac{1}{50}$, $\frac{1}{500}$, $\frac{1}{5000}$, $\frac{1}{10000}$, $\frac{1}{15000}$, $\frac{1}{30000}$, and $\frac{1}{50000}$. The absorption band is well defined and of great intensity; the second band of transmitted rays appearing at $\frac{1}{5000}$ occupies the same position as that seen in tyrosine, though there is not much resemblance between the curves of these two bodies.

It is evident that most interesting results may be anticipated from a continuation of this research, and this contribution must be accepted as rather the bare commencement of the subject than its conclusion.

The following is a summary of the chief points of interest appertaining to benzene and its derivatives :—

(1.) Benzene and the hydrocarbons, alcohols, acids, and amines derived therefrom are remarkable—first, for their powerful absorption of the more refrangible rays; secondly, for the absorption bands made visible by dissolving them in water or alcohol; and thirdly, for the extraordinary intensity of these absorption bands : that is to say, their power of resisting dilution.

(2.) Isomeric bodies containing the benzene nucleus exhibit widely different spectra, inasmuch as their absorption bands vary in position and in intensity.

(3.) The photographic absorption spectra can be employed as a means of identifying organic substances and as a most delicate test of their purity. The curves obtained by co-ordinating the extent of dilution, or in other words the quantity of substance, with the position of the rays of the spectrum transmitted by the solution, form a strongly marked and highly characteristic feature of very many substances.

IX. *On the Microrheometer.*

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[Plate 35.]

The examination of the phenomena of the flow of liquids through capillary tubes, as well as the discovery of the fact that chemical composition influenced the rate of flow, were the work of M. Poiseiulle,* who, in the paper referred to, gave the results of an extended series of carefully conducted experiments, illustrating the laws regulating the rate of flow as affected by the length and diameter of the tube, the pressure acting upon the liquid, and also, within a certain range, by the temperature. His work was carefully examined, in an experimental manner, by a commission of chemists and physicists, and found to be very exact. The formula for the rate of flow which he found was—

$$Q=1836{\cdot}724(1+0{\cdot}0336793T+0{\cdot}0002209936T^2)\frac{HD^4}{L}$$

where—

Q=quantity of liquid passed.
T=temperature in centigrade degrees.
H=pressure on liquid, measured by mercury column.
D=diameter of tube.
L=length of tube.

Poiseiulle pointed out that these results are different from those obtained by other investigators, but the results of others had been obtained by using tubes of much greater diameter, so that the laws regulating the flow of liquids through capillary tubes are different from those relating to wide tubes. The most characteristic law relating to capillary tubes appears to be that the rate of flow varies directly as the pressure; and as it is also the readiest method of testing a tube, I shall only use the term "capillary" for tubes which adhere to this law. POISEIULLE was intending to apply his studies to the action of various substances when in the capillary vessels in the living body, and hence he examined such bodies as alcohol, blood, and various saline solutions. On examining the rate of flow of various mixtures of alcohol and water, it was found that a mixture of about equal quantities of alcohol and water required about three times as much time for passage as either pure alcohol or pure water. The mixture corresponded to a hexhydrate of alcohol, so that, in this case, POISEIULLE discovered a chemical compound by purely physical means. Some results were obtained by dissolving various salts in water, and finding the rates of flow of the solutions; but as there appeared to be no connexion between the combining weights and the rates, POISEIULLE could offer no explanation of the numbers. One thing can be gathered from POISEIULLE'S work—that the flow of a liquid through a capillary tube is very seldom comparable with its physiological action in the veins. It was not for many years after this that any attention was paid to the subject, till GRAHAM,[*] in 1861, noticing the existence of the hydrate of alcohol discovered by POISEIULLE, made some experiments to find whether any hydrates of acids could be discovered by this method; and his researches showed that several acids took up water by three molecules at a time, each addition being shown by a further retardation, but always fainter than the previous one. This addition to the work of POISEIULLE showed that the capillary tube might become an important instrument in the hands of the chemist. Lately,[†] M. AUG. GUÉROUT has presented several papers on the subject to the Academy of Sciences, Paris, giving an account of some experiments on the rate of flow of various alcohols, in which he shows that the rate decreases while the molecular volume increases, but not in any definite proportion. GUÉROUT, however, overlooks the most important factor in this kind of investigation, namely, the temperature. He experiments with all his alcohols at the same temperature, while there is a difference of 60° between the boiling points. Now in such a substance as water, a difference of 60° means that the flow is three times as great at the high as at the low temperature, so that some of the alcohols which by GUÉROUT'S method seemed to offer the greatest retardation, may really, if compared under just conditions, have the least friction. As regards liquids, it would appear from the researches of KOPP that in order to obtain chemical deductions from their physical examination they must be examined when their vapour tensions are equal, as then they exist under somewhat comparable conditions as to mobility and expansion. GUÉROUT has also in the second of these papers determined the effect of variation of temperature on the rate of flow, and comes to the conclusion that the rate for water varies by a constant factor whose value is 1·025. The experiments were, however, done within a very narrow range, viz.: 10° to 17°; and GUÉROUT seems to have been ignorant of the fact that his countryman, POISEIULLE, carefully determined this some forty years earlier, and arrived at a different result. GRAHAM also published a table from 0° to 70°, but GUÉROUT does not appear to be cognisant of GRAHAM'S work. According to GUÉROUT, the curve representing the effect of temperature would be a straight line, whereas both POISEIULLE and GRAHAM'S numbers correspond with a curve. This shall be considered further on.

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* "Recherches Expérimentales sur le Mouvement des Liquids dans les Tubes de très-petits Diamètres," 'Annales de Chimie et de Physique' [3], t. vii., p. 50.

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[*] Phil. Trans., 1861, page 373.

[†] 'Compt. Rend.,' lxxix., 1201; lxxxi., 1025.

About four years ago it occurred to me that as all the work hitherto done with the capillary tube had not fairly tried its usefulness (either by reason of not considering temperature, or of using percentage instead of normal solutions), a fuller investigation of the subject might yield important results. The term "normal solutions" is used in the sense which chemists attach to it, namely, solutions in which the molecular weight of the compound in grammes is dissolved in a litre of water—or, shortly, grammes equivalent per litre. Where this could not be obtained owing to the insolubility of the salt, the strongest possible solution was taken, and its proportional effect calculated. In using the tube of POISEIULLE, which was also used by GRAHAM and GUÉROUT, I found a good deal of inconvenience. The apparatus consists of a tube bent at right angles, the horizontal lower portion being the capillary, and the vertical part consisting of the measuring bulb and tube for applying pressure. The whole was immersed in the liquid whose rate was required, and alternate suction and pressure applied to fill and empty the bulb. This requires so much of the liquid to be experimented upon that I modified the apparatus, as shown in Plate 35, fig. 1, by placing a receiving bulb at the other end of the capillary. But even this had its disadvantages, as the pressure-tube had to be placed alternately at each end, and this disturbance sometimes caused dust to fall into the bulb and choke the tube. It had, however, this advantage over POISEIULLE'S tube—that liquids like bromine could be experimented upon. After working some time with that form, I finally decided to adopt the form of apparatus shown in Plate 35, fig. 2. It consists of a capillary tube at the bottom, attached to which are two bulbs, one for the purpose of measurement and the other as a receiver. Two tubes pass up from these bulbs and end in Y-pieces. One of the Y-tubes from each tube is connected with an inverted Y-tube, and each is also provided with a clip or stop-cock. One of the inverted Y-pieces is connected with the pressure chamber, while the other is left open to the air. By this arrangement the pressure can be turned on to either side, and moreover the apparatus can be filled with any gas other than air. To introduce the liquid to be experimented upon, it is only necessary to undo the connexion of one of the Y-tubes on the measuring bulb side, and run in sufficient liquid. A thermometer is placed between the two bulbs, and its bulb is situated at the mean point of the whole system, thus registering the average temperature of the experiment. In the cases of some mixtures it was found that after some time they deposited a slight sediment which choked the tube. To obviate this another form was devised, as at Plate 35, fig. 3, where the receiving bulb is at the lower end, so that the liquid may be allowed to settle here for some time, and the upper portion forced up into the bulb free from any solid matter.

In his earlier experiments upon gases, GRAHAM gave to the passage of gases through capillary tubes the name "transpiration," which expresses admirably what is meant; but when he came to work with liquids he still used the term "transpiration," although the laws relating to liquids were quite different from those relating to gases; and as the term "transpiration" is not translatable into other languages, being already in use (as in French, for "perspiration"), I have decided to abandon the title, and substitute for it "microrheosis" (μικρός and ῥέω), and to call the instruments above described "microrheometers." The microrheosis of a liquid, then, is the time taken for the passage of a certain volume of liquid through a tube of such dimensions that the rate of flow varies as the pressure, the standard of comparison being in the meantime water.

It appeared probable at the outset that the friction in the microrheometer might be due, in a great measure, to the cohesion of the liquid; and this led me to seek for a method of determining the cohesion of all liquids (which has been published elsewhere),[*] but I soon found that the two phenomena were not comparable—in fact, the cohesion had almost no effect in retarding the flow. I need not quote the work here, as there will be plenty of proof further on to show the real cause of the retardation. In this paper I shall confine myself to the work I have done on saline solutions, and on the substance (water) in which the salts are dissolved. The tube and measuring bulb with which these experiments were done had the following dimensions. The capillary tube had an average diameter of 0·0938 millim., and deviated only ·002 from a circle. Its length was 21 millims., while the capacity of the glass bulb was 4·0530 cub. centims. Both bulb and tube were made of soda glass, whose expansion had been determined by weighing with mercury at different temperatures, with the following results:—

Coefficient from 0° to 50° . . . ·0000252 per degree.
„ „ 50° to 100° . . . ·0000259 „

In these experiments the temperature was measured by a thermometer, graduated to tenths of a degree, which had been compared with a standard. The pressure of the air in the chamber was at first regulated by an air-pump, but I found that I could regulate it much more accurately by blowing with the mouth; and this method was generally used when the pressure was not over 1 metre. In the apparatus shown in Plate 35, fig. 2, it will be seen that during one-half of the duration of the flow the liquid in the bulb from which it is flowing will be at a higher level than that in the other bulb, thus adding to the pressure; but as during the latter half of the flow it is below the level, the one effect neutralizes the other. The pressure was measured by a water-gauge, and the temperature of the water in the column noted, and when it differed from 15° a correction was made either way. In using fig. 3, Plate 35, the pressure was altered to take into account the difference in the level of the two bulbs.

At a temperature of 20°, and under a pressure of 1 metre of water, it was found that it took 131·3″ to empty the bulb as an average of the following 10 experiments. The time was marked by a stop-watch, the bulb being filled to above the mark, and the watch started when the level of the fluid passed the upper mark, and again stopped as it passed the under mark—

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[*] Trans. Roy. Soc. Edin., vol. xxviii., p. 697.

$$131{\cdot}2+131{\cdot}3+131{\cdot}2+131{\cdot}2+131{\cdot}4+131{\cdot}4+131{\cdot}3+131{\cdot}3+131{\cdot}4+131{\cdot}2'' \div 10 = 131{\cdot}29''.$$

It will be seen that this method gives very concordant results, the deviation in these experiments being only $\frac{1}{10}$th of a second, and that most likely due to personal causes. The pressure was then halved—that is, only half a metre of water was used—when the following times were obtained :—

$$263{\cdot}0+262{\cdot}9+262{\cdot}8+263{\cdot}1+263{\cdot}2+263{\cdot}1+263{\cdot}1+262{\cdot}9+263{\cdot}0+263{\cdot}1'' \div 10 = 263{\cdot}02''.$$

As this number is nearly double the former result, the tube may be said to have been capillary, according to the above definition of that term.

The probable error of the mean of the above experiments, calculated by the formula

$$\cdot 6745 \sqrt{\frac{\Sigma\epsilon^2}{n(n-1)}},$$

amounts to ·060″ in the first series, and ·256″ in the second. It appeared that before going into the examination of complex organic bodies, it would be better to attempt to gain some knowledge of the causes of the phenomena presented by the microrheosis of liquids, by thoroughly examining one liquid, to be used as a standard and as a menstruum, and then examining a series of comparable inorganic salts about whose constitution a good deal is already known, dissolved in the standard menstruum. Water was the liquid chosen.

POISEIULLE had noticed that as the temperature rose the rate of passage increased, and at first thought that the loss of density was one of the causes of this; but on examining water at temperatures below 4° he found that the retardation increased very rapidly, whereas the density decreased. GRAHAM also determined the rates for water at different temperatures, but offered no remarks on the phenomena presented. As neither of these investigators determined the rate above 70°, I undertook another examination from 0° to 100°. The temperature was regulated by a large glass water-bath, as shown in Plate 35, fig. 2; and at least 10 experiments were done at each temperature. As a pressure chamber, a large glass reservoir of about 20 litres capacity was used, placed very close to the microrheometer, and connected therewith with tubes of pure caoutchouc (such as toy balloons are made of), of only $\frac{1}{8}$th inch bore. By this arrangement no leakage occurred, and a very slight addition to the air in the reservoir was required after reversing the order of the stop-cocks. The reason for using glass as a reservoir was because it is not nearly so sensitive to slight variations of temperature as copper (used by POISEIULLE), and one has always time to regulate the pressure, even when the temperature of the laboratory is quickly rising or falling.

The following table represents the result of experiments upon water. Each number is the mean of 10 experiments, the probable error from 0° to 60° being under ·2″, and from 60° to 100° under ·4″ :—

Temperature.	Time.	Temperature.	Time.
°	″	°	″
0	235·0	30	106·0
1	220·8	35	96·8
2	211·7	40	88·7
3	205·0	45	81·8
4	198·5	50	75·5
5	192·5	60	65·0
6	187·1	70	57·5
7	181·0	80	49·8
10	167·2	85	47·1
15	146·0	90	45·5
20	131·3	95	44·3
25	115·5	100	43·8

It becomes plainly apparent from this table, as well as from curve No. I., that the retardation increases very rapidly as the temperature approaches the freezing point; but we know that the contraction of water as the temperature falls ceases at 4°, and below that becomes negative. It thus appears probable that we have two separate forces at work which act in the same direction in retarding the rate of flow in the microrheometer, but which act in opposite directions in the case of the thermal expansion of water, the one being due to the vibrations we call heat, and the other being due to the gradual arrangement of the water particles to form crystals of ice. The thermal force is much greater if we have regard to the space between 0° and 100°, but we know that water remains liquid below 0° if undisturbed, and could we cool it till its density were as low as ice the crystallic force would then be enormously greater than the thermic. An attempt was made to trace the microrheometrical curve below 0°, but the motion of the water through the tube always caused freezing. Below is a formula shorter than POISEIULLE'S, which agrees with the curve pretty well. Let $\theta=$ temp. C°. and T= time, then $\theta+6{\cdot}36=158{\cdot}93({\cdot}98618)^{T}$, which gives—

Temperature.	Found.	Calculated.
°	″	″
0	235	231
1	220·8	220·6
2	211·7	211·6
4	198·5	196·2
10	167·2	163·4
20	131·3	130·0
40	88·7	88·4
50	75·5	74·5
60	65·0	62·7
80	49·8	43·8

It is quite evident, on examining the microrheometrical table, that the crystallic force has a retarding action, but what proportion of the retardation it causes could only be determined by examining several liquids and comparing the numbers so obtained with their thermal expansion curves.

Having obtained some knowledge of the menstruum in which the salts were to be dissolved, an examination of the following salts was then commenced. As some of the substances to be examined were only slightly soluble and could not be made of normal strength, the effect of the amount of salt present was first investigated. Pure dry sodium chloride was used for this purpose, and three solutions made up—I. normal, II. half normal, III. quarter normal. The times taken for these three solutions at 20° and under 1 metre of water-pressure were—

		Probable error of mean.
	″	″
I.	140·5	0·26
II.	135·9	0·20
III.	133·6	0·22
Water.	131·3	0·06

Calling water zero, here we have an exact series in which the retardation increases in the same ratio as the salt present. No. III. retards by 2·3″, No. II. by 4·6″, and No. I. by 9·2″; the time is doubled in each case and so is the salt. The same law was found to hold good for the following salts—

Potassic nitrate,
,, chloride,
,, sulphate,
Sodic nitrate,
,, sulphate.

From this law it follows that any body, however slight may be its solubility, may be compared with salts which are soluble enough to form normal solutions.

To find whether or not the microrheosis is affected by the crystalline form of the salt, the following substances were examined:—

Formula.	Crystalline form.
KCl	Cubic.
$NaCl$	Cubic.
KNO_3	Trimetric.
K_2SO_4	Trimetric.

The salts were dissolved to form normal[*] solutions, and examined at what I will call the standard temperature and pressure, viz.: 20° and 1 metre. The following are the rates:—

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[*] By normal solutions I mean normal as to the metal, thus K_2SO_4 is halved in order to render it comparable with KCl, &c.

Salts.	Rates.	Probable error of mean.
	″	″
KCl	129	0·08
NaCl	140·5	0·14
KNO_3	126·4	0·06
$\frac{1}{2}K_2SO_4$	143·6	0·25
Water	131·3	0·06

A single glance at these numbers is sufficient to show that the crystalline form has no recognisable effect on the rate. We find two cubic salts, potassium and sodium chloride, acting in opposite directions, the one accelerating the flow by 2·3″ and the other retarding by 9·2″. These it is true are salts of different metals, but below we have potassic nitrate and sulphate both trimetric, and yet they diverge more widely than the two first, the nitrate accelerating the flow by 4·9″, and the sulphate retarding by 12·3″. Solubility, too, does not account for it, because sodic chloride is more soluble than potassic nitrate, and yet it retards while the nitre accelerates, and potassic sulphate is less soluble than sodic chloride, and retards more than it. In fact, some series go one way and some directly opposite; thus in the case of barium, strontium, and calcium the latter flows least rapidly, and at the same time is most soluble, while in the case of the chloride, bromide, and iodide of potassium the first is both least soluble and flows slowest; so that we see that neither the crystalline form nor the solubility of a salt materially affect the microrheosis of its solution; nor can the specific volume affect it, as we know that isomorphous compounds have equal specific volumes. It appeared then that the phenomena of microrheosis could only be affected by the mass or energy of the salt in solution. It may be as well here to explain that as normal solutions are used, mass is synonymous with "atomic weight," "equivalent proportion," or such other term, and that when the energy of the salt is spoken of it means the amount of work which can be obtained from it by successive combinations till it is degraded to such a state that no more work or energy can be obtained from it, which state I describe as "exhausted." The question is, Does each metal change its value when uniting with a acid, or does each metal and each acid radicle have a value of its own which is constant? To determine this four compounds were chosen, namely, the chlorides and nitrates of potassium and sodium (three of which were already known) and their values found—

Salts.	Time.	Probable error of mean.	Difference $H_2O=0$.	
	″			
KNO_3	126·4	..	−4·9	
KCl_1	129·0	..	−2·3	+4·3
NaCl	140·5	..	+9·2	+4·3
$NaNO_3$	137·9	0·10	+6·6	
Water	131·3	..	0	

When the time taken is less than water I have used a minus sign, although this may require to be changed at a future time, but I thought that if we only consider time at first it saves confusion, hence I use the above arrangement. It will be seen from the above table that each metal and metalloid or acid radicle has a value of its own, because it does not matter which way the compounds be written, the result is the same; whether we take KNO_3+NaCl or $KCl+NaNO_3$, we get a retardation of 4·3″. Thus on mixing the solutions we get as an average of 10 experiments—

Salts.	Time.	Difference.	Probable error of mean.
	″	″	″
KNO_3+NaCl	133·45	+2·15	0·08
$KCl+NaNO_3$	133·45	+2·15	0·10
Water	131·3	..	..

or as the solutions by being mixed are diluted to half strength, on doubling the value we get 4·3″ as before calculated. A series of solutions were then made up and their values determined. The following table gives the results, the numbers given being the average of 10 experiments. The time was determined by a stop-watch which beat fifths of seconds, and I found, after having conducted over 500 experiments, that my personal error was reduced from 0·6 of a second to 0·3, that being the general average of the greater bulk of the work. Each number is the mean of 10 experiments.

Salts.	Time.	Probable error of mean.	Difference.
	″	″	″
Water	131·3	..	0
Potassic nitrate	126·4	0·06	− 4·9
,, chloride	129·0	0·08	− 2·3
,, bromide	126·4	0·15	− 4·9
,, iodide	123·9	0·10	− 7·4
,, sulphate	143·6	0·25	+12·3
Sodic nitrate	137·9	0·10	+ 6·6
,, chloride	140·7	0·14	+ 9·4
,, iodide	135·7	0·06	+ 4·1
,, sulphate	155·3	0·10	+24·0
Ammonic nitrate	125·5	0·08	− 5·8
,, chloride	128·1	0·14	− 3·2
,, sulphate	142·5	0·12	+11·2
Magnesic nitrate	155·0	0·12	+23·7
,, chloride	157·3	0·26	+26·0
,, sulphate	171·9	0·30	+40·6
Baric nitrate	141·4	0·14	+10·1
,, chloride	144·0	0·15	+12·7
Strontic nitrate	143·2	0·00	+11·9
,, chloride	145·8	0·07	+14·5
Calcic nitrate	145·0	0·10	+13·7
,, chloride	147·6	0·05	+16·3
,, sulphate	152·2	1·48	+30·9
Cupric nitrate	150·2	0·12	+18·9
,, chloride	152·9	0·00	+21·6
,, sulphate	167·4	0·20	+36·1

Acids.	Time.	Probable error of mean.	Difference.
	″	″	″
Nitric acid	131·4	0·11	+ 0·1
Hydrochloric acid	136·3	0·23	+ 5·0
Sulphuric acid	141·8	0·07	10·5

Now leaving out the acids, which have evidently a special action on water (for a reason given further on), if we take the difference between the nitrate and chloride of any metal we find it has a constant value. The same holds good for the difference between chloride and sulphate or between the various salts of any two metals. Thus we find a constant value for chlorine and the acid radicles as well as for the individual metals, and these values are extremely constant, as will be seen from the subjoined table. There are a few cases in which there would appear to be a slight divergence, but these are in hydrated salts, and the reason is easily found. The following table gives the differences:—

	Metals.	″
Difference between $\frac{1}{2}N_2O_5$ and Cl in	Potassium	2·7
	Sodium	2·6
	Ammonium	2·6
	Magnesium	2·3
	Barium	2·6
	Strontium	2·6
	Calcium	2·6
	Copper	2·7

Difference between Cl and $\frac{1}{2}SO_3$ in	Potassium	14·6
	Sodium	14·8
	Ammonium	14·4
	Magnesium	14·6
	Calcium	14·6
	Copper	14·5

The total difference between the extremes of the three inorganic acids is an average of 17·2″.

The difference between chloride and bromide in the potassium salts is 2·6″, and between bromide and iodide 2·5″; the total difference between the chloride and iodide in both potassium and sodium salts being 5·1″. Two facts are brought out by these numbers: first, that the bodies containing the greatest amount of energy offer the least resistance in the tube—for instance, nitrates all flow fastest, and sulphates, which are fully exhausted compounds, flow slowest; the second fact is that the greater the mass the less resistance does it offer. We see this in the cases of the chlorides, bromides, and iodides, the last having the highest combining weight, and hence the greatest chemical mass, flow much easier than either of the others. Let us now examine the metals.

Potassium offers the least resistance of the metals (we will not consider ammonium a metal just now), and we will call it zero. Taking all the salts into consideration, we have the following differences :—

Metals.	Differences. ″
Potassium	0
Sodium	+11·6
Barium	+15·0
Strontium	+16·8
Calcium	+18·6
Copper	+23·8
Magnesium	+28·6

Here the second of the above laws comes out very clearly: the greater the chemical mass the less the retardation. In the series of barium, strontium, and calcium, which are analogous metals, this comes out with numerical exactness.

The difference between the metalloids in combination also illustrates these laws. The following table will exhibit this :—

	″
Difference between Cl and Br in K salts	2·6
,, ,, Br and I in ,,	2·5
	5·1
Difference between Cl and I in Na salt	5·1

Here the greater mass of iodine causes the salt to pass much more easily through the microrheometer. We must not forget, however, that the heat of formation of those salts varies very much, and not only has the chloride the lowest mass but it has evolved most energy in its formation. In considering the relation between mass and energy, we had perhaps better look at two series, such as the following :—

Elements.	Equivalents.	Differences.	Rates.	Differences.
Iodine	127	0	−7·4	0
Bromine	80	47	−4·9	2·5
Chlorine	35·5	44·5	−2·3	2·6
Barium	137	0	10·1	0
Strontium	87·6	49·4	11·9	1·8
Calcium	40	47·6	13·7	1·8

Now it will be seen that although the masses of the metallic bodies are considerably greater than those of the metalloids, and their differences are greater, yet the effect in the microrheometer is much less.

Again, looking at the rates of the three inorganic acids, it seems clear that water has a special reaction for each acid, as the rates are not removed from each other by the same differences as when metallic solutions are employed, thus—

Acid.	Rates.	Differences.
	$''$	$''$
HNO_3	131·4	+ 0·1
HCl	136·3	+ 5·0
$\frac{1}{2}H_2SO_4$	141·8	+10·5
Water	131·3	. .

At first thought, we would expect that the acids being salts of the monatomic metal hydrogen, and so exactly comparable with potassium and sodium salts, would show the same microrheometrical reactions; but when we consider, in the first place, the very weak saturative power of hydrogen (as a base), and also that the salt of hydrogen is dissolved in its own oxide, we see that it is probable that the results will not agree with those of a strong monatomic base like potassium. The relation of the acids require a much fuller investigation, as it has lately been shown that acid which has been allowed to stand for some time does not evolve so much heat on saturation with an alkali as acid which has been freshly prepared. Hence on further investigation the above rates may require alteration.

Lastly, as to mixtures. A large number of mixtures have been done, and as they all give exactly the calculated number, only one example need be quoted here by way of illustration. A solution was made up by mixing equal parts of the following normal solutions:—

$$KNO_3,\ K_2SO_4,\ NaNO_3,\ NaCl.$$

Each salt, therefore, exerted only one-fourth of its influence, which gives by calculation a rate of

$$\frac{-4·9}{4}+\frac{+12·3}{4}+\frac{+6·6}{4}+\frac{+9·4}{4}+131·3=136·9''.$$

	Probable error of mean.	
Average of 10 experiments	0·06″	136·85″.

Of course, as each metal and each acid has a value of its own, it does not matter how they are distributed, thus affording further conclusive proof that neither solubility, crystalline form, nor specific volume have any effect upon the microrheosis of a salt.

From the above facts we see that the microrheometer, not merely dealing with the superficial qualities of chemical matter, but bringing to light the fundamental relations of mass and energy, is likely to become the centre of a new branch of the science. My next work will consist of an examination of liquid bodies whose thermal relations are known, so as to find a fundamental relation between the energy expended by elements in forming a certain mass of substance, and the retardation offered by such substances in the microrheometer.

X. *An Experimental Determination of the Values of the Velocities of Normal Propagation of Plane Waves in different directions in a Biaxal Crystal, and a Comparison of the Results with Theory.*

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PART I.

Preliminary.—*Professor* Stokes's *Report to the British Association,* 1862, *with Outline of the Method.*

Section I.—*Review of Previous Experiments and Criticism of* Fresnel's.

In his report to the British Association in 1862, Professor Stokes says: "The exactness of the spheroidal form, assigned by Huyghens to the sheet of the wave surface within Iceland spar, does not seem to have been tested to the same degree of rigour as the ordinary refraction of the ordinary ray; for the methods applied by Wollaston (Phil. Trans., 1802, p. 381) and Malus (Mem. de l'Institut Sav. Étran., tom. ii., p. 303, 1811) for observing the extraordinary refraction can hardly bear comparison for exactness with the method of prismatic refraction adopted for the ordinary ray; and observations on the absolute velocities of propagation in different directions within biaxal crystals are almost wholly wanting.

"This has long been recognized as a desideratum, and it has been suggested to employ for the purpose the displacement of fringes of interference.

"It seems to me that a slight modification of the ordinary method of prismatic refraction would be more convenient and exact.

"Let the crystal to be examined be cut, unless natural faces or planes of cleavage answer the purpose, so as to have two planes inclined at an angle suitable for the measure of refractions; there being at least two natural faces or cleavage planes left undestroyed, so as to permit of an exact measure of the directions of any artificial faces. The prism thus formed having been mounted as usual and placed in any azimuth, let the angle of incidence or emergence (according as the prism remains fixed or turns with the telescope) be measured by observing the light reflected from the surface, and likewise the deviation for several standard-fixed lines in the spectrum; each observation furnishes us with an angle of incidence and the corresponding angle of emergence, the angle of the prism being known. For if ϕ be the angle of incidence, D the deviation, i angle of prism, ψ angle of emergence,

$$D=\phi+\psi-i$$
$$\therefore \psi=D+i-\phi$$

"But without making any supposition as to the law of double refraction, or *assuming anything beyond the truth of* HUYGHENS's *principle*, which, following directly from the superposition of small motions, lies at the base of the whole theory of undulations, we may at once deduce from the directions of incidence and emergence the direction and velocity of propagation of the wave within the crystal. For if a plane wave be incident on any plane refracting surface, it follows directly from HUYGHENS's principle that the refracted wave or waves will be plane, and if ϕ be the angle of incidence, ϕ' the inclination of the refracted wave to the surface, V the velocity of propagation in air, v the wave velocity in the crystal

$$\frac{\sin\phi}{V}=\frac{\sin\phi'}{v}.$$

"And if ϕ' ψ' be the inclination of either refracted wave to the faces of our crystal prism we have

$$v\sin\phi=V\sin\phi' \quad \ldots \ldots \quad (1)$$

$$v\sin\psi=V\sin\psi' \quad \ldots \ldots \quad (2)$$

$$\phi'+\psi'=i \quad \ldots \ldots \quad (3)$$

adding and subtracting (1) and (2) and remembering (3), we get respectively

$$v\sin\frac{\phi+\psi}{2}\cos\frac{\phi-\psi}{2}=V\sin\frac{i}{2}\cos\frac{\phi'-\psi'}{2} \quad \ldots \ldots \quad (4)$$

$$v\cos\frac{\phi+\psi}{2}\sin\frac{\phi-\psi}{2}=V\cos\frac{i}{2}\sin\frac{\phi'-\psi'}{2} \quad \ldots \ldots \quad (5)$$

"By division

$$\tan\frac{\phi+\psi}{2}\cot\frac{\phi-\psi}{2}=\tan\frac{i}{2}\cot\frac{\phi'-\psi'}{2}$$

or

$$\tan\frac{\phi'-\psi'}{2}=\tan\frac{i}{2}\tan\frac{\phi-\psi}{2}\cot\frac{\phi+\psi}{2}. \quad \ldots \ldots \quad (6)$$

"Equations (3) and (6) determine ϕ' and ψ', and v is known from

$$\frac{\sin\phi}{V}=\frac{\sin\phi'}{v}$$

or

$$\frac{\sin\psi}{V}=\frac{\sin\psi'}{v}$$

"Hence we can find the velocity of propagation of the wave, the normal to which lies in a plane perpendicular to the faces of the prism, and makes known angles with those faces, and hence with the crystallographic axes."

In accordance with these suggestions, I undertook a series of observations at the Cavendish Laboratory, Cambridge, which I propose in the present paper to describe; I also wish to discuss the results arrived at, and to compare them with those deduced from FRESNEL's and some of the rival theories of double refraction.

But previous to this it seems natural to devote some space to the consideration of the experiments that have already been made with the same object.

These we may divide into two classes: (1.) Those which have reference to Iceland spar and other uniaxal crystals; (2.) Those in which biaxal crystals were used.

BREWSTER (Thirteenth Report of the British Association) proved conclusively that one wave in Iceland spar obeys strictly the ordinary law of refraction.

SWAN (Edinburgh Trans., vol. xvi., p. 375) obtained by direct measurements with prisms placed in the position of minimum deviation values of the ordinary refractive index, which differ at most by ·00002.

To verify his construction for the extraordinary ray, HUYGHENS himself made but few experiments, and it was not till 1802 that WOLLASTON (Phil. Trans., 1802, p. 381) attempted to test the theory with any degree of exactness.

In 1810 MALUS ("Théorie de la Double Refraction," Paris Mém. des Savants Étrangers, tom. ii., p. 303) undertook a series of experiments with the same object in view.

Rather later, BIOT undertook the same task, while more recently RUDBERG (Pogg. Ann., vol. xiv., p. 45) and MASCART have measured the values of the principal indices by means of prisms cut parallel to the axis.

Since 1862 Professor STOKES has applied the method indicated above to prisms of Iceland spar, and finds the results of experiment agree with HUYGHENS's construction with a possible difference of ·0001 in the values of the refractive index. The details of the experiments are as yet unpublished.

(2.) We must consider next the case of biaxal crystals. Various experimenters have determined the values of the principal refractive indices for different crystals. FRESNEL alone has endeavoured to verify his theory by experiment ('Œuvres Complètes de FRESNEL,' tom. ii., p. 415; second supplement to the "Mémoire sur la Double Refraction"). The method adopted was to observe the displacement of the fringes of interference, formed from two parallel slits, produced by introducing two plates of topaz of the same thickness cut in different directions from the same crystal into the paths of the pencils proceeding from the two slits respectively, and to compare this with the displacement calculated on FRESNEL's theory.

The first set of observations combined with the angle between the optic axes serves to determine the principal velocities a, b, c. a being assumed to be unity, let

$$b^2=a^2-\beta=1-\beta$$
$$c^2=a^2-\gamma=1-\gamma$$

Then

$$\beta\quad=0\cdot00338$$
$$\gamma\quad=0\cdot01222$$
$$\gamma-\beta=0\cdot00884$$

In the second set of observations the plates are inclined to the incident light, so that the light when in the crystal passes along an optic axis.

To calculate the displacement of the fringes on FRESNEL'S theory, it becomes necessary to know the velocity of the light in the crystal; this is given by

$$v_1{}^2=1-\gamma\sin^2\chi$$

χ being the angle between the direction of vibration and the normal to that circular section of the wave surface whose radius is a or unity.

Hence

$$v_1=1-\tfrac{1}{2}\gamma\sin^2\chi$$

neglecting $\gamma^2\sin^4\chi$, &c.

Theory gives for the displacement measured in wave lengths

$$\mathrm{N}=12\cdot73$$

Experiment gives

$$\mathrm{N}=11\cdot87$$

For the other two waves theory gives

$$\mathrm{N}=46\cdot03$$

Experiment gives

$$\mathrm{N}=45\cdot49$$

In the third experiment the light was incident at about 60°.

In this case, ϕ being the angle between the direction of vibration and the normal to the circular section radius b

$$v_1{}^2=1-\beta\cos^2\phi-\gamma\sin^2\phi$$

and to the same approximation

$$v_1=1-\tfrac{1}{2}(\beta\cos^2\phi+\gamma\sin^2\phi)$$

Whence theory gives

$$\mathrm{N}=19\cdot57$$

Experiment gives

$$\mathrm{N}=20\cdot63$$

Thus in each case the agreement between theory and experiment is fairly close; but we must remember that the quantities to be measured are very small, and that in obtaining the theoretical results, small quantities have been neglected, without showing the effects they would produce, and which, though small, may be sufficient to constitute the difference between FRESNEL'S and some rival theory. Moreover, the values of the principal refractive indices in topaz are so nearly equal that any conceivable wave-surface will differ but little from two spheres.

To estimate then the weight which we may attach to these results, let us see how far the experiments are consistent with one of the rival theories.

Now the equation to the surface of wave slowness in the theories of GREEN and CAUCHY involves more constants than the three principal refractive indices of the crystal.

To determine these we must make the principal sections agree with experimental results in more points than the extremities of the axes; and it is probable, therefore, that they will differ but little throughout.

In fact, the existence of these unnecessary constants is a radical defect in both these theories.

Lord RAYLEIGH, however, has proposed a theory in which the only constants are the principal velocities (Phil. Mag., vol. 41, series 4, 1871).

The shape of the surface of wave slowness is also considerably different from that given by FRESNEL'S theory.

Its principal sections consist of a circle and the inverse of an ellipse instead of a circle and an ellipse.

He supposes "the density of the ether in a crystal to be a function of the direction of displacement," while the forces called into play are the same as in a homogeneous medium.

From this he deduces as the equation to determine the velocity of a wave front, of which the direction cosines of the normal are l, m, n, the equation

$$\frac{a^2l^2}{V^2-a^2}+\frac{b^2m^2}{V^2-b^2}+\frac{c^2n^2}{V^2-c^2}=0$$

a, b, c being the principal velocities.

Let us find the velocity for a wave perpendicular to the plane zx, making an angle α with Ox, $l=-\sin\alpha$, $m=0$, $n=\cos\alpha$.

$$V^2(c^2\cos^2\alpha+a^2\sin^2\alpha)=a^2c^2$$

or since $a=1$, $c^2=1-\gamma$

$$V^2\{(1-\gamma)\cos^2\alpha+\sin^2\alpha\}=1-\gamma$$

$$V^2=(1-\gamma)(1-\gamma\cos^2\alpha)^{-1}$$

$$V=(1-\tfrac{1}{2}\gamma)(1+\tfrac{1}{2}\gamma\cos^2\alpha)$$

neglecting γ^2, &c.

$$V=1-\tfrac{1}{2}\gamma(1-\cos^2\alpha)$$

$$=1-\tfrac{1}{2}\gamma\sin^2\alpha$$

But replacing α by χ, this is exactly the form found above for v_1 in FRESNEL's second experiment.

Now let us take a wave perpendicular to the plane yz inclined at an angle ϕ to Oy.

Then $l=0$, $m=\sin\phi$, $n=\cos\phi$, and the equation for V^2 becomes

$$\frac{b^2\sin^2\phi}{V^2-b^2}+\frac{c^2\cos^2\phi}{V^2-c^2}=0$$

$$V^2(b^2\sin^2\phi+c^2\cos^2\phi)=b^2c^2$$

or

$$V^2(1-\beta\sin^2\phi-\gamma\cos^2\phi)=(1-\beta)(1-\gamma)=1-(\beta+\gamma)$$

to the same approximation

$$\begin{aligned}V&=\{1-\tfrac{1}{2}(\beta+\gamma)\}\{1+\tfrac{1}{2}(\beta\sin^2\phi+\gamma\cos^2\phi)\}\\&=1-\tfrac{1}{2}(\beta\cos^2\phi+\gamma\sin^2\phi)\end{aligned}$$

and this agrees with the formula used in FRESNEL's third experiment.

Thus FRESNEL's experiments afford as much a verification of Lord RAYLEIGH's theory as of his own, and are, therefore, an insufficient test of the truth of either.

They are, however, the only attempts made to verify the theory.

Section II.—*Description of the Apparatus used and Method of making the Experiments.*

My own work was undertaken in the hope of obtaining results sufficiently accurate to decide between some of the various theories which have been propounded.

I proceed to describe the apparatus used.

The crystal itself was a piece of arragonite, very clear and of a light colour, obtained from Germany by the Demonstrator at the Cavendish Laboratory, Cambridge, through HILGER, of Tottenham Court Road, London, W.C.

Originally it was in the form of an hexagonal prism.

Four of the faces were those marked m in Professor MILLER's mineralogy, the other two were the faces marked a.

The natural ends had been cut off, and the artificial ends were approximately perpendicular to the m and a faces.

The crystal consisted of several twins, the twin planes being parallel to two of the m faces.

At this stage Professor STOKES kindly undertook to examine the crystal, to determine in which direction to cut it with the greatest advantage for the purposes of the experiment. He found that fair reflexions were obtained from three of the m faces, and decided that they should be left untouched to determine the position of any other cut faces.

Again, the mean axis of elasticity bisects the angle between the contiguous m faces;

the optic axes lie in the plane perpendicular to this axis. A prism formed with its refracting edge nearly parallel to the axis of b, would permit of observations being made in a plane passing nearly through the optic axes; and it was found that, by inclining the faces so that the axis of a nearly bisected the angle of the prism, and making that angle about 42°, light could be passed through the crystal in the direction of either optic axis. It was therefore thought right that the prism should be so cut.

Moreover, by polishing one of the a faces and cutting a face nearly coincident with the fourth m face, a second prism was formed with its refracting edge nearly parallel to c.

It was important to get observations in different zones from the same piece of arragonite, for its chemical constitution is more or less variable, and the constants—*i.e.*, the principal refractive indices—may vary in different specimens.

The work of cutting was successfully performed by HILGER, and the faces of the prisms were very fairly plane.

The crystal being thus cut, it remained to determine accurately the position of the cut faces with reference to the crystallographic axes.

For this purpose, and throughout the observations, I used a goniometer, by GRUBB, kindly lent me by Professor STOKES.

The collimator is fixed to a graduated circle of about 9 inches diameter. The graduations, which are in silver, are on the *flat face*, not the limb of the circle, and the circle is divided to arcs of 10′.

Attached to the reading telescope there are two verniers; the *graduations on the verniers being in the same plane* as those of the circle; and by means of these the circle can be read to angles of 10″.

In the centre of this circle there is a table to hold the prism; the table being attached to a second graduated circle with verniers which can be read to half minutes.

It was requisite throughout the experiments to have the edge of the prism parallel to the axis of revolution of the telescope.

To attain this the prism was attached to a stand, adjustable with set screws, which rested in slots on the table of the goniometer.

The reading telescope was fitted with a needle point instead of the usual cross wires.

In order to level the prism, I covered half the slit, so that when viewed directly the line of junction of the covered and uncovered portions coincided with the needle point.

I then moved the telescope until the images of the slit, formed by reflexion at the faces of the prism, were brought in turn into the field; and altered the levelling screws till the same coincidence was obtained.

Homogeneous light was secured by using a BUNSEN's burner with a platinum wick, one end of which was immersed in a saturated solution of salt-and-water.

By the advice of Professor STOKES, the observations were made at angles of incidence increasing in arithmetic progression.

The direct reading was taken several times during each day's work to avoid error.

The method of observing was as follows:—I set the telescope to a given scale reading, and turned the prism till the image of the slit seen by reflexion at one of its faces coincided with the needle point.

The difference between the direct reading and the known one to which the telescope was set gave the supplement of twice the angle of incidence, whence the angle of incidence is easily obtained. I then turned the telescope till one image of the slit seen by refraction through the prism coincided with the needle point and took the reading.

The difference between this and the direct reading gave the deviation for that wave. The observation was then repeated for the second wave.

I then reset the telescope, so that its axis made an angle of 4° with its former position, and took a similar set of observations. This gave a series of deviations corresponding to angles of incidence increasing by 2°.

Just in the neighbourhood of the optic axes, observations were made at every 1° of incidence.

For the first prism, each observation in the set finally chosen was repeated three times, and the mean of the results taken.

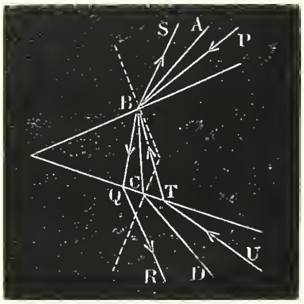
The difference between this mean and any observation was only in a few cases as great as 10″.

In the case of the second prism, the observed values of the deviation on different days for the same angle of incidence agreed so closely, rarely differing by more than 20″, that it seemed unnecessary to take a third set.

I found also that the refracted images were much more distinct for angles of incidence greater than that for minimum deviation, than for angles less.

On passing through the position of minimum deviation, I therefore reversed the prism, so that the face of incidence became that of emergence.

Fig. 1.



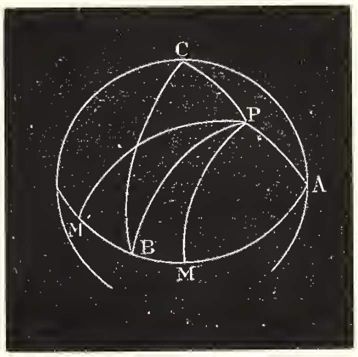
Thus, if A B C D be the ray undergoing minimum deviation, for a ray such as P B Q R, I made P B the incident ray, Q R the emergent; while for a ray like S B T U, I reversed the prism, so that U T became the incident, B S the emergent ray.

The cause of the increased distinctness lies in the fact that as the angle of incidence increases from that giving minimum deviation, the breadth of the image of the slit decreases, and *vice versâ*.

Section III.—*Determination of the Position of the Principal Plane of the First Prism with reference to the Crystallographic Axes.—Values of the Quantities observed in the Experiments, and Calculation of the Reciprocal of the Wave Velocity for the First Prism.*

We proceed now to the measurements made to determine the position of the principal plane of the first prism, of which the faces are P $P_{,}$.

Fig. 2.



Let A B C be the points in which the axes of the crystal cut a sphere, with its centre at the origin.

Let M M′ be the poles of the *m* faces of the crystal, M M′ lie in the great circle A B, and the angle M M′ is known; also B M=B M′; let P $P_{,}$ be the poles of the cut faces of the prism.

Let

$$PM=\theta,\ PM'=\theta'$$

$$P_{,}M=\theta_{,},\ P_{,}M'=\theta_{,}'$$

Let α, β, γ be the angles which O P makes with the axes.

Let $MA=\mu$.

From triangle P A M,

$$\cos PM=\cos AP\cos AM+\sin AP\sin AM\cos PAM$$

or

$$\cos\theta=\cos\alpha\cos\mu+\sin\alpha\sin\mu\cos A$$

From P A M′,

$$\cos\theta'=-\cos\alpha\cos\mu+\sin\alpha\sin\mu\cos A$$

$$\therefore\ \cos\theta-\cos\alpha\cos\mu=\cos\theta'+\cos\alpha\cos\mu$$

$$\therefore\ \cos\alpha=\frac{\cos\theta-\cos\theta'}{2\cos\mu}$$

$$=\frac{\sin\frac{\theta+\theta'}{2}\sin\frac{\theta'-\theta}{2}}{\cos\mu}\quad\ldots\ldots\ldots\ldots\quad(1)$$

Again, from triangle P M B,

$$\cos PM = \cos BP \cos BM + \sin BP \sin BM \cos PBM$$

or

$$\cos \theta = \cos \beta \sin \mu + \sin \beta \cos \mu \cos B$$

and from P B′ M,

$$\cos \theta' = \cos \beta \sin \mu - \sin \beta \cos \mu \cos B$$

$$\therefore \cos \beta = \frac{\cos \theta + \cos \theta'}{2 \sin \mu}$$

$$= \frac{\cos \frac{\theta + \theta'}{2} \cos \frac{\theta' - \theta}{2}}{\sin \mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now the angles θ θ' were determined by observation.

Each angle was observed six times on two different days.

The mean of the results was,

$$\theta = 77° \ 18' \ 44''$$
$$\theta' = 100° \ 43' \ 30''$$

Some difficulty was experienced about the angle μ.

Professor MILLER gives it as 58° 5′; but this, when substituted in the above formulæ, gives a value for P $P_{,}$ the supplement of the angle of the crystal, which differs considerably from the measured angle.

I therefore measured the angle μ and found it to be 58° 30′ nearly.

Mr. GARNETT (the Demonstrator at the Cavendish Laboratory) kindly communicated this result to the Rev. H. P. GURNEY, of Clare College, who was then lecturing for Professor MILLER, and he measured the angle with very nearly the same result as I had already arrived at. Taking, then, the value he obtained, and assuming as before that O B bisects the angle M O M′, we have

$$\mu = 58° \ 28'.^{*}$$

Whence from the formulæ (1) (2)

$$\alpha = 67° \ 10' \ 35'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\beta = 88° \ 52' \ 40'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For the position of $P_{,}$ we have similarly,

$$\theta_{,} = 80° \ 39'$$
$$\theta_{,}' = 101° \ 16' \ 23''$$

Whence,

$$\alpha_{,} = 69° \ 59' \ 20'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\beta_{,} = 91° \ 6' \ 30'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

---

[*] This value of the angle μ is given in DANA's 'Mineralogy,' for specimens of arragonite coming from Silesia. The piece used was obtained from that neighbourhood.

To determine the angles B A P, B A $P_{,}$, the triangles B A P and B′ A $P_{,}$ give $\left(\text{since } AB=\frac{\pi}{2}\right)$

$$\cos PAB=\frac{\cos\beta}{\sin\alpha}$$

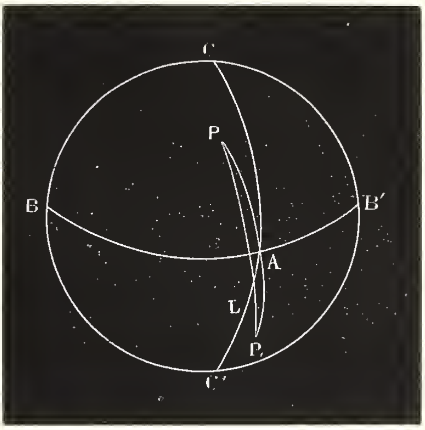
whence

$$BAP=88°\ 46'\ 40''$$

similarly from B′ A $P_{,}$

$$B'AP_{,}=88°\ 49'\ 10''$$

Fig. 3.



From triangle P A $P_{,}$

$$\cos PP_{,}=\cos AP_{,}\cos AP+\sin AP_{,}\sin AP\cos PAP_{,}$$

Whence

$$PP_{,}=137°\ 9'\ 55''$$

The mean of numerous experiments gave

$$PP_{,}=137°\ 9'\ 30''$$

The smallness of the difference affords a strong presumption in favour of the measurements.

Let P $P_{,}$ cut A C′ in L.

Then

$$\sin AP_{,}P=\frac{\sin PAP_{,}}{\sin PP_{,}}\sin AP$$

Whence

$$AP_{,}P=0°\ 3'\ 23'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (7)$$

Again

$$\cot LP_{,}\sin AP_{,}=\cos AP_{,}\cos AP_{,}L+\sin AP_{,}L\cot LAP_{,}$$

$$\therefore LP_{,}=67°\ 27'\ 25''. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8)$$

$$\sin ALP=\frac{\sin AP_{,}\sin LAP_{,}}{\sin LP_{,}}$$

$$ALP=1°\ 12'\ 4'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (9)$$

$$\sin \mathrm{AL} = \frac{\sin \mathrm{AP}_{,} \sin \mathrm{AP}_{,}\mathrm{L}}{\sin \mathrm{ALP}_{,}}$$

$$\mathrm{AL} = 2^{\circ}\ 31'\ 58'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

The solution of the triangle A L P leads to the same results.

We now come to the experimental determination of the deviation and the calculations of the refractive indices.

Throughout the work for this prism, ϕ represents the angle between the normal to the wave incident on or emerging from the face $\mathrm{P}_{,}$ when in air, ϕ' the corresponding angle in the crystal, while ψ ψ' are the angles for the face P. I propose then to give a table of the values of ϕ or ψ according as the wave was incident on $\mathrm{P}_{,}$ or P.

(2.) Of $\mathrm{D}+i$, i being the angle of the prism, $\mathrm{D}+i$ is given because the expression $\frac{\phi+\psi}{2}$, which is equal to $\frac{\mathrm{D}+i}{2}$, occurs in the formula, and it saved trouble to calculate $\mathrm{D}+i$ at once from the readings.

(3.) Of the values of ϕ' deduced from the formulæ

$$\tan \frac{\phi' \sim \psi'}{2} = \tan \frac{i}{2} \tan \frac{\phi \sim \psi}{2} \cot \frac{\phi+\psi}{2}$$

proved in the first section and $\phi'+\psi'=i$.

(4.) Of the values of μ the refractive index as the mean of the results, given by the two formulæ

$$\sin \phi = \mu \sin \phi'$$
$$\sin \psi = \mu \sin \psi'$$

Any difference in the two values of μ found from these two formulæ was due only to the errors necessarily arising from the use of proportional parts, and rarely exceeded ·00001.

I also introduce a list of errors in the calculated quantities which would occur from errors of 10″ in each of the observed quantities, taken so as to produce the maximum effect in the result.

This list was obtained by setting down from the tables the errors to which the supposed errors in D and ϕ would give rise.

The numbers in the table of errors for μ are the digits in the fifth place of decimals.

The next page gives the complete work for one observation as a sample.

Work for reducing one Observation.

Direct reading	263° 39′
Reflexion reading	180° 5′
Deviation reading	297° 34′ 12″
i=angle of prism	42° 52′ 30″

$$\begin{aligned}\phi&=\tfrac{1}{2}\{180-(\text{Direct}-\text{Reflexion})\}\\&=\tfrac{1}{2}(180^\circ\ 5'-83^\circ\ 39')\\&=48^\circ\ 13'.\quad\text{Error}-5''.\end{aligned}$$

$$\begin{aligned}\mathrm{D}+i&=\mathrm{D}-220^\circ\ 48'\ 30''\\&=76^\circ\ 45'\ 42''.\quad\text{Error}+10''.\end{aligned}$$

$$\psi=\mathrm{D}+i-\phi=28^\circ\ 32'\ 42''.\quad\text{Error}+15''.$$

$$\frac{\phi+\psi}{2}=\frac{\mathrm{D}+i}{2}=38^\circ\ 22'\ 51''.\quad\text{Error}+5'.$$

$$\frac{\phi-\psi}{2}=\frac{\phi+\psi}{2}-\psi=9^\circ\ 50'\ 9''.\quad\text{Error}-10''.$$

$$\mathrm{L}\tan\frac{i}{2}=9\cdot5936354.$$

$\mathrm{L}\cot\frac{\phi+\psi}{2}$ 10·1012497	Error. 216	$\mathrm{L}\tan\frac{\phi-\psi}{2}$ 9·2389842	Error. −1252	$\mathrm{L}\tan\frac{\phi'-\psi'}{2}$ 8·9338693	Error. −1036
$\frac{\phi'-\psi'}{2}$ 4° 54′ 30″	−4″	ϕ' 26° 19′ 45″	−4″	ψ' 16° 30′ 45″	+4″
$\mathrm{L}\sin\phi$ 9·8725466	−94	$\mathrm{L}\sin\phi'$ 9·6469205	−170	$\log\mu$ ·2256261	−76
$\mathrm{L}\sin\psi$ 9·6792905	580	$\mathrm{L}\sin\psi'$ 9·4536616	284	$\log\mu$ ·2256289	396
μ from ϕ 1·68123	−3	μ from ψ 1·68124	+15	mean 1·681235	

TABLE giving results of experiments on first prism in the first position.

Outer sheet.					Inner sheet.						
Table VI.		Table V.		Table IV.	Table I.		Table II.		Table III.		
$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.	ϕ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
		″	° ′ ″	° ′ ″	° ′	° ′ ″	° ′ ″	″			
9	1·68500	6	16 17 45	77 4 20	28 13	76 50 16	16 20 2	6	1·68119	9	1
	1·68540		17 22 26	76 39 45	30 13	76 24 53	17 25 8		1·68119		2
	1·68568		18 26 14	76 21 43	32 13	76 6 30	18 29 16		1·68125		3
	1·68578		19 29 9	76 9 10	34 13	75 54 30	19 32 21		1·68134		4
	1·68589		20 30 55	76 2 30	36 13	75 47 30	20 34 27		1·68130		5
5	1·68568	5	21 31 50	75 59 46	38 13	75 45 51	21 35 22	5	1·68129	5	6
	1·68558		22 31 23	76 2 26	40 13	75 48 58	22 35 3		1·68128		7
	1·68517		23 29 57	76 8 41	42 13	75 56 28	23 33 29		1·68119		8
	1·68467		24 27 12	76 19 8	44 13	76 8 43	24 36 22		1·68123		9
	1·68421		25 22 58	76 34 3	45 13	76 24 55	25 25 56		1·68116		10
3	1·68376	4	26 17 11	76 53 13	48 13	76 45 42	26 19 45	4	1·68123	3	11
	1·68302		27 10 5	77 15 40	50 13	77 10 5	27 12 4		1·68113		12
	1·68226		28 1 18	77 42 10	52 13	77 38 48	28 2 33		1·68114		13
	1·68190		28 26 13	77 57 3	53 13	77 54 33	28 27 10		1·68104		14
	1·68153		28 50 41	78 12 53	54 13	78 11 33	28 51 13		1·68106		15
3	1·68131	4	29 14 32	78 30 12	55 13	78 29 15	29 14 54	4	1·68103	3	16
	1·68123		29 37 41	78 49 2	56 13	78 47 35	29 38 16		1·68073		17
	1·68122		30 0 16	79 9 3	57 13	79 6 25	30 1 20		1·68032		18
	1·68122		30 22 19	79 30 8	58 13	79 26 34	30 23 47		1·67996		19
	1·68114		31 4 55	80 15 15	60 13	80 9 33	31 7 20		1·67918		20
2	1·68125	3	31 45 53	81 5 10	62 13	80 56 36	31 48 39	3	1·67841	2	21
	1·68117		32 23 6	81 58 55	64 13	81 48 26	32 27 48		1·67756		22
	1·68123		32 58 34	82 57 30	66 13	82 44 26	33 4 33		1·67674		23
	1·68122		33 31 38	84 0 23	68 13	83 44 56	33 38 51		1·67591		24
	1·68134		34 1 56	85 8 18	70 13	84 50 20	34 10 28		1·67519		25
2	1·68130	3	34 29 48	86 20 25	72 13	86 0 32	34 39 24	3	1·67450	2	26
	1·68121		34 55 0	87 37 10	74 13	87 15 43	35 5 31		1·67388		27
	1·68127		35 17 10	88 59 20	76 13	88 35 51	35 28 48		1·67328		28
	1·68121		35 36 39	90 26 0	78 13	90 1 2	35 49 8		1·67274		29

ϕ is the same for both sheets.

TABLE giving results of experiments on first prism in the second position.

Outer sheet.					Inner sheet.						
Table XII.		Table XI.		Table X.	Table VII.		Table VIII.		Table IX.		
$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.	ψ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
		″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
8	1·68502	6	23 24 20	76 8 53	33 47 5	75 16 5	23 31 25	6	1·68117	8	1
	1·68539		22 32 28	76 1 57	35 47 5	75 48 24	22 29 20		1·68122		2
	1·68570		21 31 43	75 59 53	37 47 5	75 45 32	21 28 7		1·68120		3
	1·68574		20 31 58	76 1 53	39 47 5	75 47 40	20 28 9		1·68119		4
	1·68569		19 33 28	76 8 17	41 47 5	75 54 34	19 29 32		1·68123		5
5	1·68560	5	18 36 17	76 19 0	43 47 5	76 5 45	18 32 16	5	1·68125	5	6
	1·68541		17 40 30	76 33 43	45 47 5	76 21 18	17 36 30		1·68125		7
	1·68517		16 46 12	76 52 38	47 47 5	76 41 0	16 42 17		1·68126		8
	1·68480		15 53 18	77 15 2	49 47 5	77 4 48	15 49 42		1·68126		9
	1·68439		15 2 14	77 41 52	51 47 5	77 32 47	14 58 49		1·68122		10
3	1·68386	4	14 12 45	78 12 16	53 47 5	78 4 34	14 9 48	4	1·68122	3	11
	1·68333		13 25 9	78 46 47	55 47 5	78 40 55	13 21 49		1·68131		12
	1·68277		12 39 29	79 25 24	57 47 5	79 21 4	12 37 43		1·68127		13
	1·68223		11 55 55	80 7 57	59 47 5	80 5 40	11 54 50		1·68130		14
	1·68204		11 35 2	80 31 30	60 47 5	80 29 40	11 34 15		1·68140		15
				80 55 0	61 47 5	80 55 0	11 13 55		1·68122		16

ψ is the same for both sheets.

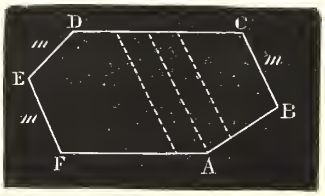
Thus Tables III., VI., IX., and XII. give the values of the refractive indices, or reciprocals of the wave velocity, for the different directions in the crystal given by II., V., VIII., and XI.

We leave here the experimental work for the first prism, and consider the second; reserving for the future the comparison of our results with theory.

Section IV.—*Description of the Second Prism: Position of its Axes.—Observed Values of the Deviations and Angles of Incidence, and Calculation of the Reciprocals of the Wave Velocity.*

It will be remembered that, in describing the cutting of the crystal, I stated that a second prism was formed with its edge nearly parallel to C, and its faces not differing much from the *m* and *a* faces of the crystal.

Fig. 4.



Let the figure, fig. 4, represent a section of the crystal by a plane perpendicular to the *m* faces.

The traces of the faces of the second prism are A B and C D.

The greater part of the face A B lies in the twin crystal, so that the results for the second prism apply to that crystal.

On examination, the *m* face B C was seen to consist of two parts, which are not in the same plane.

Observation showed that the upper half (the prism being placed as in the figure) is in the same zone as the faces D E, E F, while the lower portion is considerably removed from this zone.

I found also that when the faces D E, E F, were levelled, the line of junction of the dark and bright portions of the slit, when seen by reflexion from A B, coincided with the needle point; so that A B is in the same zone as the *m* faces; while the reflexion from C D, though in the field, was too high. Thus if Q $Q_{,}$, fig. 5, be the points where the normal to A B, fig. 4, and that to C D, fig. 4, produced backwards meet the sphere of reference, A, B, C, fig. 5, being the poles of the principal planes; Q lies on the great circle A M B; $Q_{,}$ is slightly below it.

Let M, $M_{,}'$, fig. 5, be the points where the normal to E F, and that to E D produced backwards cut the sphere. Then

$$AM=AM_{,}'=58° 28'.$$

The mean of several observations gave for MQ=103° 58'. Hence

$$AQ=45° 30' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The mean of the observations for M $Q_{,}$ and $M_{,}'$ $Q_{,}$, gave

$$MQ_{,}=57^\circ\ 51'\ 30''=\theta \text{ say};$$
$$M_{,}'Q_{,}=59^\circ\ 9'=\theta' \text{ say};$$

and the triangles M A $Q_{,}$ $M_{,}'$A $Q_{,}$ give,

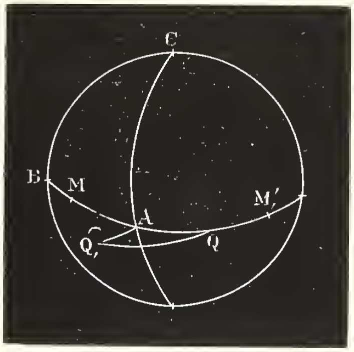
$$\cos AQ_{,}=\frac{\cos\theta+\cos\theta'}{2\cos\mu}$$

where $\mu=AM=M_{,}'A$.

Whence substituting for θ and θ'

$$AQ_{,}=2^\circ\ 43'\ 20'' \quad \ldots\ldots\ldots\ldots \quad (2)$$

Fig. 5.



In the triangle M A $Q_{,}$ the three sides are known; we can thence find the angle M A $Q_{,}$ and obtain

$$MAQ_{,}=76^\circ\ 24''\ 45' \quad \ldots\ldots\ldots\ldots \quad (3)$$

In the triangle Q A $Q_{,}$ the sides A Q, A $Q_{,}$ and the angle Q A $Q_{,}$ are known

$$AQ=\ 45^\circ\ 30'$$
$$AQ_{,}=\ 2^\circ\ 43'\ 20''$$
$$QAQ_{,}=103^\circ\ 35'\ 15''$$

Whence from the formula

$$\cos QQ_{,}=\cos AQ\cos AQ_{,}+\sin AQ\sin AQ_{,}\cos QAQ_{,}$$

we get

$$QQ_{,}=46^\circ\ 12' \text{ (or rather less)}.$$

The mean of several experiments had already given for the value of $QQ_{,}$

46° 10′

These results, differing as they do by less than 2′, afford a verification of the accuracy of the measurements.

The position of the plane Q $Q_,$ is completely determined by the arc A Q and the angle A Q $Q_,$; we have already found,

$$AQ=45° \; 30'$$

For the angle A Q $Q_,$ the triangle Q A $Q_,$ gives

$$\sin AQQ_, = \frac{\sin AQ_,}{\sin QQ_,} \sin QAQ_,$$

whence

$$AQQ_, = 4° \; 37' \; 20'' \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

Thus we see that the principal section of the second prism is inclined to the plane c at an angle of 4° 37′ 20″.

I mentioned above that the second prism lay in the twin crystal. This was proved by passing light through the crystal, in a plane nearly parallel to the principal plane B C, when four images of the slit appeared. On cutting off the end containing the second prism, two images vanished; on cutting off the other end, only the second pair vanished, and the first pair reappeared. The work for the first prism was done with the second pair, the first pair being rarely sufficiently distinct for measurement.

Throughout the work, ϕ ϕ' denote the angles which the normals to the incident or emergent wave, and the wave in the crystal make with the normal to the face Q.

The same remarks as to the reversal of the crystal hold as in the case of the first prism.

TABLE giving the results of experiments on the second prism.

Inner sheet.						Outer sheet.					
Table XV.		Table XIV.		Table XIII.		Table XVI.	Table XVII.		Table XVIII.		
$\delta\mu_1$.	μ_1.	$\delta\phi'$.	ϕ'.	$D+i$.	ϕ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_2.	$\delta\mu_2$.	
		″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
5	1·53026	5	22 45 17	73 44 40	36 17 30	82 54 20	20 35 20	5	1·68312	5	1
	1·53036		23 53 8	73 45 50	38 17 30	82 42 40	21 36 0		1·68330		2
	1·53031		24 59 51	73 50 40	40 17 30	82 37 0	22 35 30		1·68336		3
	1·53028		26 5 11	73 59 30	42 17 30	82 37 20	23 33 37		1·68347		4
	1·53029		27 9 1	74 12 20	44 17 30	82 43 10	24 30 19		1·68357		5
3	1·53021	4	28 11 24	74 28 40	46 17 30	82 53 50	25 25 35	4	1·68363	3	6
	1·53025		29 11 59	74 49 10	48 17 30	83 9 50	26 19 10		1·68378		7
	1·53033		30 10 46	75 13 30	50 17 30	83 30 10	27 11 11		1·68381		8
	1·53036		31 7 42	75 41 30	52 17 30	83 55 25	28 1 21		1·68391		9
	1·53037		32 2 41	76 13 15	54 17 30	84 25 15	28 49 40		1·68400		10
2	1·53040	3	32 55 37	76 48 50	56 17 30	84 59 15	29 36 10	3	1·68401	2	11
	1·53041		33 46 19	77 28 25	58 17 30	85 38 15	30 20 28		1·68413		12
	1·53043		34 34 40	78 12 5	60 17 30	86 21 40	31 2 40		1·68421		13
	1·53045		35 20 36	78 59 50	62 17 30	87 9 50	31 42 34		1·68437		14
	1·53049		36 3 56	79 52 0	64 17 30						15

φ is the same for both sheets.

EXPERIMENTAL results for second prism, second position.

Inner sheet.						Outer sheet.					
Table XXI.		Table XX.		Table XIX.		Table XXII.	Table XXIII.		Table XXIV.		
$\delta\mu_1$.	μ_1.	$\delta\phi'$.	ϕ'.	$D+i$.	ψ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_2.	$\delta\mu_2$.	
		″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
8	1·53027	6	26 4 53	73 59 25	31 42 30	83 53 55	27 58 54	6	1·68388	8	1
	1·53019		24 54 17	73 49 40	33 42 30	83 28 20	26 55 30		1·68376		2
	1·53032		23 44 57	73 45 20	35 42 30	83 1 10	25 53 10		1·68368		3
	1·53027		22 36 37	73 44 50	37 42 30	82 46 5	24 51 53		1·68344		4
	1·53017		21 29 28	73 48 15	39 42 30	82 38 0	23 51 53		1·68333		5
5	1·53018	5	20 23 47	73 56 10	41 42 30	82 36 10	22 53 14	5	1·68330	5	6
	1·53010		19 19 24	74 7 35	43 42 30	82 39 40	21 55 57		1·68320		7
	1·53023		18 16 48	74 23 30	45 42 30	82 48 40	21 0 7		1·68317		8
	1·53028		17 15 44	74 42 55	47 42 30	83 1 50	20 5 38		1·68295		9
	1·53013		16 16 9	75 5 30	49 42 30	83 20 45	19 13 4		1·68289		10
3	1·53017	4	15 18 40	75 32 25	51 42 30	83 43 40	18 21 59	4	1·68269	3	11
	1·53010		14 22 57	76 2 45	53 42 30	84 11 40	17 32 54		1·68264		12
	1·53013		13 29 24	76 37 15	55 42 30	84 44 10	16 45 43		1·68256		13
	1·53010		12 37 57	77 15 30	57 42 30	85 21 30	16 0 36		1·68256		14
	1·53003		11 48 45	77 57 40	59 42 30	86 3 10	15 17 33		1·68254		15
3	1·53010	4	11 2 10	78 44 25	61 42 30	86 48 50	14 36 29	4	1·68231	3	16
	1·53010		10 18 0	79 35 10	63 42 30	87 39 40	13 57 54		1·68237		17
	1·53017		9 36 35	80 30 30	65 42 30	88 35 45	13 21 57		1·68240		18
	1·53003		8 57 33	81 29 35	67 42 30	89 35 25	12 48 1		1·68220		19
	1·53014		8 21 52	82 34 10	69 42 30	90 40 30	12 16 53		1·68218		20

ψ is the same for both sheets.

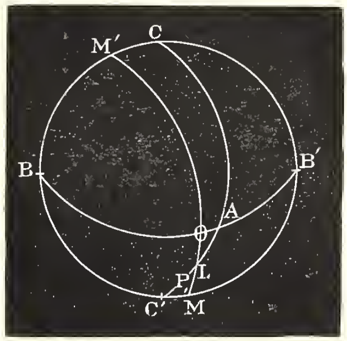
Section V.—*Determination of the Values of the Principal Indices.—Angle between the Optic Axes.*

Our next step is to determine the values of the refractive indices on Fresnel's theory.

It has been shown (Section III.) that the principal plane of the first prism cuts the plane C A C′ in a line inclined at an angle of 2° 31′ 58″ to O A, and makes with that plane an angle of 1° 12′ 4″.

Let us consider the intersection of this plane and the plane B C, B′ C′.

Fig. 6.



It will be a line, M O M′, say, inclined at 1° 12′, or rather less to C O C′.

Take O A, O B, O C, as axes of x, y, z, and let μ_a, μ_b, μ_c be the principal refractive indices, and therefore the semi-axes of the surface of wave slowness.

The sections of the surface of wave slowness on Fresnel's theory by the plane B O C are a circle of radius μ_a and an ellipse axes μ_b, μ_c in the directions of O C, O B respectively.

Hence the values of the radius vector in this direction given by experiment—*i.e.*, the values of the refractive indices for light traversing the crystal in this direction—are μ_a, and the radius vector of an ellipse axes μ_b, μ_c, inclined at about 1° 12′ to μ_b.

To determine which of the experimental values of μ is in this direction we require to find L M, L being the point of intersection of M M′ and A C′.

From the triangle C′ L M, since C′ is a right angle,

$$\cos C'LM = \tan C'L \cot LM$$

Whence

$$LM = 87° \, 28' \, 4'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

And if O $P_{,}$ be as before normal to the face $P_{,}$ of the first prism,

$$LP_{,} = 67° \, 27' \, 25'' \quad \text{[Section III. (8)].}$$

Whence

$$MP_{,} = 20° \, 0' \, 39'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and $MP_{,}$ measures the angle between the wave normal required in the crystal and the normal to the face $P_{,}$.

This is the angle we have denoted by ϕ'.

From Tables II., III., VIII., and IX. we have corresponding values of

ϕ'	μ
19° 32′ 21″	1·68134
20° 34′ 27″	1·68130
20° 28′ 9″	1·68119
19° 29′ 32″	1·68123

Other observations gave

ϕ'	μ
19° 39′ 22″	1·68126
20° 41′ 22″	1·68129

Again, none of the experimental values of μ in Table III., lines 1 to 11, and Table IX., differ greatly from 1·68125, and this is not far from the mean of the six observations recorded above.

We take, then, this as the radius vector of an ellipse axes μ_b, μ_c, inclined at 1° 12′ to μ_b.

To determine μ_c we must have recourse to the second prism.

Did the plane of the second prism coincide with that of $x\ y$, the values of μ in Table XXI. would all be equal to μ_c.

We shall consider later the effect of the inclination of the plane of the prism to that of $x\ y$, but for the present may remark that the values in Table XXI. differ little from 1·53013, the value of μ_c found by RUDBERG.

Let us take therefore

$$\mu_c = 1\cdot53013 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We have then in the plane B O C

$$\frac{1}{\mu^2} = \frac{\cos^2\theta}{\mu_b^2} + \frac{\sin^2\theta}{\mu_c^2}$$

θ being the angle between μ and μ_b.

Now

$$\mu = 1\cdot68125$$
$$\mu_c = 1\cdot53013$$
$$\theta = 1^\circ\ 12'$$

Whence

$$\mu_b = 1\cdot68132 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The value found by RUDBERG is 1·68157.

The difference is considerable, being ·00025.

We may note, however, that his values of μ_b deduced from two different pieces of crystal differed by ·0004.

Now μ_a is the value of the refractive index for the first prism for the wave E when

$$\phi'=20^\circ\ 0'\ 39''$$

The values of μ at about 30′ on either side of this are

1·68578	Table VI., 1·68589
1·68574	Table XII., 1·68569

The value

$$\mu_a=1·68580 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

seems to represent these fairly. Let us take it at present, and see how far it gives consistent results.

Rudberg gives

$$\mu_a=1·68589$$

To verify these results let us find the angle between the optic axes.

If 2δ' be this angle in the crystal

$$\tan\delta'=\sqrt{\left\{\frac{\dfrac{1}{\mu_b^2}-\dfrac{1}{\mu_a^2}}{\dfrac{1}{\mu_c^2}-\dfrac{1}{\mu_b^2}}\right\}}$$

Whence

$$\delta'=9^\circ\ 4'\ 58'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and if 2δ be the angle in air as seen through a face normal to the least axis of the crystal

$$\sin\delta=\mu_b\sin\delta'$$

whence

$$\delta=15^\circ\ 23'\ 30'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Kirchoff found 15° 27′ as the result of experiments.

The value given by Rudberg's values of the indices is

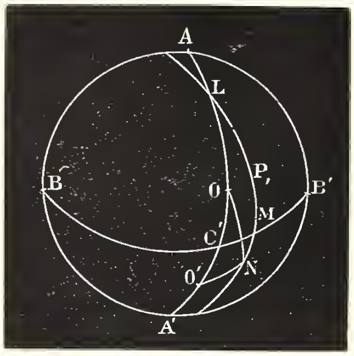
$$\delta=15^\circ\ 7'$$

which differs from Kirchoff's result by 20′, whereas the value given above as the result of my experiments differs only by 3′ 30″.

Section VI.—*Calculation of the Reciprocal of the Velocity of Wave Propagation on* FRESNEL'S *Theory for the First Prism and Comparison with Experiment.—The same for Lord* RAYLEIGH'S *Theory.*

We are now in a position to calculate the reciprocals of the velocity of wave propagation for the first prism on the theory of FRESNEL.

Fig. 7.



Let O O′, fig. 7, be the points in which the optic axes cut a unit sphere.

L M the trace of the principal plane of the prism cutting A C′ A′ in L, B C′ B′ in M.

Let N be any point in the plane, in which a wave normal meets the unit sphere.

Let

$$NO=\theta \quad NO'=\theta'.$$

$$C'LM=\chi=1^\circ\ 12' \quad [\text{Section III. (9)}].$$

$$\cos\theta=\cos LN \cos OL+\sin LN \sin OL \cos\chi=\frac{\cos OL}{\cos\lambda}\cos(LN-\lambda)$$

if $\tan\lambda=\tan OL\cos\chi$.

Similarly

$$\cos\theta'=\frac{\cos O'L}{\cos\lambda'}\cos(LM-\lambda')$$

if $\tan\lambda'=\tan O'L\cos\chi$.

Now

$$AO=90^\circ-9^\circ\ 4'\ 58'' \quad [\text{Section V. (6)}].$$

$$=80^\circ\ 55'\ 2''$$

$$AL=2^\circ\ 31'\ 58'' \quad [\text{Section III. (10)}].$$

$$OL=78^\circ\ 23'\ 4'' \quad \ldots\ldots\ldots \quad (1)$$

$$O'L=96^\circ\ 33' \quad \ldots\ldots\ldots \quad (2)$$

whence

$$\lambda=78^\circ\ 22'\ 55'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\lambda'=96^\circ\ 33'\ \ 5'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Also $\mathrm{P}_{,}$ being the point in which the normal to the face $\mathrm{P}_{,}$ meets the sphere

$$\begin{aligned}\mathrm{LN}&=\mathrm{LP}_{,}+\mathrm{P}_{,}\mathrm{N}\\&=\mathrm{LP}_{,}+\phi'\end{aligned}$$

$$\mathrm{LP}_{,}=67^\circ\ 27'\ 25'' \quad \text{[Section III. (8)].}$$

$$\begin{aligned}\mathrm{LN}-\lambda&=\mathrm{LP}_{,}+\phi'-\lambda\\&=\phi'-10^\circ\ 55'\ 30'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)\end{aligned}$$

$$\begin{aligned}\mathrm{LN}-\lambda'&=\mathrm{LP}_{,}+\phi'-\lambda'\\&=\phi'-29^\circ\ 5'\ 40'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)\end{aligned}$$

Again, if $v_1\ v_2$ are the velocities of propagation in the direction $\theta\ \theta'$, we know that

$$\begin{aligned}v_1^2+v_2^2&=a^2+c^2-(a^2-c^2)\cos\theta\cos\theta'\\v_1^2-v_2^2&=(a^2-c^2)\sin\theta'\sin\theta'\end{aligned}$$

$a\ c$ having here their usual meanings of the greatest and least principal velocities. Hence

$$\begin{aligned}2v_1^2&=a^2+c^2-(a^2-c^2)\cos(\theta+\theta')\\2v_2^2&=a^2+c^2-(a^2-c^2)\cos(\theta-\theta')\end{aligned}$$

From these formulæ we can find $v_1\ v_2$, and thence the values of $\mu_1\ \mu_2$ for

$$\mu_1=\frac{1}{v_1}$$

$$\mu_2=\frac{1}{v_2}$$

Taking

$$\mu_a=1{\cdot}68580$$

$$\mu_c=1{\cdot}53013$$

$$c^2=\frac{1}{\mu_a^2}={\cdot}351876$$

$$a^2=\frac{1}{\mu_c^2}={\cdot}427117$$

TABLE XXV.—Comparison of FRESNEL'S theory with experiment for the first prism in its first position.

	Outer sheet.				Inner sheet.			
	Excess of Experiment over Theory.	μ_2 Experiment.	μ_2 Theory.	$\theta \sim \theta'$.	$\theta + \theta'$.	μ_1 Theory.	μ_1 Experiment.	Excess of Experiment over Theory.
				° ′ ″	° ′ ″			
1	—·00006	1·68500	1·68506	7 21 20	18 21 20	1·68123	1·68119	—·00004
2	—·00002	1·68540	1·68542	5 13 20	18 20 0	1·68124	1·68119	—·00005
3	·00001	1·68568	1·68567	3 7 10	18 19 30	1·68124	1·68125	·00001
4	·00000	1·68578	1·68578	1 2 20	18 19 20	1·68125	1·68134	·00009
5	·00011	1·68589	1·68578	1 0 10	18 19 10	1·68125	1·68130	·00005
6	·00001	1·68568	1·68567	3 0 50	18 19 30	1·68124	1·68129	·00005
7	·00012	1·68558	1·68546	4 59 0	18 20 0	1·68124	1·68128	·00004
8	·00003	1·68517	1·68514	6 54 40	18 20 40	1·68123	1·68119	—·00004
9	—·00007	1·68467	1·68474	8 47 20	18 22 0	1·68122	1·68123	·00001
10	—·00004	1·68421	1·68425	10 36 40	18 24 0	1·68121	1·68116	—·00005
11	·00005	1·68376	1·68371	12 21 40	18 27 0	1·68118	1·68123	·00005
12	—·00010	1·68302	1·68312	14 1 30	18 32 30	1·68114	1·68113	—·00001
13	—·00025	1·68226	1·68251	15 32 40	18 43 40	1·68104	1·68114	·00010
14	—·00033	1·68190	1·68223	16 12 10	18 53 50	1·68096	1·68104	·00008
15	—·00046	1·68153	1·68199	16 45 20	19 9 40	1·68083	1·68106	·00023
16	—·00049	1·68131	1·68180	17 10 0	19 32 50	1·68063	1·68103	·00040
17	—·00044	1·68123	1·68167	17 26 50	20 2 50	1·68036	1·68073	·00037
18	—·00036	1·68122	1·68158	17 37 30	20 37 40	1·68005	1·68032	·00027
19	—·00030	1·68122	1·68152	17 44 50	21 15 40	1·67970	1·67996	·00026
20	—·00032	1·68114	1·68146	17 52 50	22 34 20	1·67893	1·67918	·00025
21	—·00018	1·68125	1·68143	17 56 50	23 52 40	1·67814	1·67841	·00027
22	—·00023	1·68117	1·68140	17 59 50	25 8 20	1·67731	1·67756	·00025
23	—·00016	1·68123	1·68139	18 1 30	26 19 30	1·67652	1·67674	·00022
24	—·00016	1·68122	1·68138	18 2 30	27 27 10	1·67574	1·67591	·00017
25	—·00003	1·68134	1·68137	18 3 30	28 29 30	1·67499	1·67519	·00020
26	—·00007	1·68130	1·68137	18 4 10	29 26 40	1·67428	1·67450	·00022
27	—·00016	1·68121	1·68137	18 4 40	30 18 20	1·67361	1·67388	·00027
28	—·00009	1·68127	1·68136	18 5 10	31 4 30	1·67301	1·67328	·00027
29	—·00015	1·68121	1·68136	18 5 20	31 45 40	1·67247	1·67274	·00027

TABLE XXVI.—Comparison of the results of FRESNEL'S theory with experiment for the first prism in its second position.

	Outer sheet.				Inner sheet.			
	Excess of Experiment over Theory.	μ_2 Experiment.	μ_2 Theory.	$\theta' \backsim \theta$.	$\theta + \theta'$.	μ_1 Theory.	μ_1 Experiment.	Excess of Experiment over Theory.
1	—·00008	1·68502	1·68510	7° 21′ 20″	18° 21′ 20″	1·68123	1·68117	—·00006
2	—·00008	1·68539	1·68547	5 13 20	18 20 0	1·68124	1·68122	—·00002
3	·00003	1·68570	1·68567	3 7 10	18 19 30	1·68124	1·68120	—·00004
4	—·00004	1·68574	1·68578	1 2 20	18 19 20	1·68125	1·68119	—·00006
5	—·00009	1·68569	1·68578	1 0 10	18 19 10	1·68125	1·68123	—·00002
6	—·00009	1·68560	1·68569	3 0 50	18 19 30	1·68124	1·68125	·00001
7	—·00008	1·68541	1·68549	4 59 0	18 20 0	1·68124	1·68125	·00001
8	—·00005	1·68517	1·68522	6 54 40	18 20 40	1·68123	1·68126	·00003
9	—·00009	1·68480	1·68489	8 9 22	18 21 32	1·68123	1·68126	·00003
10	—·00008	1·68439	1·68447	9 49 40	18 23 0	1·68121	1·68122	·00001
11	—·00015	1·68386	1·68401	11 26 3	18 25 13	1·68120	1·68122	·00002
12	—·00018	1·68333	1·68351	12 57 35	18 28 35	1·68116	1·68131	·00015
13	—·00021	1·68277	1·68298	14 22 48	18 34 28	1·68112	1·68127	·00015
14	—·00024	1·68223	1·68247	15 38 55	18 45 15	1·68103	1·68130	·00027
15	—·00019	1·68204	1·68223	16 11 55	18 53 55	1·68096	1·68140	·00044

The Tables XXV. and XXVI. give the results of theory for the first prism.

Let us take the outer sheet first.

In lines 1–12, Table XXV., and lines 1–10, Table XXVI., the agreement between theory and experiment is fairly close, but these series of experiments overlap and are in the immediate neighbourhood of an axis of the section.

From this point onwards, we see that the experimental curve lies entirely inside that given by FRESNEL's theory, and the difference between the two increases as we approach the neighbourhood of the optic axes, line 16, Table XXV., 14 and 15 Table XXVI., while for the rest of Table XXV. the differences diminish.

The tables extend from about 8° on one side of the axis C, to nearly 16° on the other. This is obvious, for between the optic axes $\theta \backsim \theta'$, beyond them $\theta + \theta'$ is very approximately equal to twice N C.

Now let us consider the inner sheet, as before in lines 1–14, Table XXV., lines 1–11, Table XXVI., the agreement is fairly close; but these observations extend from about 6° on one side, to 7° on the other of the maximum radius vector of the section. As in the case of the outer sheet, the differences here increase as we approach the neighbourhood of the optic axes, reaching a maximum of ·00044, line 15, Table XXVI.

After passing the optic axes, the differences decrease, and for the rest of Table XXV. average about ·00024, and in this case the curve given by experiment lies outside that given by theory.

On examining the differences given in Table XXV. for the outer sheet, I noticed that theory and experiment agree closely up to the immediate neighbourhood of the optic axis, line 16; while in the neighbourhood of the optic axis, Table XXVI., line 15, the increase in the differences is much more gradual.

In fact, if we compare the experimental values of μ_1 in Table XXV., lines 11–14, and Table XXVI., 12–15, for about the same values of $\theta + \theta'$ we find considerable differences.

This led me to calculate the results of other sets of experiments covering the same ground, which had been already made.

The following table gives the results of one such set, taken about two months previously; during the interval the goniometer had been taken partially to pieces, and afterwards refocussed and set again.

It seems worth while to give the whole set of observations which cover nearly the same ground as Table II., and form therefore a test of the degree of accuracy of the results.

The measurements were made at angles of incidence, nearly, but not quite, coincident with those of Table II.

In order to compare results, so far as the wave E (inner sheet) is concerned, a correction by interpolation is necessary therefore.

The values of ϕ', it will be observed, differ by about 5′ from those in Table II.

In the table, the first column gives the values of ϕ'; the second the values of μ;

the third those of μ (corrected by interpolation for the wave E); and the fourth the values of μ from Table III.; while the last column gives the differences between the two sets.

Comparison of values of μ deduced from two independent series of observations at an interval of about two months.

TABLE XXVII.—Wave O.

	ϕ'.	μ.
	° ′ ″	
1	16 27 16	1·68131
2	17 32 15	1·68127
3	18 36 18	1·68128
4	19 39 22	1·68126
5	20 41 22	1·68129
6	21 42 8	1·68120
7	22 41 41	1·68128
8	23 39 55	1·68125
9	24 36 48	1·68116
10	25 32 5	1·68127
11	26 25 51	1·68116
12	27 17 37	1·68130
13	28 8 10	1·68116
14	29 32 20	1·68127
15	28 56 42	1·68106

TABLE XXVII. continued.—Wave E.

ϕ'.	μ.	Values of μ reduced to the values of ϕ' in Table II. by interpolation.	Values of μ from Table III. corresponding to the same values of ϕ' as in previous column.	Excess of the first series over the second.
° ′ ″				
29 19 59	1·68108	1·68115		
29 43 27	1·68066	1·68075	1·68073	·00002
30 6 16	1·68029	1·68038	1·68032	·00006
30 28 56	1·67991	1·68000	1·67996	·00004
31 12 9	1·67905	1·67914	1·67918	—·00004
31 53 13	1·67828	1·67837	1·67841	—·00004
32 32 10	1·67743	1·67752	1·67756	—·00004
33 8 28	1·67668	1·67678	1·67674	·00004
33 42 29	1·67590	1·67600	1·67591	·00009
34 13 53	1·67508	1·67518	1·67519	—·00001
34 42 22	1·67450	1·67461	1·67450	·00011
35 8 27	1·67368	1·67380	1·67388	—·00008

Taking the observations for the wave O first, we observe that they do not differ greatly from those given for the same values of ϕ' in Table II., except in the case of those which correspond to lines 12, 13, 14, of Table III., at which points the new values of μ_1 are greater than those given in Table III., and agree better with the observations in the neighbourhood of the other optic axis.

Another set of measurements in the same neighbourhood gave as values of μ_1

1·68124	1·68131
1·68123	1·68125

Hence, on the whole, it seems likely that the results of the observations given in Table III., lines 12, 13, 14, are too small, and that an error has been made in the measurement at that point, owing to the indistinctness of the images which nearly overlapped, so that we may fairly replace Table II., lines 12, 13, 14, by Table XXVII., lines 12, 13, 14, and instead of the results of Table XXV. referring to these we shall have

μ_1 Theory.	μ_1 Experiment.	Difference.
1·68114	1·68130	·00016
1·68104	1·68116	·00012
1·68096	1·68127	·00031

This modification renders the results similar in the neighbourhood of the two optic axes.

Let us now consider the results given in the latter part of the table for the wave E.

The two sets of observations there recorded were made under entirely different circumstances. The telescope and collimator of the goniometer were removed in the interval between them, and both reset and refocussed; and yet in but one case is the difference between the results of the two observations as great as ·0001, while the average difference, irrespective of sign, is about ·00005.

This comparison forms a strong test of the accuracy of the experiments, and the close agreement of the results seems to show that the error in them is at any rate not greater than ·00005.

It would appear that more trustworthy results might have been obtained by combining the series of results registered in this last table with those discussed throughout.

But experiments covering the whole ground embraced in the series we have chosen had not been made at the same time previously.

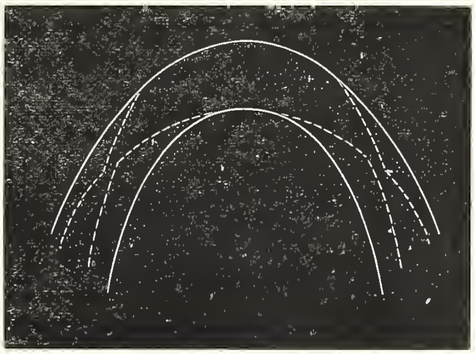
Moreover, we must recollect that each value of ϕ and $D+i$, given in Tables I., IV., &c., is the mean of three for the first prism and of two for the second.

And, again, this comparison assures us that the difference between the results of experiment and theory amounting, as it does in some cases, to ·0005 in the value of μ, is far too great to have arisen from experimental errors or errors of observation.

It seems, then, that assuming the position of the plane of the prism, and the values of the principal axes to be accurately determined, we may assert that in a central section of the wave surface inclined at a small angle to the plane of the optic axes, there is considerable difference between FRESNEL'S theory and experiment: that the differences between the two are most marked in the neighbourhood of the optic axes, and amount there to ·0005 about. For the outer sheet of the surface (except just in the neighbourhood of the principal axes), the theoretical values of the radius vector are uniformly greater than the experimental, while for the inner sheet the reverse is the case.

In fact, the curves of section of the surface, as given by experiment, approach more nearly to one another than those of the surface of wave slowness on FRESNEL'S theory.

Fig. 8.



In the above figure, fig. 8, the dotted line gives the result of experiments, while the strong line gives the form of the section on FRESNEL'S theory.

To compare the results with those given by Lord RAYLEIGH'S theory, we have the equation to the surface, a, b, c being the principal velocities,

$$\frac{l^2}{\frac{V^2}{a^2}-1}+\frac{m^2}{\frac{V^2}{b^2}-1}+\frac{n^2}{\frac{V^2}{c^2}-1}=0$$

Put $V=\frac{1}{r}$, r is a radius vector of the surface of wave slowness

$$\frac{a^2l^2}{1-a^2r^2}+\frac{b^2m^2}{1-b^2r^2}+\frac{c^2n^2}{1-c^2r^2}=0$$

Let us suppose as a first approximation that $m=0$, *i.e.*, that the plane of the prism is coincident with a principal plane of the crystal,

$$a^2l^2(1-c^2r^2)+c^2n^2(1-a^2r^2)=0$$

$$r^2a^2c^2=a^2l^2+{}^2cn^2$$

$$r^2=\frac{l^2}{c^2}+\frac{n^2}{a^2}$$

Also $l=\cos\theta$, $n=\sin\theta$

θ being the angle between r and the maximum radius vector

$$r^2=\frac{1}{c^2}-\left(\frac{1}{c^2}-\frac{1}{a^2}\right)\sin^2\theta$$

$$\frac{1}{c^2}=2{\cdot}84192$$

$$\frac{1}{c^2}-\frac{1}{a^2}={\cdot}50062.$$

From these and the known values of θ, μ or r can be calculated. The table gives the result.

Comparison between Lord RAYLEIGH'S Theory and Experiment.

θ.	μ Theory.	μ Experiment.	Excess of Theory over Experiment.
° ′ ″			
1 31 10	1·68570	1·68568	·00002
8 25 33	1·68261	1·68190	·00071
11 6 40	1·68028	1·67918	·00110
14 9 48	1·67688	1·67519	·00169
15 48 28	1·67476	1·67274	·00202

The differences between theory and experiment are so marked that it seems unnecessary to calculate the values of μ for smaller intervals in the values of θ.

And though some of the apparent difference may be due to the error made in assuming the principal plane of the prism to coincide with one of the crystal, that cannot account for the whole; for we have seen that in FRESNEL'S surface the error made by the same assumption appears only in the fourth place of decimals, in the value of the refractive index, while the differences between Lord RAYLEIGH'S theory and experiment show themselves in the third place, and tend to increase with θ.

Thus it seems clear that Lord RAYLEIGH'S theory will not account for the phenomena of double refraction in arragonite. This result agrees with that arrived at by Professor STOKES for Iceland spar.

Section VII.—*Effect of Varying Constants of Theory and Position of Plane of Prism with reference to Axes of Crystal.*

It remains now to discuss the effects of variations in the values of the axes a, b, c.

The values of c and b are determined directly from observation, and in the neighbourhood of these axes the agreement is most close. Any alteration in them would affect especially that part of the section which lies between the optic axes, and for which the differences between theory and experiment are least; let us consider, therefore, a variation in the value of a.

Such a variation would produce an alteration in the angle between the optic axes, and therefore in the values of θ, θ' for any wave normal.

If, however, we take a wave normal at some distance from the optic axes, the values of θ, θ' will be altered by nearly equal amounts in opposite directions, and $\theta+\theta'$ will be nearly constant.

Let us then find the value for a in order that the experimental value in line 29 of Table XXV. may agree with theory.

We have

$$\begin{aligned} 2v_1^2 &= a^2+c^2-(a^2-c^2)\cos(\theta+\theta') \\ &= a^2\{1-\cos(\theta+\theta')\} \\ &\quad +c^2\{1+\cos(\theta+\theta')\} \end{aligned}$$

$$\therefore a^2=\frac{v_1^2-c^2\cos^2\dfrac{\theta+\theta}{2}}{\sin^2\dfrac{\theta+\theta'}{2}}$$

$$v_1=\frac{1}{\mu_1}$$

$$\mu_1=1{\cdot}67274$$

$$\theta+\theta'=31^\circ\ 45'\ 40''$$

$$c^2={\cdot}351876$$

From these we get

$$a^2={\cdot}447553$$

as against ·427117, and the value of μ_c is

$$\mu_c=1{\cdot}49478$$

instead of $\qquad 1{\cdot}53013$

A reference to the tables for the second prism shows at once the impossibility of such a change.

It is true that any decrease in μ_c would produce a decrease in the angle between the optic axes. This would increase θ' and decrease θ by nearly equal amounts, and on the whole produce a small decrease in $\theta+\theta'$.

This change would decrease v_1 and therefore increase μ_1, and hence to produce the required change in μ_1—*i.e.*, to make μ_1 equal to the experimental value in line 29—we must take a value for μ_c larger, but only slightly larger, than that given above.

There remains now for discussion the effect of a variation in the position of the plane of the prism with reference to the crystallographic axes.

We shall consider this by discussing separately the effect of a rotation about each of three lines through the centre approximately at right angles. These lines are

(1.) The normal to the principal plane of the prism; (2.) The intersection of this plane with the plane B C B′; (3.) The intersection of the principal plane with the plane A C A′.

(1.) Rotation about a normal to the principal plane of the prism.

This cannot produce the required effect, for though we can thus decrease the differences on one side of the maximum radius vector of the section, we increase those on the other side by about the same amounts.

(2.) Rotation about the intersection with B C B′.

This is of no use, for the effects produced on opposite sides of the maximum radius vector are again of an opposite kind.

(3.) There remains, therefore, only the rotation about the line of intersection with the plane C A C′, or a variation in the angle χ. This change is to be such that the sections of the outer and inner sheets may be brought into closer proximity; χ must therefore be decreased.

Now on referring to the tables of the experimental results, it will be seen that none of those in Table III., lines 1–11, Table VI., lines 16–29, and Table IX., lines 1–16, differ greatly from 1·68125.

If it were then permissible to neglect the angle χ, this would be the value of μ_b, and the other section would be on FRESNEL'S theory an ellipse of μ_a, μ_c.

In the abstract published in the 'Proceedings of the Royal Society,' No. 188, 1878, this has been done, and a limit assigned to the error thus introduced.

Professor STOKES has since pointed out that near the optic axes the limit is considerably too small, and thus led me to the accurate calculation of the theoretical values of μ_1, μ_2 given above.

The effect of a decrease in the angle χ may, therefore, be estimated by referring to the calculations for the case in which we put $\chi=0$.

The accompanying table gives the results. θ being the angle between the wave normal and the major axis of the ellipse, which is given by

$$\theta=\phi'\backsim\theta_0$$

where θ_0 is the angle between the normal to the face of the prism and major axis, and is

$$=20^\circ\ 0'\ 40''\quad\text{[Section V. (2)]}.$$

Table XXVIII.

Tables from which experimental result is taken.	θ.	r or μ from Theory.	r or μ from Experiment.	Excess of Experiment over Theory.
	° ′ ″			
XI.	8 25 38	1·68193	1·68204	
,,	8 4 45	1·68224	1·68223	− 1
,,	7 21 11	1·68284	1·68277	− 7
,,	6 35 31	1·68342	1·68333	− 9
,,	5 47 55	1·68396	1·68386	−10
,,	4 58 26	1·68444	1·68439	− 5
,,	4 7 22	1·68486	1·68480	− 6
V.	3 42 55	1·68504	1·68500	− 4
XI.	3 14 28	1·68522	1·68517	− 5
V.	2 38 14	1·68542	1·68540	− 2
XI.	2 20 10	1·68550	1·68541	− 9
V.	1 34 26	1·68566	1·68570	+ 4
XI.	1 23 23	1·68570	1·68560	−10
V.	0 31 31	1·68578	1·68578	− 0
XI.	0 27 12	1·68578	1·68569	− 9
V.	0 30 15	1·68578	1·68589	+11
XI.	0 31 18	1·68577	1·68574	− 3
XI.	1 31 3	1·68567	1·68570	+ 3
V.	1 31 10	1·68567	1·68568	+ 1
,,	2 30 43	1·68544	1·68560	..
XI.	2 31 48	1·68544	1·68539	− 5
V.	3 29 17	1·68515	1·68517	+ 2
XI.	3 33 40	1·68510	1·68502	− 8
V.	4 26 32	1·68472	1·68467	− 5
,,	5 22 18	1·68422	1·68421	− 1
,,	6 16 31	1·68367	1·68376	+ 9
,,	7 9 25	1·68300	1·68302	+ 2
,,	8 0 38	1·68230	1·68226	− 4
,,	8 25 33	1·68194	1·68190	− 4
,,	8 50 1	1·68156	1·68153	− 3
II.	9 14 14	1·68118	1·68131	..
,,	9 37 36	1·68077	1·68073	− 4
,,	10 0 40	1·68037	1·68032	− 5
,,	10 23 7	1·67997	1·67996	− 1
,,	11 6 40	1·67914	1·67918	+ 4
,,	11 47 59	1·67831	1·67841	+10
,,	12 27 8	1·67748	1·67756	+ 8
,,	13 3 53	1·67666	1·67674	+ 8
,,	13 38 11	1·67587	1·67591	+ 4
,,	14 9 48	1·67511	1·67519	+ 8
,,	14 38 44	1·67439	1·67450	+11
,,	15 4 51	1·67373	1·67388	+15
,,	15 28 8	1·67310	1·67328	+18
,,	15 48 28	1·67250	1·67274	+24

For about 10° on either side of the major axis (the passage across which is indicated by the dark line in the table) the results of theory and experiment agree fairly well.

The average difference irrespective of sign is (if we neglect the values for $\theta=2° \; 30' \; 43''$ and for $\theta=9° \; 14' \; 14''$, which gives a point on the circle of radius 1·68125) ·000047, and only in two cases does the difference amount to ·0001.

But when we come to values for θ greater than 11° we see that the experimental value of μ is, as before, always greater than the theoretical, and that the difference increases with θ, and at last becomes nearly the same as in Table XXV.

Thus this change does not produce agreement between the two.

We may, however, consider shortly what alteration in μ_c would in this case bring theory and experiment more closely into agreement.

We have

$$\frac{1}{\mu^2}=\frac{\cos^2\theta}{\mu_a^2}+\frac{\sin^2\theta}{\mu_c^2}$$

$$\therefore \delta\mu=\frac{\mu^3}{\mu_c^3}\sin^2\theta\delta\mu_c$$

Taking the last values on Table XXVIII.

$$\delta\mu=\cdot00024$$

$$\theta=15° \; 48'$$

$$\mu=1\cdot67274$$

$$\mu_c=1\cdot53013$$

Whence

$$\delta\mu_c=\cdot00248$$

so that the minor axis of the ellipse, which having the same major axis as the experimental curve would pass through the extreme experimental point, is

$$\mu_c=1\cdot53261$$

instead of 1·53013

And on referring to Table XXI., which gives the values of μ_c very approximately, we see that this value is quite out of the question.

Thus a decrease in the angle χ will not produce the required effect.

Neither will an increase. For the first part of the inner sheet in Table III., and throughout Table IX., we must have the theoretical value of μ_1 and therefore of $\theta+\theta'$ nearly constant; and this is clearly impossible along a plane section inclined at a finite angle to the plane of the optic axes.

Thus a change in χ will not render theory and experiment consistent.

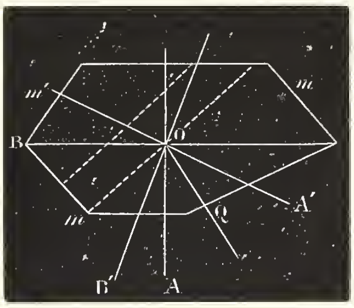
Thus no change in the position of the plane of the prism will bring FRESNEL'S theory into agreement with experiment.

Section VIII.—*The Theoretical Investigations for the Second Prism on* FRESNEL'S *Theory with Comparison with Experiment.*

Our next step is the theoretical investigation for the second prism.

Now experiment has shown that this prism lies in the twin crystal; we must therefore find the axes of this crystal.

Fig. 9.



Let the figure represent a section of the crystal by a plane perpendicular to the axis O C.

The twinning takes place about an axis perpendicular to m', and consists of a rotation through 180° about that axis.

Its effect, therefore, is to bring O B, fig. 9, into the position O B', where

$$\text{angle } m\text{OB}=m\text{OB}'$$

that is, to turn the axes in the direction B to A, through an angle,

$$\begin{aligned}&=2(m\text{OB})=2\left(\frac{\pi}{2}-\text{AO}m\right)\\&=\pi-2\mu=180^\circ-116^\circ\ 56'\\&=63^\circ\ 4'\end{aligned}$$

and if Q is the point where the perpendicular on the face Q of the second prism meets the sphere, we know that

$$\text{AQ}=45^\circ\ 30'\quad[\text{Section IV. (1)}].$$

Hence since $\text{AA}'=63^\circ\ 4'$ and Q lies in plane A O B,

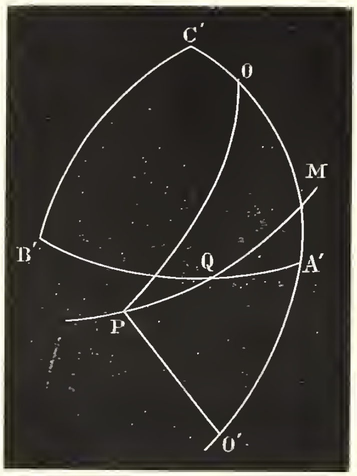
$$\text{A}'\text{OQ}=17^\circ\ 34'. \qquad (1)$$

Now let A' B' C', fig. 10, be the points in which the new axes cut the sphere, and let

P Q M be the plane of the prism; P being any point in the plane; Q the normal to the face, and M the point in which the plane cuts B′ C′, O, O′ the optic axes.

Then if P is the pole of any wave in the crystal, $PQ=\phi'$.

Fig. 10.



Let $PO=\theta$, $PO'=\theta'$, we shall require the values of θ, θ'.

Now in triangle A′ M Q

$$\text{angle } Q=4^\circ\ 37'\ 20''\quad [\text{Section IV. (4)}].$$

$$A'Q=17^\circ\ 34'$$

$$\cot MQ=\cos Q\cot A'Q$$

whence

$$MQ=17^\circ\ 37'\ 14''\quad\ldots\ldots\ldots\ldots\quad(2)$$

Let $A'MQ=\chi$.

Then from triangle A′ M Q,

$$\cos M=\cos QA'\sin Q.$$

whence

$$\chi=85^\circ\ 35'\ 40''\quad\ldots\ldots\ldots\ldots\quad(3)$$

Now,

$$\cos\theta'=\cos PM\cos MO'+\sin PM\sin MO'\cos\chi$$

$$=\cos MO'(\cos PM+\sin PM\tan MO'\cos\chi)$$

put

$$\tan MO'\cos\chi=\tan\lambda'$$

$$\cos\theta'=\frac{\cos MO'\cos(PM-\lambda')}{\cos\lambda'}\quad\ldots\ldots\ldots\ldots\quad(4)$$

Similarly

$$\cos\theta=\frac{\cos \mathrm{MO} \cos(\mathrm{PM}+\lambda)}{\cos\lambda} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (5)$$

where

$$\tan\lambda=\tan \mathrm{MO}\cos\chi$$

Now

$$\mathrm{A'O}=\mathrm{A'O'}=90^\circ-9^\circ\ 4'\ 58'' \quad [\text{Section V. (6)}].$$
$$=80^\circ\ 55'\ 2''$$

From triangle A′ M Q

$$\sin \mathrm{A'M}=\sin \mathrm{MQ}\sin \mathrm{Q}$$

Whence

$$\mathrm{A'M}=1^\circ\ 23'\ 52'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (6)$$

$$\therefore\ \mathrm{MO}=79^\circ\ 31'\ 10''. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (7)$$

$$\mathrm{MO'}=82^\circ\ 18'\ 54''. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (8)$$

Hence

$$\lambda=22^\circ\ 33'\ 2''. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (9)$$

$$\lambda'=29^\circ\ 39'\ 5'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (10)$$

$$\mathrm{PM}+\lambda=\phi'+\mathrm{QM}+\lambda$$
$$=\phi'+40^\circ\ 10'\ 16'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (11)$$

$$\mathrm{PM}-\lambda'=\phi'+\mathrm{QM}-\lambda'$$
$$=\phi'-12^\circ\ 1'\ 51'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (12)$$

Substituting these values of λ, MO, PM+λ, &c. (the values of ϕ' being taken from Tables XIV., XVII., and XX.), in equations (4), (5), we can get a series of values for θ θ'.

And as before [Section VI.], remembering that θ' here is $\pi-\theta'$ in Section VI.,

$$2v_1^2=a^2+c^2+(a^2-c^2)\cos(\theta-\theta')$$
$$2v_2^2=a^2+c^2+(a^2-c^2)\cos(\theta+\theta')$$

Thus we obtain a series of values for v_1, v_2, and hence for μ_1, μ_2, given in Table XXIX.

To obtain the distance of P from the minimum radius vector, we have to add to ϕ' the angle M Q=17° 37′ 14″ [Section VIII. (2)].

TABLE XXIX.—Comparison of FRESNEL'S theory and experiment for second prism.

	Outer sheet.					Inner sheet.				
	ϕ' from Tables XVII. and XXIII.	Excess of Experiment over Theory.	μ_2 Experiment.	μ_2 Theory.	$\theta+\theta'$.	$\theta-\theta'$.	μ_1 Theory.	μ_1 Experiment.	Excess of Experiment over Theory.	ϕ' from Tables XIV. and XX.
	° ′ ″				° ′ ″	° ′ ″				° ′ ″
1	12 16 53	−·00024	1·68218	1·68242	164 15 15	1 17 0	1·53014	1·53014	..	8 21 52
2	12 48 1	−·00026	1·68220	1·68246	164 20 17					
3	13 21 57	−·00010	1·68240	1·68250	164 25 43					
4	13 57 54	−·00027	1·68227	1·68254	164 31 42					
5	14 36 29	−·00028	1·68231	1·68259	164 38 7			1·53010		
6	15 17 33	−·00009	1·68254	1·68263	164 45 9	1 57 45	1·53016	1·53003	−·00013	11 48 45
7	16 0 36	−·00013	1·68256	1·68269	164 52 32			1·53010		
8	16 45 43	−·00016	1·68256	1·68272	165 0 34					
9	17 32 54	−·00015	1·68264	1·68279	165 8 56					
10	18 21 59	−·00015	1·68269	1·68286	165 18 10					
11	19 13 4	−·00003	1·68289	1·68292	165 27 46	2 35 0	1·53019	1·53013	−·00006	16 16 9
12	20 5 38	−·00003	1·68295	1·68298	165 37 52					
13	20 35 20	+·00010	1·68312	1·68302	165 43 32					
14	21 36 0	+·00020	1·68330	1·68310	165 55 42					
15	22 35 30	+·00019	1·68336	1·68317	166 7 38					
16	23 33 27	+·00022	1·68347	1·68325	166 19 42	4 2 10	1·53028	1·53028	..	26 5 11
17	24 30 19	+·00025	1·68357	1·68332	166 31 32					
18	25 25 35	+·00024	1·68363	1·68339	166 43 48					
19	26 19 10	+·00032	1·68378	1·68346	166 55 6					
20	27 11 11	+·00028	1·68381	1·68353	167 6 38					
21	28 1 21	+·00031	1·68391	1·68360	167 17 56	4 43 34	1·53035	1·53036	+·00001	31 7 42
22	28 49 40	+·00036	1·68400	1·68366	167 28 54					
23	29 36 10	+·00029	1·68401	1·68372	167 39 32					
24	30 20 28	+·00035	1·68413	1·68378	167 50 0					
25	31 2 40	+·00038	1·68421	1·68383	167 59 56					
26	31 42 34	+·00048	1·68437	1·68389	167 9 29	5 24 56	1·53043	1·53045	+·00002	35 20 36

Considering first the inner sheet, we see that theory and experiment agree closely.

The greatest difference is ·00013; and in this case, by referring to the adjacent experimental values, it is clear that the value 1·53003 is too small.

We must remember, however, that the investigations are concerned with that sheet of the surface of wave slowness of which the section by the plane, $x\,y$, is a circle, and that the section considered is inclined at an angle less than 5° to this plane; so that the form of the curve of section differs but little from a circle, and we may therefore reasonably suppose that any theory in which one of the principal sections of the surface of wave slowness is a circle, would also agree fairly closely with our results.

We therefore proceed to discuss the results for the outer sheet given in the table just preceding.

And here we see at once that there is considerable difference between theory and experiment.

Moreover, from $\phi'=12^\circ\ 16'\ 53''$ to $\phi'=20^\circ\ 5'\ 38''$, the theoretical values of μ are uniformly greater than the experimental; while from $\phi'=20^\circ\ 35'\ 20''$ to $\phi'=31^\circ\ 42'\ 34''$ the reverse is the case.

The difference between the two increases on the whole uniformly, and in the last case is as great as ·0005.

For the values $\phi'=19^\circ\ 13'\ 4''$ and $\phi'=20^\circ\ 5'\ 38''$, theory and experiment agree closely.

To find the angle between any radius vector and the minimum one of the curve, we have to add to the value of ϕ' the angle M Q, which is 17° 37′ 14″.

Let us call the angle Q O P, θ.

Then we see that for the angles $\theta=0$, $\theta=37^\circ$ (about), and $\theta=90^\circ$, the curves given by theory and experiment agree.

From $\theta=29^\circ\ 54'\ 7''$ to $\theta=37^\circ$ (about), the theoretical curve lies outside the experimental.

From $\theta=37^\circ$ (about) to $\theta=49^\circ\ 19'\ 48''$ the reverse is the case.

In the neighbourhood of the axes the two must agree more closely.

These conditions would all be satisfied by a curve which agrees with FRESNEL's section at the extremity of the minimum radius vector; lies inside it for about 40°; then cuts FRESNEL's curve again, and lies outside for the rest of the quadrant, agreeing again at the extremity of the maximum radius vector.

At first sight there seems some discrepancy between this result and that arrived at already in Table XXV. for the first prism.

But closer examination shows that the two results are corroborative.

For, in the first place, the observations there tabulated apply to the neighbourhood of the major axis of the elliptic section.

For about 6° on either side of this principal axis, the differences between theory and experiment are sometimes positive, sometimes negative; that is, between these limits FRESNEL's section satisfies the results of experiment; but in the last eighteen observations there recorded, the experimental value of μ_1 is always greater than the theoretical.

Thus the experimental results are represented by a curve, which near the major axis agrees with FRESNEL'S section, but which, as we go from 6° to 16° away from that axis, differs from that section, and lies outside the section.

This result, viz.: that on going away from the maximum radius vector the curve given by experiment should lie outside that given by theory, is exactly that to which we have been led by the work for the second prism.

The cutting of the crystal would not allow of observations being taken near the minor axis of the ellipse, in the case of the first prism.

Of course, it has here been assumed that the sections of the surface of wave slowness, by the principal planes, are curves of the same kind; or rather the results of experiment lead to that as a fact.

Lord RAYLEIGH'S theory needs no further discussion here, for his curve being the inverse of an ellipse, always lies outside the ellipse with the same axes, and therefore cannot agree with experiment.

Section IX.—*Possibility of an Error in the Position of the Crystallographic Axes discussed.—No Section of the Surface of Wave Slowness in* FRESNEL'S *Theory can agree with the experimental Results.*

But the possibility remains that there is an error in the determination of the position of the faces of the prisms, especially in the second case, with respect to the crystallographic axes.

The probability of any error is small, as the observations were repeated on several different occasions with nearly coincident results.

To make certain, however, let us solve the inverse problem. Having given the two values, μ_1 μ_2, of the radii vectores of the surface of wave slowness drawn in the same direction l m n in the crystal to find l, m, n.

We have as the equation to find v on FRESNEL'S theory

$$\frac{l^2}{v^2-a^2}+\frac{m^2}{v^2-b^2}+\frac{n^2}{v^2-c^2}=0$$

Whence

$$v^4-v^2\{l^2(b^2+c^2)+m^2(c^2+a^2)+n^2(a^2+b^2)\}+l^2b^2c^2+m^2c^2a^2+n^2a^2b^2=0$$

∴ if v_1 v_2 are the roots of this equation, that is, the velocities for the same wave front

$$v_1^2v_2^2=l^2b^2c^2+m^2c^2a^2+n^2a^2b^2$$
$$v_1^2+v_2^2=l^2(b^2+c^2)+m^2(c^2+a^2)+n^2(a^2+b^2)$$

but $n^2=1-m^2-l^2$

Whence we get

$$b^2l^2(c^2-a^2)+a^2m^2(c^2-b^2)=v_1^2v_2^2-a^2b^2$$
$$l^2(c^2-a^2)+m^2(c^2-b^2)=v_1^2+v_2^2-a^2-b^2$$

Whence

$$l^2(a^2-c^2)(a^2-b^2)=(a^2-v_1^2)(a^2-v_2^2)$$
$$m^2(b^2-c^2)(b^2-a^2)=(b^2-v_1^2)(b^2-v_2^2)$$
$$n^2(c^2-a^2)(c^2-b^2)=(c^2-v_1^2)(c^2-v_2^2)$$

From the tables for the second prism, we see that for the inner sheet for

$$\phi'=28° \ 11' \ 21'' \frac{1}{v_1}=1·53021$$

For the outer sheet,

$$\phi'=28° \ 1' \ 21'' \frac{1}{v_2}=1·68391$$

Since the value of v_1 changes very slowly, we may reasonably treat these two values for ϕ' as coincident, and substitute in the above formulæ.

We find

$$\cos^{-1} n_1=49° \ 35'$$
$$\cos^{-1} m_1=40° \ 27'$$

Again, for inner sheet for

$$\phi'=38° \ 2' \ 41'' \frac{1}{v_1}=1·53037$$

For outer sheet

$$\phi'=31° \ 42' \ 34'' \frac{1}{v_2}=1·68437$$

Treating these two as coincident, we find

$$\cos^{-1} n_2=55° \ 44'$$
$$\cos^{-1} m_2=34° \ 23'$$

For inner sheet for

$$\phi'=26° \ 5' \ 11'' \frac{1}{v_1}=1·53028$$

For outer sheet

$$\phi'=26° \ 19' \ 10'' \frac{1}{v_2}=1·68378$$

Whence

$$\cos^{-1} n_3=47° \ 56'$$
$$\cos^{-1} m_3=42° \ 8'$$

We may justify the treating of the two values of ϕ' as the same, by the consideration that μ_1 varies at the rate of ·00001 (about) for 1°, and therefore the difference for 20′ is inappreciable.

Now the angle between the radii vectores in the first and second case cannot be less than

$$\cos^{-1} n_2 - \cos^{-1} n_1$$

i.e.,

$$\text{not less than } 55^\circ\ 44' - 49^\circ\ 35'$$
$$\text{or } 6^\circ\ 9'$$

But the angle between these radii is the difference between the two corresponding values of ϕ', or

$$= 31^\circ\ 42'\ 34'' - 28^\circ\ 1'\ 21''$$
$$= 3^\circ\ 41'\ 13''$$

Again, taking the second and third, the angle deduced from theory is not less than

$$\cos^{-1} n_2 - \cos^{-1} n_3$$
$$\text{or } 55^\circ\ 44' - 47^\circ\ 56'$$
$$\text{or } 7^\circ\ 48'$$

but from experiment it is

$$31^\circ\ 42'\ 34'' - 26^\circ\ 19'\ 10''$$
$$= 5^\circ\ 23'\ 24''$$

In both these cases the angle, as given by theory, must be very much larger than that given by experiment.

Thus, in order that the three pairs of values of $\mu_1\ \mu_2$ may correspond to three radii of the surface of wave slowness on FRESNEL'S theory, the angles between them must be half as large again as they are found to be.

Hence no section of the surface of wave slowness on FRESNEL'S theory can agree with the results of the experiments. We may remark, in addition, that the differences between consecutive values of μ for the outer sheet given by experiment are about double those given by theory.

After this investigation it seems needless to discuss possible alterations in the position of the plane of the second prism, with a view to bringing the results of experiment more nearly into accordance with theory. While as regards the constants a, b, c, it has already been seen that they have received their most probable values.

Thus our results, so far as they go, point to the fact that FRESNEL'S construction does not represent Nature, and that some other theory must be sought to explain the phenomena of double refraction.

And as regards the rival theories already proposed, Lord RAYLEIGH'S, we have seen, is even more than FRESNEL'S at variance with observed appearances, while I can hardly think that sufficient data have yet been collected to repay the labour of comparison with the theories of GREEN and CAUCHY, for on account of the undetermined constants it would be necessary to force theory and experiment to agree in so many points that they must, perforce, almost agree exactly along the small arcs investigated.

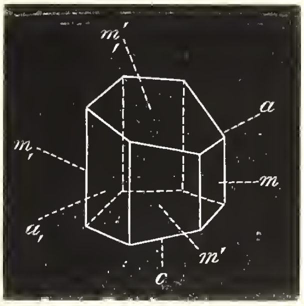
PART II.

Section I.*—*Description of a Second Crystal and Results of Experiments.*

Before asking permission to lay the results obtained in the previous part of the paper before the Society, I thought it would be better, if possible, to repeat the investigation with a second piece of the crystal. With some trouble I obtained a suitable piece last February from A. Hilger, of Tottenham Court Road.

Like the former piece, it was an hexagonal prism in shape, but of considerably smaller cross-section.

Fig. 1.

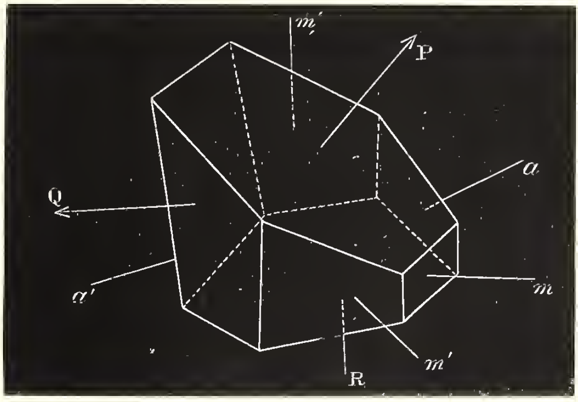


The axis of the prism was parallel to the axis of c, the base being perpendicular to the axis, while the top was broken off obliquely.

The planes $m_,$ m' gave the best reflexions, and were therefore chosen to determine the position of artificial faces.

My chief aim in having the prism cut was to get as long a continuous arc on the wave surface as possible. To attain this I had a face R, fig. 2, polished as nearly coincident with c as possible.

Fig. 2.



The oblique end was cut (P, fig. 2) so as to be inclined at about 37° to R, the edge of the prism thus formed between P and R being nearly parallel to the intersection of m and c.

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* N.B. The two Parts being quite distinct, the numbering of the Sections and Figures is commenced afresh with Part II.

The face $m_{,}$ was cut so as to be inclined at about 35° 20′ to P, the line of intersection of this face (Q, fig. 2) and P being also nearly parallel to that of P and R.

So that the principal planes of the prisms formed by P and R, P and Q respectively, were nearly coincident, and as my observations extended almost from perpendicular incidence on Q to perpendicular incidence on R: they embraced an arc of over 70°.

The instrument used and the methods adopted for levelling, reading, &c., have been fully described in the previous part of the paper.

The results of the experiments are contained in Tables I., II., III., and IV.

Tables I. and II. refer to the prism R P.

ϕ is the angle made by the wave normal in air with the normal to P.

ϕ' the corresponding angle in the crystal.

D is the observed deviation.

i the angle of prism.

$D+i$ is given, for it occurred in the calculation, and was just as easily found as D.

The value taken for i is

$$i=37° \ 2' \ 56''$$

This value is the mean of twelve observations, none of which differed from the mean by 10″.

ϕ' is found from the formulæ $\phi'+\psi'=i$

$$\text{and } \tan\frac{\phi'-\psi'}{2}=\tan\frac{i}{2}\cot\frac{\phi+\psi}{2}\tan\frac{\phi-\psi}{2}$$

which has already been proved (Part I., Section I.).

In Table I. ϕ is the same for both outer and inner sheet, being the angle of incidence.

In Table II. ϕ has different values for the two sheets, being the angle of emergence, for as already explained, on passing through the position of minimum deviation, the prism was reversed so that the face of incidence became that of emergence.

Throughout the work μ_1 refers to the inner, μ_2 to the outer sheet.

The values of $\delta\phi'$, $\delta\mu$ are calculated as before on the assumption of errors of 10″ in 2ϕ and $D+i$, the observed quantities taken; these errors being combined so as to produce the greatest possible effect in the result.

Tables III. and IV. refer to P Q, for which $i=35° \ 18' \ 50''$.

The symbols have the same meaning, but while in I. and II. the wave normal falls on the same side of the normal to P as the axis O C, in III. and IV. it is on the other side of O P.

In III. ϕ is the angle of emergence, and therefore different for the inner and outer sheets; in IV. it is the angle of incidence, and the same for both.

Each observation was repeated twice on different occasions; the results of two measurements rarely differed by 20′.

TABLE I.

Outer sheet.					ϕ.	Inner sheet.					
$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.		$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
		″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
·00009		9	35 59 41	83 53 56	82 7 10	83 42 26	36 6 16	9	1·68102	·00009	1
	1·68559		35 45 55	82 17 1	80 7 10	82 5 36	35 52 30		1·68112		2
	1·68558		35 29 24	80 44 41	78 7 10	80 33 21	35 35 59		1·68106		3
	1·68553		35 10 1	79 17 31	76 7 10	79 6 26	35 16 20		1·68115		4
	1·68546		34 47 54	77 54 51	74 7 10	77 44 6	34 53 57		1·68114		5
·00007	1·68539	8	34 22 47	76 37 16	72 7 10	76 26 36	34 28 43	8	1·68115	·00007	6
	1·68532		33 55 3	75 24 6	70 7 10	75 13 26	34 0 54		1·68106		7
	1·68526		33 24 38	74 15 31	68 7 10	74 4 51	33 30 24		1·68099		8
	1·68519		32 51 40	73 11 16	66 7 10	73 0 36	32 57 21		1·68089		9
	1·68505		32 16 18	72 11 11	64 7 10	72 0 31	32 21 52		1·68074		10
·00006	1·68514	8	31 38 14	71 15 51	62 7 10	71 4 51	31 43 51	8	1·68069	·00006	11
	1·68497		30 58 13	70 23 51	60 7 10	70 12 26	31 3 56		1·68030		12
	1·68489		30 15 49	69 36 6	58 7 10	69 24 31	30 21 28		1·68016		13
	1·68485		29 31 16	68 52 16	56 7 10	68 39 31	29 37 20		1·67962		14
	1·68465		28 44 50	68 11 51	54 7 10	67 58 26	28 51 1		1·67917		15
·00004	1·68473	7	27 56 13	67 35 36	52 7 10	67 20 41	28 2 53	7	1·67859	·00004	16
	1·68463		27 5 54	67 2 36	50 7 10	66 46 36	27 12 48		1·67806		17
	1·68454		26 13 50	66 33 6	48 7 10	66 15 16	26 21 14		1·67721		18
	1·68457		25 19 58	66 7 16	46 7 10	65 47 16	25 27 58		1·67634		19
	1·68457		24 24 34	65 44 56	44 7 10	65 22 21	24 33 11		1·67532		20
·00002	1·68459	6	23 27 40	65 26 1	42 7 10	65 0 31	23 36 55	6	1·67421	·00002	21
	1·68451		22 29 27	65 10 11	40 7 10	64 41 51	22 39 13		1·67303		22
	1·68448		21 29 52	64 57 56	38 7 10	64 25 56	21 40 18		1·67161		23
	1·68441		20 29 5	64 49 6	36 7 10	64 13 16	20 40 3		1·67015		24
	1·68446		19 27 3	64 44 6	34 7 10	64 3 36	19 38 38		1·66854		25
	1·68461	6	18 23 54	64 42 56	32 7 10	63 57 26	18 36 0	6	1·66694		26
	1·68446		17 19 54	64 45 1	30 7 10	63 54 1	17 32 26		1·66502		27
	1·68448		16 14 53	64 51 36	28 7 10	63 54 6	16 27 50		1·66300		28

TABLE II.

Outer sheet.						Inner sheet.						
ϕ.	$\partial\mu_2$.	μ_2.	$\partial\phi'$.	ϕ'.	$D+i$.	ϕ.	$D+i$.	ϕ'.	$\partial\phi'$.	μ_1.	$\partial\mu_1$.	
° ′ ″			″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
32 49 16		1·68433		18 46 18	64 42 26	32 3 51	63 57 1	18 34 20		1·66678		1
30 50 31		1·68458		17 43 17	64 43 41	30 0 46	63 53 56	17 29 1		1·66490		2
28 55 16		1·68446		16 41 4	64 48 26	28 1 21	63 54 31	16 24 36		1·66302		3
27 3 31		1·68445		15 40 5	64 56 41	26 5 1	63 58 11	15 21 1		1·66095		4
25 15 21		1·68448		14 40 21	65 8 31	24 12 1	64 5 11	14 18 26		1·65883		5
23 30 46	3	1·68458	4	13 41 58	65 23 56	22 22 21	64 15 31	13 16 57	4	1·65667	3	6
21 49 11		1·68453		12 44 51	65 42 21	20 36 11	64 29 21	12 16 45		1·65459		7
20 11 11		1·68448		11 49 16	66 4 21	18 53 6	64 46 16	11 17 45		1·65240		8
18 36 46		1·68455		10 55 19	66 29 56	17 13 41	65 6 51	10 20 18		1·65037		9
17 5 26		1·68446		10 2 51	66 58 36	15 37 16	65 30 26	9 24 11		1·64818		10
15 37 50	4	1·68452	5	9 12 17	67 31 6	14 4 21	65 57 31	8 29 41	5	1·64603	4	11
14 13 41		1·68446		8 23 25	68 6 51	12 35 11	66 28 21	7 37 1		1·64400		12
12 53 26		1·68451		7 36 37	68 46 36	11 9 41	67 2 51	6 46 13		1·64203		13
11 36 41		1·68450		6 51 45	69 29 51	9 47 16	67 40 26	5 57 2		1·63984		14
10 23 56		1·68454		6 9 3	70 17 6	8 29 1	68 22 11	5 10 4		1·63782		15
9 14 51	4	1·68450	5	5 28 28	71 8 1	7 15 26	69 8 36	4 25 41	5	1·63615	4	16
8 9 51		1·68445		4 50 10	72 3 1	6 5 26	69 58 36	3 43 20		1·63433		17
7 9 6		1·68445		4 14 19	73 2 16	4 59 41	70 52 51	3 3 25		1·63258		18
6 12 51		1·68453		3 41 3	74 6 1	3 58 41	71 51 51	2 26 16		1·63100		19
5 20 21		1·68437		3 10 1	75 13 31	3 2 11	72 55 21	1 50 46		1·62956		20
4 33 1	5	1·68447	6	2 41 58	76 26 11	2 10 6	74 3 16	1 19 54	6	1·62807	5	21
3 49 36		1·68428		2 16 15	77 42 46	1 23 31	75 16 41	0 51 20		1·62691		22
3 12 1		1·68451		1 53 58	79 5 11	0 41 51	76 35 1	0 25 44		1·62583		23

TABLE III.

Outer sheet.						Inner sheet.						
ϕ.	$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.	ϕ.	$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
° ′ ″			″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
1 32 40		1·68464		0 55 0	73 39 50	0 49 20	71 17 50	0 30 20		1·62619		1
2 19 15		1·68450		1 22 39	72 26 25	0 4 30	70 2 40	0 2 46		1·62500		2
3 10 50		1·68454		1 47 28	71 18 0	0 44 25	68 51 35	0 27 21		1·62360		3
4 6 30		1·68450		2 26 15	70 13 40	1 37 25	67 44 35	1 0 3		1·62208		4
5 6 45		1·68458		3 1 56	69 13 55	2 34 45	66 41 55	1 35 28		1·62058		5
6 10 50	9	1·68456	9	3 30 52	68 18 0	3 35 50	65 43 0	2 13 26	9	1·61904	9	6
7 19 10		1·68464		4 20 14	67 26 20	4 40 40	64 47 50	2 53 26		1·61715		7
8 31 10		1·68462		5 2 41	66 38 20	5 49 5	63 56 15	3 35 53		1·61525		8
9 46 40		1·68449		5 47 9	65 53 50	7 0 50	63 8 0	4 20 28		1·61320		9
11 6 20		1·68462		6 33 56	65 13 30	8 16 20	62 23 30	5 7 23		1·61120		10
12 29 5	8	1·68453	8	7 22 24	64 36 15	9 34 45	61 41 55	5 56 12	8	1·60897	8	11
13 55 45		1·68462		8 12 55	64 2 55	10 56 30	61 3 40	6 47 4		1·60671		12
15 25 25		1·68457		9 5 2	63 32 35	12 21 35	60 28 45	7 39 59		1·60450		13
16 58 30		1·68454		9 58 50	63 5 40	13 49 10	59 56 20	8 34 30		1·60201		14
18 35 5		1·68462		10 54 19	62 42 15	15 19 40	59 26 50	9 30 46		1·59943		15
20 14 50	8	1·68466	7	11 51 14	62 22 0	16 53 35	59 0 45	10 29 0	7	1·59710	8	16
21 57 25		1·68459		12 49 27	62 4 35	18 29 20	58 36 30	11 28 25		1·59427		17
23 43 35		1·68465		13 49 6	61 50 45	20 8 30	58 15 40	12 29 39		1·59173		18
25 32 55		1·68469		14 49 58	61 40 5	21 50 30	57 57 40	13 32 24		1·58911		19
27 25 30		1·68475		15 51 58	61 32 40	23 35 15	57 42 25	14 36 32		1·58651		20
29 21 0	7	1·68457	6	16 54 56	61 28 10	25 22 10	57 29 20	15 41 47	6	1·58365	7	21
31 20 30		1·68464		17 59 3	61 27 40	27 12 0	57 19 10	16 48 24		1·58086		22
33 23 30		1·68465		19 4 4	61 30 40	29 4 50	57 12 0	17 56 15		1·57815		23
35 30 0		1·68458		20 9 52	61 37 10	31 0 20	57 7 30	19 5 10		1·57536		24
37 40 55		1·68465		21 16 33	61 48 5	32 59 5	57 6 15	20 15 13		1·57269		25
39 56 20		1·68473		22 23 57	62 3 10	35 0 50	57 8 0	21 26 12		1·56994		26

TABLE IV.

Outer sheet.					ϕ.	Inner sheet.					
$\delta\mu_2$.	μ_2.	$\delta\phi'$.	ϕ'.	$D+i$.		$D+i$.	ϕ'.	$\delta\phi'$.	μ_1.	$\delta\mu_1$.	
		″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	″			
7	1·68444	6	18 16 33	61 27 40	31 53 10	57 6 50	19 36 20	6	1·57426	7	1
	1·68456		19 19 40	61 31 40	33 53 10	57 6 35	20 46 52		1·57144		2
	1·68464		20 21 44	61 38 55	35 53 10	57 9 40	21 56 27		1·56879		3
	1·68454		21 22 47	61 49 5	37 53 10	57 15 45	23 5 3		1·56616		4
	1·68454		22 22 32	62 2 40	39 53 10	57 25 0	24 12 31		1·56383		5
6	1·68448	7	23 21 2	62 19 20	41 53 10	57 37 15	25 18 47	7	1·56153	6	6
	1·68458		24 18 0	62 39 35	43 53 10	57 52 25	26 23 47		1·55929		7
	1·68452		25 13 38	63 2 45	45 53 10	58 10 40	27 27 22		1·55716		8
	1·68440		26 7 44	63 29 0	47 53 10	58 32 0	28 29 26		1·55512		9
	1·68452		27 0 1	63 59 0	49 53 10	58 57 5	29 29 33		1·55342		10
8	1·68452	8	27 50 39	64 32 10	51 53 10	59 24 50	30 28 13	8	1·55157	8	11
	1·68451		28 39 27	65 8 45	53 53 10	59 56 20	31 24 47		1·54996		12
	1·68454		29 26 17	65 49 0	55 53 10	60 31 20	32 19 19		1·54846		13
	1·68450		30 11 9	66 32 45	57 53 10	61 10 0	33 11 39		1·54708		14
	1·68459		30 53 49	67 20 25	59 53 10	61 52 0	34 1 58		1·54561		15
8	1·68448	9	31 34 29	68 11 35	61 53 10	62 38 45	34 49 19	9	1·54460	8	16
	1·68454		32 12 39	69 7 5	63 53 10	63 29 5	35 34 26		1·54348		17
	1·68464		32 48 23	70 6 50	65 53 10	64 23 50	36 16 45		1·54251		18
	1·68461		33 21 47	71 10 25	67 53 10	65 22 55	36 56 17		1·54163		19
	1·68452		33 52 43	72 18 15	69 53 10	66 26 35	37 32 52		1·54082		20
9	1·68465	10	34 20 43	73 31 5	71 53 10	67 35 10	38 6 15	10	1·54019	9	21
	1·68442		34 46 29	74 47 40	73 53 10	68 48 15	38 36 45		1·53948		22
	1·68447		35 9 5	76 9 35	75 53 10	70 6 45	39 3 42		1·53900		23
	1·68452		35 28 49	77 36 20	77 53 10	71 30 5	39 27 32		1·53846		24
	1·68446		35 45 48	79 7 45	79 53 10	72 59 10	39 47 36		1·53821		25
9	1·68444	11	35 59 45	80 44 15	81 53 10	74 33 10	40 4 31	11	1·53774	9	26

Section II.—*Determination of the Position of the Planes of the Prism* P R′ *with reference to the Axes of Elasticity.*

In order to compare these results with theory we require to know accurately the position of the principal planes of the two prisms. We will consider the prism, faces P R, first.

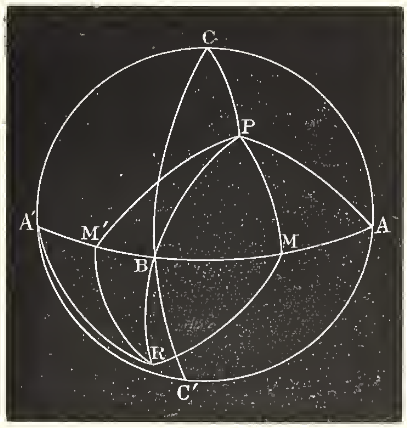
Let O A, O B, O C be the directions of the principal axes of the wave surface, O B bisects the angle between the faces *m m′*, O C is parallel to the intersection of these faces.

Let the normals to the faces *m m′* cut a unit sphere, centre O, in M M′, the normal to the face P in P, and the normal to R in R.

Let the direction angles of P and R with reference to A, B, C be $\alpha_1\ \beta_1\ \gamma_1$, $\alpha_2\ \beta_2\ \gamma_2$ respectively.

[P lies in quadrant A B C, R in A′ B C′.]

Fig. 3.



Let

$$PM=\theta_1,\ PM'=\theta_1'$$
$$RM=\theta_2,\ RM'=\theta_2'$$

Observations on the angles between P and *m*, P and *m′* give θ_1, θ_1' respectively.

While observations on the angles between R and *m*, R and *m′* give θ_2, θ_2' respectively.

Let

$$MM'=2\mu=2MB.$$

Then, as in the first part, from triangles M′ B P, M P B we get

$$\cos\beta_1=\frac{\cos\frac{\theta_1'+\theta_1}{2}\cos\frac{\theta_1'-\theta_1}{2}}{\cos\mu} \quad \ldots\ldots\ldots \quad (1)$$

$$\cos\alpha_1=\frac{\sin\frac{\theta_1'+\theta_1}{2}\sin\frac{\theta_1'-\theta_1}{2}}{\cos\mu} \quad \ldots\ldots\ldots \quad (2)$$

with similar formulæ for β_2, α_2.

All the angles θ_1, θ_1', θ_2, θ_2', and μ were observed several times throughout the work.

The mean of the observations give

$$\theta_1 = 54^\circ\ 17'\ 6'' \tag{3}$$

$$\theta_1' = 75^\circ\ 16'\ 20'' \tag{4}$$

$$\theta_2 = 88^\circ\ 40'\ 35'' \tag{5}$$

$$\theta_2' = 88^\circ\ 20'\ 20'' \tag{6}$$

$$\mu = 31^\circ\ 53'\ 30'' \tag{7}$$

In no case was the difference between the mean and any one of the observations so great as 20″.

Substituting these values we find

$$\left.\begin{aligned}\alpha_1 &= 71^\circ\ 49'\ 30''\\ \beta_1 &= 60^\circ\ 25'\ 49''\end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned}\alpha_2 &= 90^\circ\ 19'\ 10''\\ \beta_2 &= 88^\circ\ 14'\ 32''\end{aligned}\right\} \tag{11}$$

Also

$$\sin^2\gamma_1 = \cos^2\alpha_1 + \cos^2\beta_1$$
$$\sin^2\gamma_2 = \cos^2\alpha_2 + \cos^2\beta_2$$

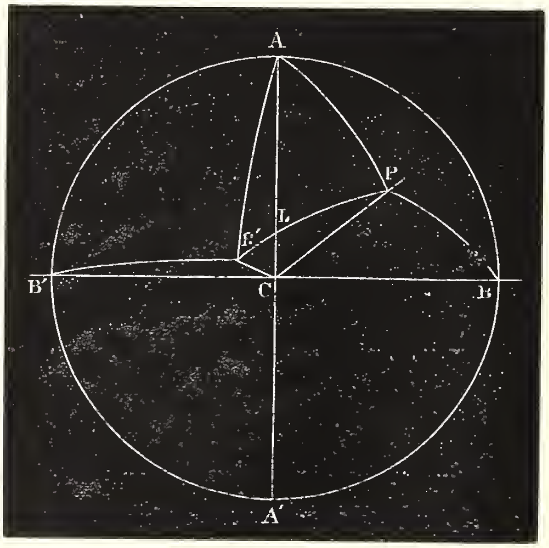
whence

$$\gamma_1 = 35^\circ\ 42'\ 57'' \tag{12}$$

$$\pi - \gamma_2 = 1^\circ\ 47'\ 11'' \tag{13}$$

Let R O meet the sphere again in R′, and let the figure represent the sphere as seen by an eye on O C produced.

Fig. 4.



We have now sufficient data to calculate the angle R′ P.

$$\begin{aligned} CP &= \gamma_1 \\ AP &= \alpha_1 \\ BP &= \beta_1 \\ CR' &= \pi-\gamma_2 \\ AR' &= \pi-\alpha_2 = 89° \; 40' \; 50'' \\ B'R' &= \beta_2. \end{aligned}$$

From triangle B′ A R′

$$\cos B'R' = \cos B'AR' \sin AR'$$

$$\therefore \cos B'AR' = \frac{\cos B'R'}{\sin AR'}$$

$$= \frac{\cos \beta_2}{\sin \alpha_2}$$

whence

$$B'AR' = \;\; 88° \; 14' \; 32''$$

$$\therefore R'AC = \;\; 1° \; 45' \; 28''$$

From triangle C A P

$$\cos CP = \cos CAP \sin AP$$

$$\therefore \cos CAP = \frac{\cos CP}{\sin AP}$$

$$\therefore CAP = \;\; 31° \; 17' \; 20''$$

$$\therefore R'AP = 33° \;\; 2' \; 48''$$

From triangle R′ A P

$$\cos R'P = \cos R'A \cos AP + \sin R'A \sin AP \cos R'AP$$

$$= \cos(\pi-\alpha_2) \cos \alpha_1 + \sin(\pi-\alpha_2) \sin \alpha_1 \cos R'AP$$

whence

$$R'P = 37° \; 2' \; 53''$$

But R′P can be measured directly, the mean of twelve observations, none of which differed from the mean by 10″, gave

$$R'P = 37° \; 2' \; 56''$$

The close agreement between these results confirms the accuracy of the values of α_1, β_1, &c.

Let P R′ cut C A in L, we proceed to determine C L, R′ L, and the angle A L P.

From triangle A R′ L

$$\cot AL \sin AR' = \cot AR'P \sin R'AC + \cos AR' \cos R'AC$$

$$\therefore \cot AL \sin \alpha_2 = \cot AR'P \sin R'AC + \cos \alpha_2 \cos R'AC$$

we have seen already that

$$R'AC = 1^\circ\ 45'\ 28''$$

For A R′ P we have

$$\sin AR'P = \frac{\sin AP}{\sin R'P} \sin R'AP$$

$$\therefore AR'P = 59^\circ\ 18'\ 37''$$

On substituting we get

$$AL = 88^\circ\ 38'\ 18''$$

$$\therefore CL = 1^\circ\ 21'\ 42'' \quad \ldots\ldots\ldots\ldots \quad (15)$$

For R′ L the triangle R′ A L gives

$$\sin R'L = \frac{\sin R'AL \sin AL}{\sin AR'L}$$

whence

$$R'L = 2^\circ\ 2'\ 37''$$

For A L P the triangle A L P gives

$$\sin ALP = \frac{\sin AP}{\sin LP} \sin LAP$$

$$LP = R'P - R'L = 35^\circ\ 0'\ 19'' \quad \ldots\ldots\ldots\ldots \quad (16)$$

Whence

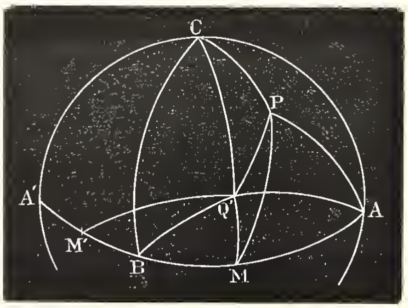
$$ALP = 59^\circ\ 20'\ 11'' \quad \ldots\ldots\ldots\ldots \quad (17)$$

Thus we have determined the position of the plane R′ P completely.

Section III.—*Determination of the Position of the Plane* P Q *with reference to the Principal Planes.*

Our next step will be to determine the position of P Q′, Q′ being the point in which Q O produced backwards cuts the sphere.

Fig. 5.



Let the direction angles of O Q′ be α_3 β_3 γ_3

Let

$$\begin{gathered} \mathrm{MQ}'=\theta_3 \\ \mathrm{M}'\mathrm{Q}'=\theta_3' \end{gathered}$$

then

$$\cos\beta_3=\frac{\cos\dfrac{\theta_3'+\theta_3}{2}\cos\dfrac{\theta_3'-\theta_3}{2}}{\cos\mu}$$

$$\cos\alpha_3=\frac{\sin\dfrac{\theta_3'+\theta_3}{2}\sin\dfrac{\theta_3'-\theta_3}{2}}{\cos\mu}$$

But θ_3, θ_3' are the angles between the faces Q and m, Q and m' respectively. And observation shows that

$$\left.\begin{aligned} \theta_3&=19^\circ\ \ 0'\ 15'' \\ \theta_3'&\ \ 64^\circ\ 37'\ \ 0'' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Whence

$$\beta_3=35^\circ\ 58'\ 41'' \quad \ldots\ldots\ldots\ldots \quad (4)$$

$$\alpha_3=60^\circ\ 42'\ 56'' \quad \ldots\ldots\ldots\ldots \quad (5)$$

$$\cos \mathrm{Q}'\mathrm{BA}=\frac{\cos\alpha_3}{\sin\beta_3}$$

$$\mathrm{Q}'\mathrm{BA}=33^\circ\ 37'\ 51''$$

$$\cos \mathrm{CQ}'=\sin \mathrm{BQ}'\sin \mathrm{Q}'\mathrm{BA}$$

$$\therefore \gamma_3=\mathrm{CQ}'=71^\circ\ 0'\ 44'' \quad \ldots\ldots\ldots\ldots \quad (6)$$

To corroborate these values let us find P Q′ and compare with experiment

$$\cos \mathrm{Q}'\mathrm{AB}=\frac{\cos\beta_3}{\sin\alpha_3}$$

$$\therefore \mathrm{Q}'\mathrm{AB}=21^\circ\ 54'\ 12''$$

But we know from the previous work (Section II.) that

$$\mathrm{PAB}=90^\circ-\mathrm{PAC}=58^\circ\ 42'\ 42''$$

$$\therefore \mathrm{PAQ}'=36^\circ\ 48'\ 30''$$

$$\cos \mathrm{PQ}'=\cos \mathrm{AP}\cos \mathrm{AQ}'+\sin \mathrm{AP}\sin \mathrm{AQ}'\cos \mathrm{PAQ}$$

Whence

$$\mathrm{PQ}'=35^\circ\ 18'\ 35''$$

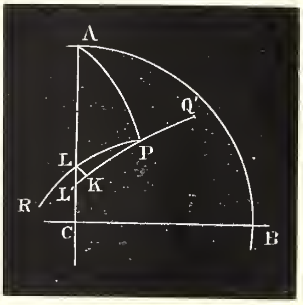
The mean of a large series of experiments gave

$$\mathrm{PQ}'=35^\circ\ 18'\ 50''$$

and again we have strong evidence in favour of the correctness of the work.

Let the figure, as before, represent the sphere as seen by an eye on O C produced.

Fig. 6.



Let Q′ P produced cut C A in L′ we proceed to determine C L′, P L′, and P L′ A. From the triangle L′ A P

$$\cot \mathrm{AL}' \sin \mathrm{AP} = \cot \mathrm{APL}' \sin \mathrm{L'AP} + \cos \mathrm{AP}' \cos \mathrm{L'AP}$$

From triangle P Q′ A

$$\sin \mathrm{APL}' = \frac{\sin \mathrm{AQ}'}{\sin \mathrm{PQ}'} \sin \mathrm{PAQ}'$$

$$\pi - \mathrm{APL}' = 64^\circ\ 41'\ 28''$$

Also

$$\mathrm{L'AP} = \mathrm{CAP} = 31^\circ\ 17'\ 20''$$

[Section II. from previous work.]

Substituting these values we get

$$\mathrm{AL}' = 88^\circ\ 44'\ 20''$$

$$\mathrm{CL}' = 1^\circ\ 15'\ 40'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The triangle A L′ P gives

$$\sin \mathrm{AL'P} = \frac{\sin \mathrm{AP}}{\sin \mathrm{AL}'} \sin \mathrm{APL}'$$

whence

$$\mathrm{AL'P} = 59^\circ\ 13'\ 2'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Also

$$\sin \mathrm{L'P} = \frac{\sin \mathrm{AL}'}{\sin \mathrm{APL}'} \sin \mathrm{L'AP}$$

whence

$$\mathrm{L'P} = 35^\circ\ 3'\ 14'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

To corroborate these results draw L K perpendicular to P L′, then L K L′ is approximately a right-angled triangle, and angle LL′K=60° nearly.

Hence

$$\mathrm{L'K} = \tfrac{1}{2}\mathrm{LL}' \text{ approximately}$$

$$\text{Now } \mathrm{L'K} = \mathrm{PL}' - \mathrm{PL} = 3' \text{ approximately}$$

$$\mathrm{LL}' = \mathrm{CL} - \mathrm{CL}' = 6' \text{ approximately}$$

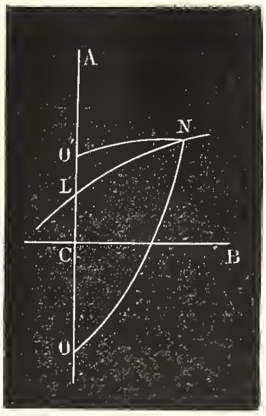
$$\therefore \mathrm{L'K} \text{ is equal to } \tfrac{1}{2}\mathrm{LL}'$$

Section IV.—*General Theory.*

To apply these results to the theoretical calculation.

Taking the same figure

Fig. 7.



Let the optic axes cut the sphere in O O′; consider a plane cutting the plane A O C between C and O′ in L say; let N be any point on it; let NLA=χ, NO=θ, NO′=θ'.

Then we know that if v_1, v_2 are the velocities along the wave normal through N, a, b, c the principal velocities, a being the greatest

$$v_1^2-v_2^2=(a^2-c^2)\sin\theta\sin\theta'$$

$$v_1^2+v_2^2=(a^2+c^2)-(a^2-c^2)\cos\theta\cos\theta'$$

$$2v_1^2=(a^2+c^2)-(a^2-c^2)\cos(\theta+\theta') \quad \ldots\ldots\ldots \quad (1)$$

$$2v_2^2=(a^2+c^2)-(a^2-c^2)\cos(\theta-\theta'). \quad \ldots\ldots\ldots \quad (2)$$

We require, therefore, the values of θ and θ'.

Now

$$\begin{aligned}\cos\theta &= \cos \text{OL} \cos \text{LN} - \sin \text{OL} \sin \text{LN} \cos\chi \\ &= \cos \text{OL}(\cos \text{LN} - \tan \text{OL} \sin \text{LN} \cos\chi) \\ &= \frac{\cos \text{OL}}{\cos\lambda}\cos(\text{LN}+\lambda) \quad \ldots\ldots\ldots \quad (3)\end{aligned}$$

if $\tan\lambda=\tan \text{OL} \cos\chi$

Similarly

$$\cos\theta'=\frac{\cos \text{O}'\text{L}}{\cos\lambda'}\cos(\text{LN}-\lambda') \quad \ldots\ldots\ldots \quad (4)$$

where

$$\tan\lambda'=\tan \text{O}'\text{L} \cos\chi.$$

Thus to use the formulæ for v_1, v_2 we require to know accurately the position of the optic axes.

These can only be determined by finding the values of a, b, c.

The results of the experiments enable us to do this with considerable exactness.

Let μ_a, μ_b, μ_c be the principal refractive indices, μ_a being the greatest, μ_c the least

$$\mu_a=\frac{1}{c}$$

$$\mu_b=\frac{1}{b}$$

$$\mu_c=\frac{1}{a}$$

Section V.—*Values of the Principal Refractive Indices.*

We are now in a position to determine from our experimental results the values of a, b, c the principal velocities.

The plane P R′ cuts the plane A C at a short distance from C.

The section of the surface of wave slowness by the plane A O C consists of a circle of radius μ_b and an oval curve of axes μ_a and μ_c in the directions O C, O A respectively, which for a small distance on either side of a principal axis may be treated as an ellipse; as in FRESNEL's theory.

Also, if L be the point in which P R′ cuts the plane A O C

$$LP=35^\circ\ 0'\ 19''$$

[*vide* Section II. (16)].

The value of μ corresponding to a value of $\phi'=35^\circ\ 0'\ 19''$ in Table I. for the inner sheet will thus be the value of μ_b.

But we have as corresponding values

ϕ'	μ
34° 28′ 43″	1·68115
34° 53′ 57″	1·68114
35° 16′ 20″	1·68115

we may therefore take

$$\underline{\mu_b=1\cdot 68115.}$$

The values of the radius vector to the outer sheet in the same neighbourhood are, Table I., lines 4 and 5,

ϕ'	μ
34° 47′ 54″	1·68546
35° 10′ 1″	1·68553.

For

$$\phi'=35^\circ\ 0'\ 19''$$

we may take therefore

$$\mu=1{\cdot}68550$$

This then must be the length of the radius vector of an ellipse, axes μ_a and μ_c, which passes through the point L—*i.e.*, which is inclined at an angle CL=1° 21′ 42″ to the axis μ_a [*vide* Section II. (15)].

Now if θ be the angle which a radius vector r makes with the axis μ_a

$$\frac{1}{r^2}=\frac{\cos^2\theta}{\mu_a{}^2}+\frac{\sin^2\theta}{\mu_c{}^2}$$

in the case considered

$$\theta=1^\circ\ 21'\ 42''$$

$$\sin^2\theta={\cdot}000565$$

and the error made in giving to μ_c an approximate value will be inappreciable.

Let us take, therefore, for μ_c the value found in the previous part which agrees with RUDBERG's value

$$\mu_c=1{\cdot}53013$$

$$\frac{\sin^2\theta}{\mu_c{}^2}={\cdot}000241$$

Now

$$r=1{\cdot}68550$$

$$\frac{1}{r^2}={\cdot}351919$$

$$\frac{1}{\mu_a{}^2}=\frac{1}{\cos^2\theta}\left(\frac{1}{r^2}-\frac{\sin^2\theta}{c^2}\right)$$

whence

$$\mu_a=\underline{1{\cdot}68560}$$

It now remains to determine μ_c accurately.

The natural faces $a_{,}\, m_{,}'$ (fig. 1) formed a prism whose edge was parallel to the axis O C.

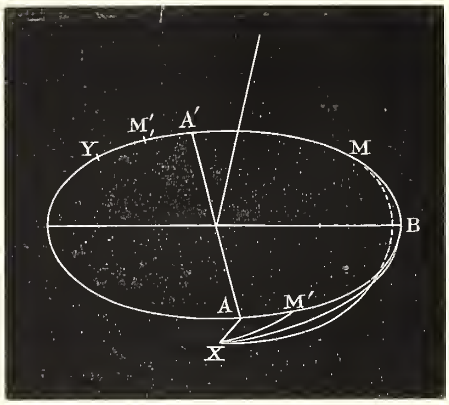
By passing light through these faces and observing the incidence and deviation as before, I was enabled to calculate μ_c.

To pass light through these faces it was necessary, however, to polish them, and this operation altered their position appreciably. The values obtained by direct measurement required a small correction, and to calculate this it was necessary to determine the position of these new faces.

Let the faces approximately coincident with $a_{,}$ and $m_{,}'$ be called X Y respectively. On levelling the faces of reference $m\ m'$, I found that Y remained very nearly in the same zone as before, viz.: that of $m\ m'\ m_{,}\ m_{,}'$; while X fell rather below this zone. Observations on the angles between X and Y and the faces $m\ m'$ respectively gave

$$\begin{aligned} \text{XM} \ (=\theta_4 \text{ say}) &= 123^\circ \ 32' \\ \text{XM}'(=\theta'_4 \text{ say}) &= \ 59^\circ \ 52' \\ \text{arc YA}'\text{M} \ (=\theta_5 \text{ say}) &= 117^\circ \ 17' \\ \text{arc YA}'\text{M}'(=\theta_5 \text{ say}) &= 181^\circ \ 3' \end{aligned}$$

Fig. 8.



Since Y, M, M', are in one zone with A and B, and B bisects arc M M'

$$\begin{aligned} 2\text{BA}'\text{Y} &= \text{M}'\text{A}'\text{Y} + \text{MA}'\text{Y} \\ \therefore \text{ arc BA}'\text{Y} &= 149^\circ \ 10' \\ \text{A}'\text{Y} &= 149^\circ \ 10' - 90^\circ \\ &= 59^\circ \ 10' \end{aligned}$$

Let $\alpha_4 \ \beta_4 \ \gamma_4$ be the direction angles of X.
Then, as before, [Section II. (1)]

$$\cos\alpha_4 = \frac{\sin\dfrac{\theta'_4+\theta_4}{2}\sin\dfrac{\theta'_4-\theta_4}{2}}{\cos\mu} \quad \&\text{c.}$$

whence

$$\begin{aligned} \alpha_4 &= \ 3^\circ \ 28' \ 50'' \\ \beta_4 &= 91^\circ \ 42' \end{aligned}$$

To test these values let us calculate the angle X Y; the triangle X A B gives

$$\cos \text{XAB} = \frac{\cos \text{XB}}{\sin \text{AX}}$$

whence

$$\pi - \text{XAB} = 60^\circ \ 43' \ 50''$$

In the triangle X A Y we now know A X, A Y and the angle X A Y.

Solving, we get

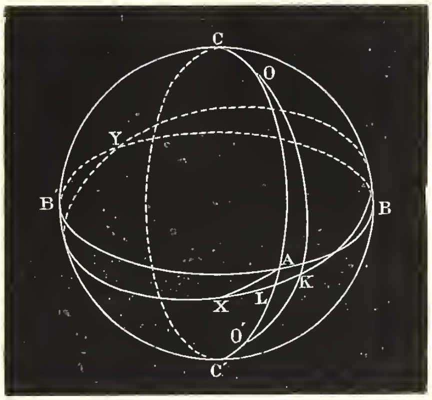
$$XY=119^\circ\ 5' \tag{1}$$

Experiment gave

$$119^\circ\ 4'$$

Here again we have considerable confirmation.

Fig. 9.



Let the plane Y X L be the plane of the prism cutting C A C′ in L; it will be necessary to find A L, X L, and the angle A L Y.

The right angled triangle L A Y gives

$$\sin AY = \tan AL \cot AYL.$$

The triangle X A Y gives

$$\sin LYA = \frac{\sin AX \sin YAX}{\sin XY}$$

whence

$$LYA = 3^\circ\ 28'\ 30''$$

Substituting this value, we get

$$AL = 2^\circ\ 59'\ 10'' \tag{2}$$

From the same triangle we get

$$\cos YL = \cos AY \cos AL$$

whence

$$YL = 120^\circ\ 47'\ 10''$$

but

$$XY = 119^\circ\ 4'$$

$$\therefore XL = 1^\circ\ 43'\ 10'' \tag{3}$$

From the same triangle again

$$\cos ALY = \tan AL \cot LY$$

whence

$$ALY = 91^\circ\ 47''$$

Now we know that if O, O′ are the points where the optic axes cut the unit sphere K, the point where any wave normal meets the sphere if

$$\mathrm{KO}=\theta$$

$$\mathrm{KO}'=\theta'$$

$$2v^2=\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}-\left(\frac{1}{\mu_a^2}-\frac{1}{\mu_c^2}\right)\cos(\theta-\theta')$$

$$\therefore \frac{1}{\mu_c^2}(1+\cos(\theta-\theta'))$$

$$=\frac{2}{\mu^2}-\frac{1}{\mu_a^2}(1-\cos(\theta-\theta'))$$

$$\frac{1}{\mu_c^2}\cos^2\left(\frac{\theta-\theta'}{2}\right)=\frac{1}{\mu^2}-\frac{1}{\mu_a^2}\sin^2\left(\frac{\theta-\theta'}{2}\right) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

If then we can determine $\theta-\theta'$ since μ is given by experiment, this formula determines the value of μ_c.

Now along the plane X Y, $\theta-\theta'$ is very small, and a considerable change in the positions of O O′ will produce but small variations in the value of $\theta-\theta'$ and ∴ of μ_c.

Let us, therefore, take the value of the angle between the optic axes given by the values already determined for μ_a and μ_b, viz. :

$$\mu_a=1{\cdot}68560$$

$$\mu_b=1{\cdot}68115$$

and the approximate value of μ_c

$$\mu_c=1{\cdot}53013$$

If $2\phi'$ is the angle between the optic axes

$$\tan\phi'=\frac{\mu_c}{\mu_a}\sqrt{\left\{\frac{\mu_a^2-\mu_b^2}{\mu_b^2-\mu_c^2}\right\}}$$

whence

$$\phi'=9^\circ\ 4'\ 5''$$

Hence

$$\mathrm{AO}=\mathrm{AO}'=80^\circ\ 55'\ 55''$$

$$\mathrm{AL}=\ 2^\circ\ 59'\ 10''$$

$$\mathrm{OL}=83^\circ\ 55'\ 5''$$

$$\mathrm{O'L}=77^\circ\ 56'\ 45''$$

$$\chi=\mathrm{ALK}=88^\circ\ 13'$$

$$\mathrm{XL}=1^\circ\ 43'\ 10''$$

Let K be on the side of L remote from X. Experiment shows that this is the actual position of the wave normal in the case considered.

Let

$$\mathrm{KX}=\psi'$$

$$\therefore\ \mathrm{LK}=\mathrm{KX}-\mathrm{XL}=\psi'-1^\circ\ 43'\ 10''$$

$$\cos\theta=\cos\mathrm{OL}\cos\mathrm{LK}+\sin\mathrm{OL}\sin\mathrm{LK}\cos\chi=\frac{\cos\mathrm{OL}}{\cos\lambda}\cos(\mathrm{LK}-\lambda)$$

if

$$\tan\lambda=\tan\mathrm{OL}\cos\chi$$

Substituting the values of O L and χ

$$\lambda=16^\circ\ 17'\ 10''$$

$$\log\frac{\cos\mathrm{OL}}{\cos\lambda}=\bar{1}\cdot0427912$$

$$\mathrm{LK}-\lambda=\psi'-18^\circ\ 0'\ 20''$$

Similarly

$$\cos\theta'=\frac{\cos\mathrm{O'L}}{\cos\lambda}\cos(\mathrm{LK}+\lambda')$$

where

$$\tan\lambda'=\tan\mathrm{O'L}\cos\chi$$

$$\therefore\ \lambda'=8^\circ\ 17'\ 30''$$

$$\log\frac{\cos\mathrm{O'L}}{\cos\lambda'}=\bar{1}\cdot3243204$$

$$\mathrm{LK}+\lambda'=\psi'+6^\circ\ 34'\ 20''$$

A table on the next page gives the values of ψ, the angle made by the incident wave normal with the normal to X; $\mathrm{D}+i$ the deviation + the angle X Y; ψ' the angle made by the wave normal in the crystal with the normal to X; and $\frac{1}{\mu^2}$, μ being the reciprocal of the wave velocity.

TABLE to determine the Value of μ_c.

ψ.	D+i.	ψ'.	$\frac{1}{\mu^2}$.
° ′ ″	° ′ ″	° ′ ″	
51 16 7	101 47 15	30 39 0	·427076
49 41 7	101 48 45	29 53 13	·427061
47 49 7	101 57 45	28 57 43	·427049
46 5 22	102 13 0	28 6 3	·427039
44 26 7	102 34 45	27 14 5	·427029
42 55 27	103 3 5	26 25 31	·427018

Substituting the values of ψ' from this table in the formulæ for cos θ and cos θ' we can find θ and θ'.

The values of $\theta-\theta'$ are

° ′ ″
3 31 50
3 37 30
3 42 20
3 47 50
3 52 50
3 57 40

Substituting these values in the formula for μ_c, the mean of the results agrees closely with

$$\underline{\mu_c=1\cdot53013}$$

Thus we may take

$$\underline{\mu_a=1\cdot68560}$$

$$\underline{\mu_b=1\cdot68115}$$

$$\underline{\mu_c=1\cdot53013}$$

And the angle between O C and the optic axes is

$$9^\circ\ 4'\ 5''. \qquad (5)$$

If ϕ is the angle as seen in air

$$\underline{\phi=15^\circ\ 21'\ 50''}$$

The value KIRCHHOFF found by experiment was

$$\underline{15^\circ\ 27'}$$

The close agreement is noteworthy.

As in the first part, the agreement is much closer than it would be taking RUDBERG'S values of μ_a, μ_b, μ_c.

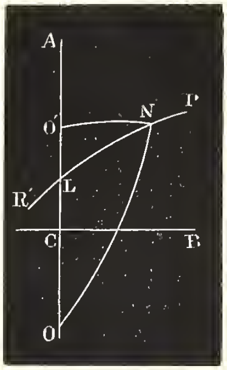
Section VI.—*Application of the Theory to Experiment.—Tables giving Theoretical Values of μ.—Discussion of Results.*

We can now continue the work of calculating the theoretical values of the reciprocal of the velocity.

Consider first the plane P R′.

With the same notation we have [Section IV. (3 and 4)].

Fig. 10.



$$\cos\theta = \frac{\cos \mathrm{OL}}{\cos\lambda}\cos(\mathrm{NL}+\mathrm{L}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\tan\lambda = \tan \mathrm{OL} \cos \mathrm{ALN}$$

$$\cos\theta' = \frac{\cos \mathrm{O'L'}}{\cos\lambda'}\cos(\mathrm{NL}-\lambda') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\tan\lambda' = \tan \mathrm{O'L} \cos \mathrm{ALN}$$

$$\mathrm{OC} = \mathrm{O'C} = 9^\circ\ 4'\ 5'' \quad \text{[Section V. (5)]}$$

$$\mathrm{LC} = 1^\circ\ 21'\ 42'' \quad \text{[Section II. (15)]}$$

$$\therefore\ \mathrm{OL} = 10^\circ\ 25'\ 47'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\mathrm{O'L} = 7^\circ\ 42'\ 23'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\mathrm{ALN} = \chi = 59^\circ\ 20'\ 11'' \quad \text{[Section II. (17)]}.$$

Whence

$$\lambda = 5^\circ\ 21'\ 47'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\log\frac{\cos \mathrm{OL}}{\cos\lambda} = \bar{1}\cdot 9946698 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\lambda' = 3^\circ\ 56'\ 52'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\log\frac{\cos \mathrm{O'L}}{\cos\lambda'} = \bar{1}\cdot 9970915 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$LN = LP - PN = LP - \phi' = 35^\circ\ 0'\ 19'' - \phi' \quad \ldots \ldots \quad (9)$$

$$LN + \lambda = 40^\circ\ 22'\ 6'' - \phi' \quad \ldots \ldots \quad (10)$$

$$LN - \lambda' = 31^\circ\ 3'\ 27'' - \phi' \quad \ldots \ldots \quad (11)$$

Substituting these values we can calculate θ and θ', and hence find the values of v_1 and v_2 or their reciprocals μ_1 and μ_2.

Tables V. and VI. give the results of this substitution: on the right hand for the inner sheet, on the left hand for the outer.

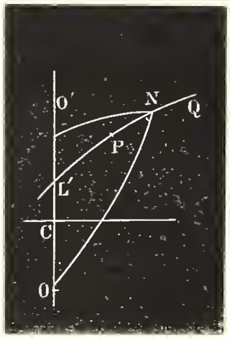
The centre columns give the values of $L\,N + \lambda$, $L\,N - \lambda'$.

Then on either side, $\theta + \theta'$ for the inner sheet, and $\theta - \theta'$ for the outer. Then the theoretical values of $\mu_1\ \mu_2$, then the experimental values taken from Tables I. and II., and finally, in the outside column, the excess of theory above experiment.

The next step is to discuss the theory for the plane P Q.

With the same figure and notation as before, if this plane cut the plane A C in L′ we have [Section IV. (3 and 4)] $\lambda_{,}\ \lambda'_{,}$ being values of $\lambda\ \lambda'$ in this case.

Fig. 11.



$$\cos\theta = \frac{\cos OL'}{\cos\lambda_{,}} \cos(L'N + \lambda_{,}) \quad \ldots \ldots \quad (12)$$

$$\tan\lambda_{,} = \tan OL' \cos AL'N$$

$$\cos\theta' = \frac{\cos O'L'}{\cos\lambda_{,}'} \cos(L'N - \lambda_{,}') \quad \ldots \ldots \quad (13)$$

where

$$\tan\lambda_{,}' = \tan O'L' \cos AL'N$$

$$CO = CO' = 9^\circ\ 4'\ 5'' \quad \text{[Section V. (5)]}$$

$$CL' = 1^\circ\ 15'\ 40'' \quad \text{[Section III. (7)]}$$

$$\therefore OL' = 10^\circ\ 19'\ 45'' \quad \ldots \ldots \quad (14)$$

$$O'L' = 7^\circ\ 48'\ 25'' \quad \ldots \ldots \quad (15)$$

$$AL'N = \chi' = 59^\circ\ 13'\ 2'' \quad \text{[Section III. (8)]}$$

whence

$$\lambda_{,}=5° \ 19' \ 45'' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (16)$$

$$\lambda_{,}'=4° \ 0' \ 50''. \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (17)$$

$\mathrm{L'N}=\mathrm{L'P}+\mathrm{PN}$ (N being now on the side of P remote from L′) $=35° \ 3' \ 14''+\phi'$.

$$\mathrm{L'N}+\lambda_{,}=40° \ 22' \ 59''+\phi' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (18)$$

$$\mathrm{L'N}-\lambda_{,}'=31° \ 2' \ 24''+\phi' \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (19)$$

$$\log \frac{\cos \mathrm{OL'}}{\cos \lambda_{,}}=\bar{1}\cdot 9947864 \ . \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (20)$$

$$\log \frac{\cos \mathrm{O'L'}}{\cos \lambda_{,}'}=\bar{1}\cdot 9970226 \ . \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (21)$$

and as before, we can calculate from these data the values of θ, θ' and hence of μ_1 μ_2.

Tables VII. and VIII give the results of this.

The fifth column gives the values of L N+λ for the inner sheet, the next L N−λ′, the next $\theta+\theta'$, the eighth the value of the reciprocal of the velocity calculated on Fresnel's theory [by means of Section IV. (1 and 2)],

$$\frac{2}{\mu_1^2}=a^2+c^2-(a^2-c^2)\cos(\theta+\theta')$$

$$\frac{2}{\mu_2^2}=a^2+c^2-(a^2-c^2)\cos(\theta-\theta')$$

the ninth, the same quantity from experiment, and the last, the excess of the theoretical over the experimental value.

The values of $\theta-\theta'$ given in the fourth column are calculated for the inner sheet.

Now as the values of ϕ' corresponding to the same incident wave are different for the two sheets, the values of θ and θ' will be different; but on referring to the values it will be seen that after the first twenty observations, $\theta-\theta'$ is very nearly constant throughout the arc considered; so that μ_2 varies very slowly, and except just at first, we may, to the degree of approximation required, use the value of θ and θ' for the inner sheet in calculating the theoretical value of the reciprocal of the velocity for the outer sheet.

In the case of the first twenty observations for which $\theta-\theta'$ varies appreciably with ϕ', the experimental values of μ_2 are obtained by interpolation from those values in Table I., to which they correspond.

Table V. contains the theory for I.
,, VI. ,, ,, II.
,, VII. ,, ,, III.
,, VIII. ,, ,, IV.

TABLE V.

Outer sheet.				LN + λ.	LN − λ′.	Inner sheet.				
Difference.	μ_2 Experiment.	μ_2 Theory.	$\theta - \theta'$.			$\theta + \theta'$.	μ_1 Theory.	μ_1 Experiment.	Difference.	
			° ′ ″	° ′ ″	° ′ ″	° ′ ″				
				4 15 50	5 2 49	18 14 0	1·68111	1·68102	09	1
				4 29 36	4 49 3	18 11 50	1·68113	1·68112	01	2
− 4	1·68558	1·68554	2 6 50	4 46 7	4 32 32	18 9 50	1·68114	1·68106	08	3
− 2	1·68553	1·68551		5 5 46	4 12 53	18 8 30	1·68115	1·68115	00	4
+ 1	1·68546	1·68547		5 28 9	3 50 30	18 8 10	1·68115	1·68114	01	5
+ 5	1·68539	1·68544		5 53 23	3 25 16	18 9 40	1·68114	1·68115	−01	6
+ 9	1·68532	1·68541		6 21 12	2 57 27	18 13 10	1·68111	1·68106	05	7
+10	1·68527	1·68537	4 12 20	6 51 42	2 26 57	18 19 40	1·68106	1·68099	07	8
+ 9	1·68520	1·68529		7 24 45	1 53 54	18 29 50	1·68097	1·68089	08	9
+20	1·68502	1·68522		8 0 14	1 18 25	18 44 30	1·68085	1·68074	09	10
+ 0	1·68515	1·68515		8 38 15	0 40 24	19 4 30	1·68068	1·68069	−01	11
+ 9	1·68498	1·68507		9 18 10	0 0 29	19 30 50	1·68046	1·68030	16	12
+ 8	1·68490	1·68498	6 44 30	10 0 38	0 41 59	20 3 50	1·68017	1·68016	01	13
+ 3	1·68486	1·68489		10 44 46	1 26 7	20 44 10	1·67980	1·67962	18	14
+16	1·68465	1·68481		11 31 5	2 12 26	21 32 50	1·67936	1·67917	19	15
+ 1	1·68473	1·68474		12 20 13	3 1 34	22 29 0	1·67880	1·67859	21	16
+ 4	1·68463	1·68467		13 9 18	3 50 39	23 31 30	1·67817	1·67806	11	17
− 8	1·68454	1·68462	8 28 10	14 0 52	4 42 13	24 42 10	1·67743	1·67721	22	18
+ 1	1·68457	1·68458		14 54 8	5 35 29	26 0 30	1·67656	1·67634	22	19
− 3	1·68457	1·68454		15 48 55	6 30 16	27 23 50	1·67560	1·67532	28	20
− 8	1·68459	1·68451		16 45 11	7 26 32	28 53 10	1·67463	1·67421	42	21
− 3	1·68451	1·68448		17 42 53	8 24 14	30 28 20	1·67331	1·67303	27	22
− 3	1·68448	1·68445	9 10 40	18 41 48	9 23 9	32 7 0	1·67200	1·67161	39	23
	1·68441			19 42 3	10 23 24	33 52 0	1·67054	1·67015	39	24
	1·68446			20 43 28	11 24 49	35 40 30	1·66896	1·66854	42	25
	1·68461			21 46 6	12 27 27	37 33 20	1·66726	1·66694	32	26
	1·68446			22 50 40	13 31 1	39 26 50	1·66547	1·66502	45	27
− 9	1·68448	1·68439	9 26 20	23 54 16	14 35 37	41 26 0	1·66352	1·66300	52	28

TABLE VI.

Outer sheet.				LN + λ.	LN − λ′.	Inner sheet.				
Difference.	μ_2 Experiment.	μ_2 Theory.	$\theta-\theta'$. ° ′ ″	° ′ ″	° ′ ″	$\theta+\theta'$. ° ′ ″	μ_1 Theory.	μ_1 Experiment.	Difference.	
	1·68433			21 47 46	12 29 7	37 35 30	1·66702	1·66678	24	1
	1·68458			22 53 5	13 34 26	39 34 50	1·66534	1·66490	44	2
	1·68446			23 58 30	14 38 51	41 32 40	1·66341	1·66302	39	3
	1·68445			25 1 5	15 42 26	43 29 30	1·66143	1·66095	48	4
	1·68448			26 3 40	16 45 1	45 26 0	1·65939	1·65883	56	5
−21	1·68458	1·68437	9 29 20	27 5 9	17 46 30	47 21 20	1·65730	1·65667	63	6
	1·68453			28 5 21	18 46 42	49 14 40	1·65520	1·65459	61	7
	1·68448			29 4 21	19 45 42	51 7 20	1·65306	1·65240	66	8
	1·68445			30 1 48	20 43 9	52 54 40	1·65098	1·65037	61	9
	1·68446			30 57 55	21 39 16	54 42 40	1·64884	1·64818	66	10
−15	1·68452	1·68437	9 30 40	31 52 25	22 33 46	56 26 50	1·64674	1·64603	71	11
	1·68446			32 45 5	23 26 26	58 8 30	1·64466	1·64400	66	12
	1·68451			33 35 53	24 17 14	59 45 20	1·64264	1·64203	61	13
	1·68450			34 25 4	25 6 25	61 20 10	1·64055	1·63984	69	14
	1·68454			35 12 2	25 53 23	62 50 40	1·63872	1·63782	90	15
−13	1·68450	1·68437	9 30 40	35 56 25	26 37 46	64 16 40	1·63678	1·63615	63	16
	1·68445			36 38 46	27 20 7	65 38 40	1·63509	1·63433	76	17
	1·68445			37 18 41	28 0 2	66 55 50	1·63340	1·63258	82	18
	1·68453			37 55 50	28 37 11	68 7 50	1·63182	1·63100	82	19
	1·68437			38 31 20	29 12 41	69 16 30	1·63030	1·62956	74	20
−10	1·68447	1·68437	9 30 0	39 2 12	29 43 33	70 16 40	1·62897	1·62807	90	21
	1·68428			39 30 46	30 12 7	71 12 20	1·62776	1·62691	85	22
	1·68451			39 56 22	30 37 43	72 2 0	1·62662	1·62583	79	23

TABLE VII.

Outer sheet.				LN + λ.	LN − λ′.	Inner sheet.				
Difference.	μ_2 Experiment.	μ_2 Theory.	$\theta - \theta'$.			$\theta + \theta'$.	μ_1 Theory.	μ_1 Experiment.	Difference.	
			° ′ ″	° ′ ″	° ′ ″	° ′ ″				
−27	1·68464	1·68437	9 29 50	39 52 39	30 32 4	71 52 50	1·62682	1·62619	63	1
	1·68450			40 20 13	30 59 38	72 46 30	1·62563	1·62500	63	2
	1·68454			40 50 20	31 29 45	73 45 10	1·62430	1·62360	70	3
	1·68450			41 23 2	32 2 27	74 48 50	1·62268	1·62208	60	4
	1·68458			41 58 27	32 37 52	75 58 0	1·62131	1·62058	74	5
−19	1·68456	1·68437	9 29 30	42 36 25	33 15 50	77 12 10	1·61965	1·61904	61	6
	1·68464			43 16 25	33 55 50	78 30 20	1·61786	1·61715	71	7
	1·68462			43 58 52	34 38 17	79 53 30	1·61600	1·61525	75	8
	1·68469			44 43 27	35 22 52	81 20 20	1·61400	1·61320	80	9
	1·68462			45 30 22	36 9 47	82 52 0	1·61192	1·61120	72	10
−15	1·68453	1·68438	9 28 30	46 19 11	36 58 36	84 27 50	1·60974	1·60897	77	11
	1·68462			47 10 3	37 49 28	86 7 40	1·60748	1·60671	77	12
	1·68457			48 2 58	38 42 23	87 51 10	1·60514	1·60450	64	13
	1·68454			48 57 29	39 36 54	89 38 10	1·60285	1·60201	84	14
	1·68462			49 53 45	40 33 10	90° + 1 28 40	1·60025	1·59943	82	15
−28	1·68466	1·68438	9 27 20	50 51 59	41 31 24	90° + 3 23 0	1·59769	1·59710	59	16
	1·68549			51 51 24	42 30 49	,, 5 19 50	1·59510	1·59427	83	17
	1·68465			52 52 38	43 32 3	,, 7 20 20	1·59246	1·59173	73	18
	1·68469			54 55 23	44 34 48	,, 9 23 40	1·58977	1·58911	66	19
	1·68475			54 59 31	45 38 56	,, 11 30 0	1·58705	1·58651	54	20
−18	1·68457	1·68439	9 26 20	56 4 46	46 44 11	90° + 13 38 30	1·58432	1·58365	67	21
	1·68464			57 11 23	47 50 48	,, 15 49 30	1·58155	1·58086	69	22
	1·68465			58 19 14	48 58 39	,, 18 3 30	1·57880	1·57815	65	23
	1·68458			59 28 9	50 7 34	,, 20 19 30	1·57605	1·57536	69	24
	1·68465			60 38 12	51 17 37	,, 22 37 40	1·57329	1·57269	60	25
−36	1·68475	1·68439	9 24 30	61 49 11	52 28 36	90° + 24 57 50	1·57058	1·56994	64	26

TABLE VIII.

Outer sheet.				LN + λ.	LN − λ′.	Inner sheet.				
Difference.	μ_2 Experiment.	μ_2 Theory.	$\theta-\theta'$.			$\theta+\theta'$.	μ_1 Theory.	μ_1 Experiment.	Difference.	
			° ′ ″	° ′ ″	° ′ ″	° ° ′ ″				
− 5	1·68444	1·68439	9 25 10	59 59 19	50 38 44	90+21 20 50	1·57482	1·57426	56	1
	1·68456			61 9 51	51 49 16	„ 23 40 0	1·57209	1·57144	65	2
	1·68464			62 19 26	52 58 51	„ 25 57 30	1·56944	1·56879	65	3
	1·68454			63 28 2	54 7 27	„ 28 13 0	1·56690	1·56616	74	4
	1·68454			64 35 30	55 14 55	„ 30 26 20	1·56446	1·56383	63	5
− 8	1·68448	1·68440	9 22 30	65 41 46	56 21 11	90+32 36 10	1·56215	1·56153	62	6
	1·68458			66 46 46	57 26 11	„ 34 45 50	1·55991	1·55929	62	7
	1·68452			67 50 21	58 29 46	„ 36 51 10	1·55779	1·55716	63	8
	1·68440			68 52 25	59 31 50	„ 38 54 20	1·55580	1·55512	68	9
	1·68452			69 52 32	60 31 57	„ 40 52 50	1·55393	1·55342	51	10
−12	1·68452	1·68440	9 21 50	70 51 12	61 30 47	90+42 48 50	1·55216	1·55157	59	11
	1·68451			71 47 46	62 27 11	„ 44 40 50	1·55051	1·54996	55	12
	1·68454			72 42 18	63 21 43	„ 46 28 40	1·54899	1·54846	53	13
	1·68450			73 34 38	64 14 3	„ 48 12 0	1·54756	1·54708	48	14
	1·68459			74 24 57	65 4 22	„ 49 51 40	1·54623	1·54561	62	15
− 7	1·68448	1·68441	9 20 0	75 11 18	65 51 43	90+51 24 20	1·54515	1·54460	55	16
	1·68454			75 57 25	66 36 50	„ 52 54 40	1·54393	1·54348	45	17
	1·68464			76 39 44	67 19 9	„ 54 18 20	1·54293	1·54251	42	18
	1·68461			77 19 16	67 58 41	„ 55 36 40	1·54203	1·54163	40	19
	1·68452			77 55 51	68 35 16	„ 56 49 0	1·54123	1·54082	41	20
−23	1·68465	1·68442	9 19 50	78 29 14	69 8 39	90+57 55 10	1·54051	1·54019	32	21
	1·68442			78 59 44	69 39 9	„ 58 55 30	1·53988	1·53948	40	22
	1·68447			79 26 41	70 6 6	„ 59 49 0	1·53934	1·53900	34	23
	1·68452			79 50 31	70 29 56	„ 60 36 10	1·53887	1·53846	41	24
	1·68446			80 10 35	70 50 0	„ 61 16 0	1·53848	1·53821	27	25
− 2	1·68444	1·68442	9 19 20	80 27 39	71 6 55	90+61 49 20	1·53817	1·53774	33	26

Let us consider first the inner sheet.

In Table V. we see the differences are small to begin with, and increase fairly regularly throughout the whole of the table, the experimental value being almost always less than the theoretical.

Lines 4, 5, and 6 were those chosen to determine the value of μ_b which explains the coincidence of the two curves in that neighbourhood. The table covers an arc of the surface of nearly 20°, as is seen by referring to the values of ϕ' in Table I.

Throughout this arc the theoretical section lies outside the experimental, the difference between the two increasing as we proceed from the axis μ_b.

This continues throughout Table VI.

The first three lines of Table VI. just overlap the last three of Table V. The difference reaches a maximum value towards the end of Table VI. at a point for which $\phi'= 1^\circ 19' 54''$ (*vide* Tables II., 21, and VI., 21), that is at about 34° away from the axis μ_b.

After this the difference begins to decrease at first slowly throughout Table VII. which refers to the other prism P Q, so that the arc of section has changed slightly between Tables VI. and VII., both arcs passing through the point P.

Table VII. covers an arc of 22° (Table III.).

The first two lines in Table VII. refer to a wave normal lying on the same side of P as L. In the rest of the table the wave normal is on the side of P remote from L.

This arc is continued throughout Table VIII.; the theoretical values of μ are still greater than the experimental, but the difference diminishes as we proceed along the arc.

The last three observations in Table VII. and the first three in Table VIII. overlap.

Table VIII. covers an arc of 20°.

Thus the observations extend over two arcs, each passing through the point P inclined to one another at that point at so small an angle as to be almost continuous, the one arc being 35° 40′, and the other 40° 40′ in length.

The results of theory and experiment agree at the extremity of the first arc, differ most widely in the neighbourhood of the point where the two arcs meet, and tend towards equality again throughout the second arc.

It is worth noticing that the experimental results at the end of Table VI., the end of the first arc, agree closely with those at the beginning of Table VII., the beginning of the second arc.

The experimental results for the inner sheet are therefore represented by a curve which coincides with FRESNEL'S curve at the extremity of one axis, and lies inside it throughout the next 75° of its length.

For the outer sheet the differences are less than for the inner sheet.

$\theta-\theta'$ varies so slowly that it seemed sufficient for the purposes of comparison to calculate every fifth theoretical value, with the exception of that portion of Table V. for which the variations of $\theta-\theta'$ are sensible.

The section cuts the principal section A O C between lines 4 and 5, Table V. Just at that point the theoretical value is greater than the experimental. It continues so until we reach line 18, Table V., that is for an arc of about 9°.

From that point onwards the theoretical value of μ_2 is uniformly less than the experimental.

Thus the result of experiment for the outer sheet is represented by a curve which coincides with the theoretical section at the point in which it cuts the plane A O C, then lies inside it for an arc of about 9°.

At this point the two again coincide, and for the rest of the arc experimented on, the section given by experiment lies outside that given by theory; the difference between the two increasing throughout Tables VI. and VII., and diminishing again in Table VIII.

Section VII.—*Effect of possible Errors in the Values of* μ_a, μ_b, μ_c *discussed.*

The next task will be to determine the effect of any small errors which may have occurred:

(1.) In the determination of μ_a, μ_b, μ_c;

(2.) In the angles which fix the position of the planes P R, P Q relatively to the axes of the surface.

(1.) μ_a. To determine μ_a we considered the length of the radius vector common to the principal section A O C and the section by the plane P R. This radius vector is fixed in length, and is on FRESNEL'S theory the radius vector of an ellipse, axes μ_a and μ_c, inclined at a small angle to μ_a.

The increase in the value of μ_a will therefore increase the value of this angle, that is, it will alter the position of the plane P L R′ and be more properly considered under (2).

μ_b is determined by the intersection of the section of the inner sheet by the plane P R, with a circle of radius μ_b, and is therefore fixed at least within the limits of experimental error.

μ_c is found from an independent and, on the whole, less trustworthy series of observations—less trustworthy because the faces were much less plane than P, Q, R.

Let us consider the effect of decreasing the value of μ_c.

If ϵ be the angle between the optic axes

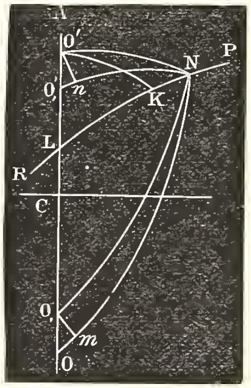
$$\tan\epsilon = \sqrt{\left\{\frac{\frac{1}{\mu_b^2}-\frac{1}{\mu_a^2}}{\frac{1}{\mu_c^2}-\frac{1}{\mu_b^2}}\right\}}$$

As μ_c decreases $\tan\epsilon$ decreases; therefore ϵ decreases.

Let us consider how this affects θ and θ'.

Taking the same figure as previously.

Fig. 12.



Let O′ K be an arc perpendicular to A C.

Let $O_{,}$ $O_{,}'$ be the new positions of O O′.

Then

$$O_{,}O=O_{,}'O'=\delta\epsilon$$

Let $O_{,}$ m, O′ n be perpendicular on N O, N $O_{,}'$ respectively.

$$O'L=7^{\circ} \text{ approximately}$$
$$O'LP=60^{\circ} \text{ approximately}$$
$$\therefore\ LK=14^{\circ} \text{ approximately}$$

Let N fall between P and K

$$\delta\theta=-Om=-\delta\epsilon\cos NOL$$
$$\delta\theta'=O_{,}'n=\delta\epsilon\cos NO_{,}'A$$
$$\delta(\theta+\theta')=-\delta\epsilon\,(\cos NOL-\cos NO_{,}'A)$$

Thus $\delta(\theta+\theta')$ is negative and decreases numerically as we approach P; for N O L, N $O_{,}'$ A become more nearly equal.

Thus between K and P $\theta+\theta'$ decreases with ϵ.

Between L and K both θ and θ' decrease, therefore $\theta+\theta'$ decreases *à fortiori*; but $\delta(\theta+\theta')$ is always numerically less than $2\delta\epsilon$.

Let us see how these changes affect μ_1.

We have [Section IV. (1)]

$$\frac{2}{\mu_1^2}=\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta').$$

If μ_c is decreased, $\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}$ is increased, $\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}$ is increased, $\theta+\theta'$ is decreased.

Therefore $\cos(\theta+\theta')$ is increased so long as $\theta+\theta'$ is less than 90°.

Therefore $\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ is increased so long as $\theta+\theta'$ is less than 90°.

And the increase in $\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ is greatest when $\theta+\theta'$ is least.

Also the increase in $\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ is less than the increase in $\left(\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}\right)$.

Therefore $\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ is increased, and this increase is greatest when $\theta+\theta'$ is greatest.

Therefore, as $\theta+\theta'$ increases up to 90° μ_1 decreases.

When $\theta+\theta'$ is greater than 90°, $\cos(\theta+\theta')$ is negative.

As $\theta+\theta'$ increases, $\cos(\theta+\theta')$ increases numerically.

Therefore $\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ increases numerically and is negative.

Therefore $\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$ increases *à fortiori*.

Therefore μ_1 decreases.

Thus the effect of decreasing μ_c is to produce a decrease in the value of μ_1, and this decrease continually increases as we go from L to Q.

Now if we compare the differences between the theoretical and experimental values of μ_1, given in Tables V. and VI., we see that they increase with $\theta+\theta'$, the theoretical value being in excess of the experimental.

The effect of the assumed alteration in the value of μ_c would be to decrease the theoretical value of μ_1 by an amount which constantly increases, that is, to bring it more nearly into agreement with experiment.

But the effect in Tables VII. and VIII. would be contrary.

On referring to them we see that the differences between theory and experiment continually decrease.

The effect of the proposed alteration would be to subtract from the theoretical value of μ_1 given in those tables: a quantity which increases rapidly as we get further from L.

The result would be that the theoretical value would soon become less than the experimental, and the difference between the two would continually increase.

We might, it is true, choose the decrement of μ_c so as to make the extremities of the arcs considered coincide, and then the differences would be diminished throughout. The effect of this would be to make the theoretical section lie outside of the experimental for about 75°, then cut it and lie inside it.

Whether the two could again be made to coincide when cutting the plane A O B experiment alone could settle.

Let us now consider the effect of the supposed change on μ_2.

We have

$$\frac{2}{\mu_2^2}=\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta').$$

We have seen that θ is decreased by the change.

θ' is at first decreased, though by less than θ, afterwards it is increased.

Therefore, for any point on the curve L P the value of $\theta-\theta'$ is decreased by this alteration.

Therefore $\cos(\theta-\theta')$ is increased.

$\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$ is increased, though by less than $\left(\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}\right)$.

Therefore $\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$ is increased.

Therefore μ_2 is decreased.

Or the variation in μ_2 is in the same direction as in μ_1.

But since $\theta-\theta'$ is $<\theta+\theta'$ $\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$ is $>\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$.

And the whole change in

$$\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$$

is much less than the whole change in

$$\left(\frac{1}{\mu_c^2}+\frac{1}{\mu_a^2}\right)-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta').$$

Or the decrease in μ_2 is small compared with that in μ_1.

Again, as the variation in $\theta-\theta'$ varies from zero at L to a maximum about K, while $\cos(\theta-\theta')$ also varies considerably, the variation in μ_2 will be considerably different for different points between L and K.

After passing K, however, $\theta-\theta'$ is nearly constant all along the curve.

The change in the value of μ_2, therefore, will not differ much for different points, that is, for different waves.

The whole effect of the alteration, therefore, will be to decrease throughout the theoretical value of μ_2, this decrease being greatest about Table V., line 23.

In the greater part of Table V. the result will be to produce a closer agreement between theory and experiment.

But in the rest of the work, since the theoretical value of μ_2 is less than the experimental, the effect will be to increase the difference between the two and so widen the discrepancy between theory and experiment.

Thus taking both sheets we cannot produce on the whole a closer agreement between theory and experiment by decreasing the value of μ_c. Neither can we by increasing it.

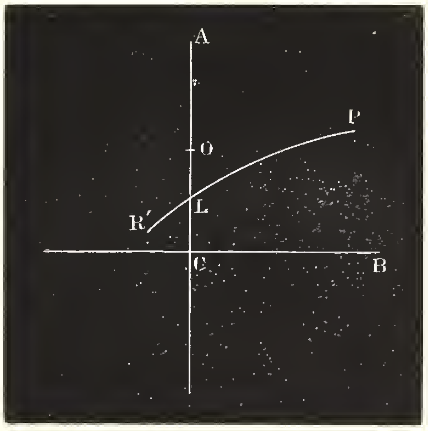
For though we might produce closer agreement for the outer sheet by thus increasing the theoretical value of μ_2, we should add greatly to the error in the case of the inner sheet.

Section VIII.—*Effect of Change in Position of Planes of Prisms with reference to the Principal Planes of the Crystal.*

We have now to discuss the effect of small variations in the position of the planes P R and P Q.

We take P R first, and we consider its position as determined by C L and the angle A L P, while the position of the normals to P and R is fixed by P L (*vide* Section II.).

Fig. 13.



Now variations in C L and A L P produce variations in O L, O′ L, λ λ′, &c., and hence in θ θ', while variations in L P only change θ and θ', O L, O′ L, λ and λ′ remaining constant.

Let us consider, therefore, the variations of L P first.

A reference to our tables shows us that through the greater part of the arcs L P, P Q′, $\theta-\theta'$ is constant, so that $\delta\theta$ considered as dependent on L P is equal to $\delta\theta'$ nearly.

Again,

$$\frac{2}{\mu_1^2}=\left(\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}\right)-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$$

$$\frac{2}{\mu_2^2}=\left(\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}\right)-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$$

and since the change in $(\theta-\theta')$ due to a small increment of L P is very small, such an increment will not alter the value of μ_2.

But if $\theta-\theta'$ is nearly constant for a small motion along L P, $\theta+\theta'$ will be nearly so for a small motion perpendicular to L P.

Thus a small motion perpendicular to P L will not alter appreciably the value of μ_1;

we may therefore obtain corrections to μ_1 μ_2 independently, to μ_1 by supposing an increase in the value of L P, to μ_2 by supposing the planes P R. P Q to turn through a small angle about some line in them.

Moreover, we shall consider that line to be in each case the line of intersection with the plane A O C, and treat this line as the same for both planes, the two lines thus considered as coincident are really inclined at an angle of 3′.

To find then the variation which we must assume in L P to bring the results of theory and experiment into closer agreement for the inner sheet, we proceed as follows—

If in the equation

$$\frac{2}{\mu_1^2}=\left(\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}\right)-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta+\theta')$$

we substitute an experimental value for μ_1, we can determine a value of $(\theta+\theta')$. Subtracting this from the calculated value, we get $\delta(\theta+\theta')$.

But for the displacement considered $\delta\theta=\delta\theta'$.

Hence we have found $\delta\theta$; but $\cos(\mathrm{LN}-\lambda)=\frac{\cos\lambda}{\cos \mathrm{OL}}\cos\theta$. O L and λ are unaltered by the displacement supposed; therefore we get δ(L N), and L P=L N+ϕ'. ϕ' is constant, therefore we have found δ L P.

To put this plan into execution, I chose as the experimental value for μ_1 the 21st in Table VI.

$$\mu_1=1\cdot62807$$

the differences between theory and experiment having at that point a maximum value and being fairly regular.

The resulting value for $(\theta+\theta')$ is

$$70°\ 57'\ 20''$$

The calculated value is

$$70°\ 16'\ 40''$$

The difference is

$$0°\ 40'\ 40''$$

$$\therefore\ \delta\theta=0°\ 20'\ 20''$$

This gives for δ(L P)

$$\delta(\mathrm{L\,P})=0°\ 21'\ 20''$$

Thus at this point the result of theory and experiment could be brought into coincidence by supposing the line O P normal to the face P to turn through an angle of 0° 21′ 20″ in the plane L P away from L.

Before discussing the effect of this change on the values of μ_1 in other directions, let us consider the variation it will be necessary to make in the angle A L N to bring the theoretical results for the outer sheet into accordance with experiment. Taking the experimental value for μ_2 corresponding to the incident wave which gave rise to the value of μ_1 we have just been discussing, we have

$$\mu_2=1{\cdot}68447$$

Whence by means of the formula

$$\frac{2}{\mu_2^2}=\left(\frac{1}{\mu_a^2}+\frac{1}{\mu_c^2}\right)-\left(\frac{1}{\mu_c^2}-\frac{1}{\mu_a^2}\right)\cos(\theta-\theta')$$

we find

$$\theta-\theta'=9^\circ\ 15'$$

The value of $\theta-\theta'$ given by theory is Table VI., line 21,

$$\theta-\theta'=9^\circ\ 30'$$

We have to find the change in the value of χ or A L N which will produce this.

In this variation $\theta+\theta'$ is constant

$$\therefore\ \delta\theta=-\delta\theta'$$

but

$$\delta(\theta-\theta')=-15'$$

$$\therefore\ \delta\theta=-7'\ 30''$$

$$\delta\theta'=\ \ 7'\ 30''$$

From these results we can find the increment in χ from the equations

$$\cos\theta=\frac{\cos \mathrm{OL}}{\cos\lambda}\cos(\mathrm{LN}-\lambda)$$

$$\tan\lambda=\tan \mathrm{OL}\cos\chi$$

$$\cos\theta=\cos \mathrm{OL}\cos \mathrm{LN}+\sin \mathrm{OL}\sin \mathrm{LN}\cos\chi$$

$$\therefore\ \delta\chi=\frac{\sin\theta\delta\theta}{\sin \mathrm{OL}\sin \mathrm{LN}\sin\chi}$$

Substituting values we are led to a value of $\delta\chi$, nearly equal to 1°. So that the new value of $\chi=60^\circ\ 20'$ [Section II. (17)].

This change in the value of χ produces a corresponding change in the values of λ, λ'. The new values are for the prism P R

$$\lambda=5^\circ\ 12'\ 20'' \qquad (1)$$

$$\lambda'=3^\circ\ 50'\ \ 0'' \qquad (2)$$

while

$$\log\frac{\cos \mathrm{OL}}{\cos\lambda}=\bar{1}\cdot9945594 \quad \ldots\ldots\ldots\ldots \quad (3)$$

$$\log\frac{\cos \mathrm{OL}}{\cos\lambda'}=\bar{1}\cdot9970325 \quad \ldots\ldots\ldots\ldots \quad (4)$$

for the prism P Q

$$\lambda_{,}=5^\circ\ 10'\ 20'' \quad \ldots\ldots\ldots\ldots \quad (5)$$

$$\lambda_{,}=3^\circ\ 54' \quad \ldots\ldots\ldots\ldots \quad (6)$$

$$\log\frac{\cos \mathrm{OL}'}{\cos\lambda_{,}}=\bar{1}\cdot9946764 \quad \ldots\ldots\ldots\ldots \quad (7)$$

$$\log\frac{\cos \mathrm{O'L'}}{\cos\lambda_{,}'}=\bar{1}\cdot9969629 \quad \ldots\ldots\ldots\ldots \quad (8)$$

The ensuing Table gives the values of μ_1, μ_2 for nine different incident waves taken from the previous work, calculated on the supposition that both these changes have been made in the position of the plane P R.

The column headed Difference gives the excess of theory over experiment; that headed α gives the excess of the theory before it was modified; β gives the change in the theoretical value of μ_1 due to this displacement.

TABLE IX.

Outer sheet.			Inner sheet.			
Where taken from Experiment Table.	$\theta-\theta'$.	LN + λ.	LN − λ'.	$\theta+\theta'$.		
	° ′ ″	° ′ ″	° ′ ″	° ′ ″		
Table V., line 8	4 25 20	7 3 30	1 57 30	18 25 40		1
„ „ „ 18	8 20 50	14 12 40	5 10 20	25 14 50		2
„ „ „ 28	9 12 0	24 6 10	15 3 50	42 6 0		3
„ VI. „ 11	9 15 20	22 4 20	23 2 0	57 7 40		4
„ „ „ 21	9 14 30	39 14 0	30 11 40	70 57 30		5
„ VII. „ 11	9 12 30	46 31 0	37 26 50	85 8 30		6
„ „ „ 21	9 9 50	56 16 30	47 12 20	90° + 14 18 50		7
„ VIII. „ 11	9 5 40	71 3 0	61 59 0	„ 43 29 0		8
„ „ „ 26	9 4 20	80 39 20	71 35 10	„ 62 30 40		9

Difference.	μ_2 New Theory.	μ_2 Experiment Value.	μ_2 Former Theory.	μ_1 Former Theory.	μ_1 Experiment Value.	μ_1 New Theory.	Difference.	
+ 7	1·68533	1·68526	1·68537	1·68106	1·68099	1·68012	+ 3	1
+11	1·68465	1·68454	1·68462	1·67743	1·67721	1·67707	−14	2
− 3	1·68445	1·68448	1·68437	1·66352	1·66300	1·66285	−15	3
− 9	1·68443	1·68452	1·68437	1·64674	1·64603	1·64590	−13	4
− 4	1·68443	1·68447	1·68437	1·62897	1·62807	1·62806	− 1	5
− 9	1·68444	1·68453	1·68438	1·60974	1·60897	1·60882	−15	6
−12	1·68445	1·68457	1·68439	1·58432	1·58365	1·58346	−19	7
− 5	1·68447	1·68452	1·68440	1·55216	1·55157	1·55138	−19	8
− 4	1·68448	1·68444	1·68442	1·53793	1·53774	1·53779	+ 5	9

α.	β.
+ 7	4
+22	37
+52	67
+71	86
+90	91
+77	92
+67	86
+59	78
+29	24

It will be noticed that the differences for this modified theory are much less than for the original, but they have nearly all changed sign. In fact, the correction applied, calculated so as to produce the closest possible agreement in line 5, has been too great, and the differences can be reduced by applying a similar but smaller correction in the opposite direction.

Now this variation in the value of μ_1 has been caused almost entirely by the increase of the angle L P; we must therefore decrease this angle again, and as the numbers in the difference column are each about one-fifth of the corresponding number in the column β, we must decrease L P by one-fifth of the amount by which it was previously increased.

But this increase was

$$\delta(\mathrm{LP})=0^\circ\ 21'\ 20''$$

we must therefore decrease this by

$$0^\circ\ 4'\ 15'$$

leaving the total increase

$$0^\circ\ 17'\ 5\,'$$

In consequence of this, we must add to the modified theoretical values of μ_1 respectively, quantities equal to one-fifth of the corresponding numbers in column β.

The result gives us a new set of theoretical values, which agree with experiment throughout to a remarkable degree.

These results are expressed in Table X.

Column 1 gives the values of μ_1 already modified.

Column 2 gives one-fifth of quantities in β, which, when added to the values of μ_1 in Column 1 respectively, give Column 3 the resulting theoretical values.

Column 4 gives the experimental values, and Column 5 the excess of theory over experiment.

TABLE X.

Modified value of μ_1.	$\frac{1}{5}\beta$.	Final value of μ_1.	Experimental value of μ_1.	Difference.	
1·68102	1	1·68103	1·68099	4	1
1·67707	7	1·67714	1·67721	−7	2
1·66285	13	1·66298	1·66300	−2	3
1·64590	17	1·64607	1·64603	4	4
1·62806	18	1·62824	1·62807	17	5
1·60882	18	1·60900	1·60897	3	6
1·58346	17	1·58363	1·58365	−2	7
1·55138	16	1·55154	1·58157	−3	8
1·53779	5	1·53784	1·53774	10	9

These differences are exceedingly small—sometimes positive, sometimes negative—but rarely greater than the possible error of experiment as appears from the tables of error in I., II., III., and IV.

Thus we have found a plane section of the surface which agrees closely with the results of experiment.

This section passes through L, a point on the principal section A C at an angular distance of 1° 21′ 42″ from C, is inclined to that section at an angle of 60° 20′, while the normal to the face P lies at an angular distance from L equal to

$$35^\circ\ 0'\ 19''+0^\circ\ 17'\ 15''=35^\circ\ 17'\ 34''$$

Under these circumstances it will be necessary to consider the probability of making such an error as this result indicates in the position of the normal to P with reference to the principal axes.

But before doing this we must say a few words about the variations in the values of μ_2 produced by the correction now considered.

The final correction will not affect any of them except the first, which will be slightly increased by it.

This follows from the fact that between lines 1 and 2 of Table IX. $\theta-\theta'$ varies considerably, and increases or decreases with $\theta+\theta'$, so that μ_1, μ_2 increase or decrease together.

The change considered has increased μ_1, it will therefore increase μ_2.

The difference in line 2 is considerable, but it may be noticed that the experimental value is clearly too small, falling as it does (Table I., line 18) between two considerably greater values. In the other cases the differences between theory and experiment have been diminished but remain of the same sign as before.

A still further displacement in the same direction would therefore produce still further agreement.

We have now to consider the effect of varying C L.

Let us trace the effect of increasing C L by 10′.

The value of μ_a depends on that of C L being found from it by the formula (Section V.)

$$\frac{1}{\mu_a^2}=\sec^2\mathrm{CL}\left(\frac{1}{r^2}-\frac{\sin^2\mathrm{CL}}{\mu_c^2}\right)$$

where

$$\mu_c^2=1\cdot53013$$

and r is given from experiment

$$r=1\cdot68550$$

The new value of a will be found to be

$$\mu_a=1\cdot68571$$

instead of $1\cdot68560$

This increase in the value of a will increase the angle betwen the optic axes from 9° 4′ 5″, the value already found, to 9° 9′ 20″.

We have, therefore, to discuss the effect of these changes in the theoretical values of μ_1, μ_2.

We will further suppose that the angle χ is so altered that the section still passes through the former position of P—*i.e.*, so that the only error made in the previous determination of its position has been in the position of R′.

This is probable, for C R′ is small and A R′, B R′, nearly right angles.

The new value of χ will be

$$\chi = 59^\circ\ 32'$$

These changes will produce variations in the values of O L, O′L, $\lambda\ \lambda'$, for P R′, and O L′, O′ L′, $\lambda_{,}$ $\lambda_{,}'$, for P Q (we assume that L L′ is unaltered).

The complexity of the changes rendered it very difficult to discuss the effects on μ_1, μ_2 from general considerations. I have therefore calculated the new values in five different positions taken from the previous tables.

The table following gives the general results.

TABLE XI.

Whence taken.	Difference.	μ_1 Modified Theory.	μ_1 Experiment.	μ_1 Former Theory.	Difference.	
Table V., line 28	36	1·66386	1·66300	1·66352	52	1
,, VI., ,, 11	49	1·64652	1·64603	1·64674	71	2
,, VI., ,, 21	65	1·62872	1·62807	1·62897	90	3
,, VII., ,, 21	41	1·58406	1·58365	1·58432	67	4
,, VIII., ,, 11	43	1·55200	1·55157	1·55216	59	5

Whence taken.	Difference.	μ_2 Modified Theory.	μ_2 Experiment.	μ_2 Former Theory.	Difference.	
Table V., line 28	−1	1·68447	1·68448	1·68437	−11	1
,, VI., ,, 11	−6	1·68446	1·68452	1·68437	−15	2
,, VI., ,, 21	+1	1·68448	1·68447	1·68437	−10	3
,, VII., ,, 21	−7	1·68450	1·68457	1·68439	−18	4
,, VIII., ,, 11	0	1·68452	1·68452	1·68440	−12	5

The difference table on the left is between the modified theory and experiment, that on the right between the former theory and experiment.

The upper table refers to the inner, the lower to the outer sheet.

It will be seen that the differences for μ_1, μ_2 are both decreased, though still remaining of the same sign.

The differences for μ_1 are about two-thirds of their former value.

If we then increase C L still further by twice as much as previously, so that the total increase is 30′, we shall get still closer agreement for μ_1.

TABLE XII.

α.	β.	μ_1 Final Theory.	μ_2 Experiment.	Difference.
16	48	1·66304	1·66300	4
22	66	1·64608	1·64603	5
25	75	1·62822	1·62807	15
26	78	1·58354	1·58365	−11
16	48	1·55268	1·55157	11

γ.	δ.	μ_2 Final Theory.	μ_2 Experiment.	Difference.
10	30	1·68467	1·68448	19
9	27	1·68464	1·68452	12
11	33	1·68470	1·68447	23
11	33	1·68472	1·68457	15
12	36	1·68475	1·68452	23

Column α gives the change in μ_1 due to a change of 10′ in C L.

Column β gives three times that change which, assuming the theory of proportional parts, will be the change in μ_1 due to a change of 30′ in C L.

Subtracting this from the values of μ_1 given by the original theory, we get the final theoretical values of μ_1. These agree much more closely with experiment than the original values.

But the effect of this further change will be to over correct μ_2.

The first change has reduced the differences for μ_2 by the amounts given by γ, Table XII.

Multiplying these by 3 we get δ, the changes produced in μ_2 by a change of 30′ in C L, and adding these values to the original theoretical values of μ_2 we get the final theory, which differs from experiment by rather more than the original theory and in the other direction.

Thus an increase of C L will not, on the whole, produce the required effect. Similarly a decrease will not produce it either.

Section IX.—*Possibility of an Error in the Positions of the Faces discussed.*

We have seen, however, that by increasing L P by 17′ 5″ and A L P by 1°, we can make the agreement between theory and experiment extremely close.

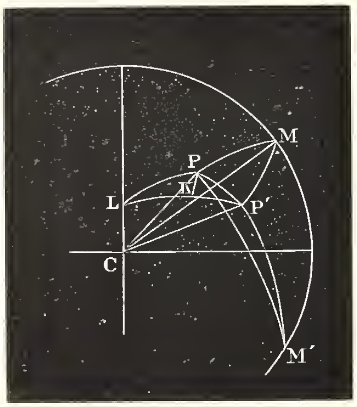
This alteration in the values of L P and A L P may be effected in two ways.

We may suppose either that, the axes of elasticity retaining their position with reference to the planes m, m', we have made an error in the observed angles P M, P M′ (Section II.), and hence in the position of P referred to the axes.

Or, secondly, that the position of P relatively to m m' has been accurately found, but that the axes of elasticity differ slightly in direction from their assumed positions, so that O C is not parallel to the intersection of m m', and O B does not bisect the angle M O M′.

We will take the first hypothesis first, and consider what must be the error in P M, P M′ to give rise to the required change in the position of P.

Fig. 14.



Let P′ be the new position of P, we require M P′ M′ P′.

Let P K be perpendicular on L P′

$$
\begin{aligned}
LK &= LP \text{ approximately}\\
KP' &= 0^\circ\ 17' \text{ approximately}\\
KP &= PLK \sin 35^\circ \text{ approximately}\\
&= 60' \times {\cdot}57\\
&= 34' \text{ approximately}
\end{aligned}
$$

Now P K P′ is approximately a small plane triangle with a right angle at K.

$$
\begin{aligned}
PP' &= \sqrt{PK^2 + P'K^2}\\
&= \sqrt{34^2 + 17^2}\\
&= 17'\sqrt{5}
\end{aligned}
$$

whence

$$PP' = 38'$$

Again

$$\cos CP' = \cos CL \cos LP' - \sin CL \sin LP' \cos ALP'$$

whence

$$
\begin{aligned}
CP' &= 35^\circ\ 58'\\
CP\ &= 35^\circ\ 43' \quad \text{[Section II. (12)]}\\
PP' &= \ 0^\circ\ 38'
\end{aligned}
$$

Hence from the triangle P C P′ we get

$$PCP'=0^\circ\ 58$$

But from the triangle A C P

$$ACP=57^\circ\ 42'\ 30''\ \text{[Section II.]}$$

$$\therefore\ ACP'=58^\circ\ 40'\ 30''$$

but

$$ACM=58^\circ\ 7'\ 30''\ \text{[Section II. (7)]}$$

$$\therefore\ MCP=\ 0^\circ\ 33'$$

$$\cos MP'=\cos MCP \sin CP'$$

whence

$$MP'=54^\circ\ 58'\ 50''$$

The original value of M P′ was as given by the mean of a large number of observations

$$MP'=54^\circ\ 17'\ 6''\ \text{[Section II. (3)]}.$$

The difference, amounting as it does to over 40′, is far beyond any possible error of experiment.

Let us find further the change in M′ P,

$$\cos M'P'=\sin CP' \cos M'CP'$$

$$M'CP'=63^\circ\ 47'-33'$$

$$=63^\circ\ 14'$$

whence

$$M'P'=74^\circ\ 39'\ 40''$$

The observed value was

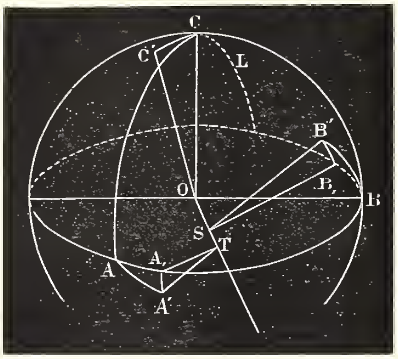
$$M'P=75^\circ\ 16'\ 20''\ \text{[Section III. (4)]}.$$

The difference is 0° 36′ 40″. This, again, is far beyond any possible experimental error. Thus, so long as we assume the position of the axes of elasticity to be definitely fixed in the crystal so that O C is parallel to the intersection of *m m′* while O B bisects the angle between them, the displacement of the plane P R′ necessary to bring the results of experiment and theory into agreement is far too great to be possible.

But there remains the supposition that the axes of elasticity have not exactly the same position in all crystals of arragonite. So that the displacement of P R relatively to the axes might be effected by changing slightly their position with reference to the faces of the crystal, P R retaining their position, relatively to those faces, unchanged. The possibility of this is a question for the mineralogist. I have been as yet unable to find data for a satisfactory answer. It seems, however, plausible to suppose that in a substance like arragonite which is not chemically pure, but contains foreign substances to a variable degree in different specimens, some slight variation such as that indicated might occur.

We can, without much difficulty, calculate the amount of the change.

Fig. 15.



We may very approximately treat the rotation round O L as if it were about O C, C L being very small.

The effect of a rotation of 1° about O C will be to bring A and B to $A_{,}$ $B_{,}$ respectively in the same plane, where

$$AA_{,}=BB_{,}=1^\circ$$

$A_{,}$ $B_{,}$ being to the right of $A_{,}$ and B.

The increment of 17′ in L P may be effected by turning the axes through an angle of 17′ about a normal to the plane L P passing through O, let O S T be this normal, let A′ B′ C′ be the final positions of A B C.

$A_{,}A'$, $B_{,}B'$ are arcs of small circles whose centres lie on O S T at T and S respectively, so that

$$A_{,}TA'=B_{,}SB'=17'$$

C C′ is an arc of a great circle, and is $=17'$.

$$B_{,}B'=SB_{,}\times B'SB_{,}$$

$$SOB_{,}=60^\circ \text{ approximately}$$

$$\therefore\ SB'=\sin 60^\circ=\frac{\sqrt{3}}{2}$$

$$\therefore\ B_{,}B'=\frac{17'\sqrt{3}}{2}$$

$$BB'=\sqrt{(BB_{,}^2+B'B_{,}^2)}$$

$$=\sqrt{\left(60'^2+\frac{3\times 17^2}{4}\right)}$$

$$=1^\circ\ 2' \text{ approximately}$$

$$A_{,}A'=ATA'\sin 30$$

$$=8'\ 30''$$

$$AA'^2=\sqrt{\left(60'^2+\frac{17'^2}{4}\right)} \text{ approximately}$$

$$=1^\circ\ 1' \text{ approximately.}$$

Thus the new positions of the axes are inclined to the old at angles of 1° 1′, 1° 2′, and 17′ respectively for O A, O B, O C. So that the axis of C in its new position would be inclined to the line of junction of $m\ m'$ at an angle of 17′, while that of b would be inclined to the bisector of these planes at an angle of 1° 2′.

If variations such as here considered be possible, so that we cannot be certain *à priori* of the position of the axes of elasticity, the only method of testing FRESNEL's or any other theory will be by trial.

Section X.—*General Results of the Investigation.*

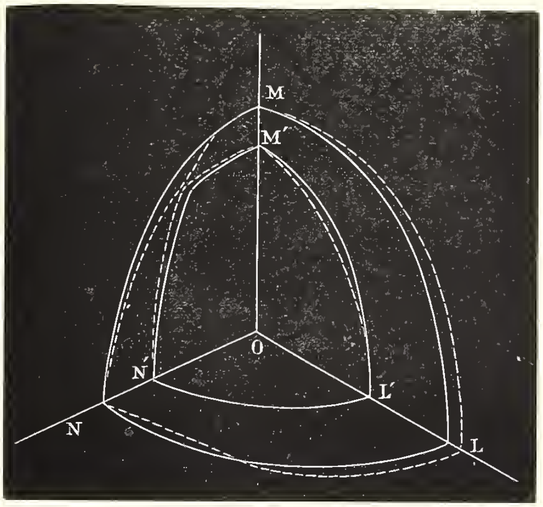
Combining the results of the two series of experiments, it seems to me most probable that FRESNEL's theory is only true as a first approximation.

Both series of observations have led to the discovery of considerable deviation from the theory, unless we assume errors in the experiments, especially in the determination of the position of the planes of the prisms with reference to the axes of the crystals, which are greatly in excess of the amounts we can reasonably expect.

It will be noticed that the values taken for the constants μ_a and μ_b in the two parts of the paper differ by ·0002 and ·00017 respectively. This is due to the fact that the two crystals used were different. RUDBERG (POGG. Annalen, vol. xvii., p. 1) found differences of as much as ·0004 in the values of μ_b deduced from two specimens of arragonite.

It may be objected that the variations between theory and experiment are not in exactly the same direction in the two Parts; but we must remember (1) that the arcs investigated are taken from entirely different portions of the surface; (2) in the first Part, the approximately elliptic section belongs to the outer sheet; in the second, to the inner sheet of the wave surface.

Fig. 16.



The accompanying figure will illustrate the results of the investigation.

M O N is the principal plane of the first prism nearly coincident with the plane through the optic axes A O C.

N O L the principal plane of the second prism nearly coincident with the plane B O A, so that O N is nearly coincident with O A.

M O L the principal planes of the two prisms in the second crystal here treated as coincident, inclined at about 60° to B O C, and cutting it in a line nearly coincident with O C.

Hence O M is not far removed from O C. O L is nearly in the plane A O B and inclined at about 60° to O A or O N.

The strong lines give approximately the form of the sections of FRESNEL'S surface by these planes, the dotted lines the results of experiment.

In the case of the arc N′ L′ the results of theory and experiment agreed closely.

For the arcs M N, M′ N′ the experiments covered an arc of about 16° from M and M′.

For N L the experiments covered an arc extending from L to about 10° on the side nearest to N of the point where the two arcs intersect.

For M L the experiments extended over an arc of about 70° measured from M.

Section XI.—*Effect of Dispersion considered.*

The theory of dispersion appears to me to afford a more probable explanation of these small variations from FRESNEL'S construction.

FRESNEL himself remarked ('Second Supplément au premier Mémoire sur la double réfraction,' Œuvres Complètes de FRESNEL, tom. ii.) that in the case of the vibrations which constitute light the radius of the sphere of action of the molecular forces brought into play by the vibration is not necessarily very small compared with the wave length.

And, consequently, it is incorrect to suppose that the propagation of each of the disturbances of which a vibration is composed is uninfluenced by the disturbances which precede and follow it, and that the velocity of propagation is independent of the manner in which they proceed and follow it.

This supposition is the basis of FRESNEL'S work on double refraction.

Let us consider the effect of dispersion in a doubly refracting medium.

In an isotropic medium the relation between V, the velocity of wave propagation, and λ, the wave length, is generally allowed to be

$$\frac{1}{V}=a+\frac{b}{\lambda^2}+\frac{c}{\lambda^4}+, \&c.,$$

a, b, c, &c., being constant, the values of the terms continually decreasing, so that except in highly dispersive media we may put $\frac{1}{V}=a+\frac{b}{\lambda^2}$ with sufficient exactness.

Let us suppose that an equation of this form holds in crystalline media also, only that a, b, c, &c., instead of being constants are functions of the directions of propagation and vibration; and, further, let us suppose that FRESNEL'S construction is true for waves of infinite length, so that the equation $\frac{1}{v}=a$ gives us a FRESNEL'S wave surface.

From the known values of the constants of the wave surface for different rays of the spectrum the constants of the surface for infinite wave lengths can be found, and hence the value of a calculated in any given direction.

If from experiment we find the value of $\frac{1}{v}$ or μ for any wave length, the difference $\mu-a$ ought on this theory to be equal to $\frac{b}{\lambda^2}$.

And if we find the values of μ for different wave lengths (μ_1, μ_2, μ_3 say) in the same direction, we have

$$\mu_1-a=\frac{b}{\lambda_1^2}$$

$$\mu_2-a=\frac{b}{\lambda_2^2}$$

$$\mu_3-a=\frac{b}{\lambda_3^2}$$

whence

$$\frac{\mu_1-a}{\mu_2-a}=\frac{\lambda_2^2}{\lambda_1^2}$$

$$\frac{\mu_2-a}{\mu_3-a}=\frac{\lambda_3^2}{\lambda_2^2}$$

To verify this I observed the values of μ in two directions for the rays C, D, and F, with the following results :—

$$\left(\frac{\lambda_C}{\lambda_D}\right)^2=1\cdot2403$$

$$\frac{\mu_D-a}{\mu_C-a}=1\cdot2875 \text{ (first experiment)}$$

$$1\cdot2770 \text{ (second experiment)}$$

$$\left(\frac{\lambda_D}{\lambda_F}\right)^2=1\cdot46978$$

$$\frac{\mu_F-a}{\mu_D-a}=1\cdot47208 \text{ (first experiment)}$$

$$=1\cdot47348 \text{ (second experiment).}$$

The numbers, especially in the last case, are sufficiently close to make it seem worth while continuing the investigations.

XI. *On certain Definite Integrals occurring in Spherical Harmonic Analysis and on the Expansion, in Series, of the Potentials of the Ellipsoid and the Ellipse.*

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Communicated by J. W. L. Glaisher, *M.A., F.R.S.*

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§ 1. The object of this paper is to explain a method of dealing with a class of results, some of which are of frequent occurrence and some of considerable importance, in the solution of Laplace's equation in series.

The fundamental theorem on which the method depends is expressed by the integration of

$$\iint e^{\alpha x+\beta y+\gamma z}dS$$

over a sphere whose centre is the origin of coordinates and whose radius is R. This sphere we shall call the sphere of reference.

By change of axes the above integral takes the form

$$2\pi R\int_{-R}^{R} e^{x\sqrt{(\alpha^2+\beta^2+\gamma^2)}}dx$$

and its value is

$$2\pi R\frac{e^{R\sqrt{(\alpha^2+\beta^2+\gamma^2)}}-e^{-R\sqrt{(\alpha^2+\beta^2+\gamma^2)}}}{\sqrt{(\alpha^2+\beta^2+\gamma^2)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or, in series,

$$4\pi R^2\left\{1+\frac{R^2}{3!}(\alpha^2+\beta^2+\gamma^2)+\ \dots\ +\frac{R^{2i}}{2i+1!}(\alpha^2+\beta^2+\gamma^2)^i+\ \dots\right\} \quad . \quad . \quad . \quad (2)$$

It follows that if V be any function whose value may be expressed for all points within the sphere by a convergent series, according to Taylor's theorem, or by the symbolical form

$$e^{x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}}V_0$$

then the integral $\iint VdS$ taken over the surface of the sphere is

$$4\pi R^2\sum_0^\infty\frac{R^{2i}}{2i+1!}\left(\frac{d^2}{dx}+\frac{d^2}{dy}+\frac{d^2}{dz}\right)^i V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The corresponding theorem in two dimensions, which we will also employ, is this :—The integral

$$\int e^{ax+\beta y}ds$$

taken round the periphery of a circle of reference, is

$$2\pi \mathrm{R}\sum_0^\infty \frac{\mathrm{R}^{2i}}{2^{2i}i!\ i!}(\alpha^2+\beta^2)^i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence it follows that

$$\int \mathrm{V}ds,$$

taken round the same periphery, is

$$2\pi \mathrm{R}\sum_0^\infty \frac{\mathrm{R}^{2i}}{2^{2i}i!\ i!}\left(\frac{d^2}{dx}+\frac{d^2}{dy}\right)^i \mathrm{V}_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The theorems expressed by (3) and (5) are employed below in a variety of cases connected with spherical harmonics. It is therefore necessary to specify at the outset the notation to be used.

We denote, according to custom, the colatitude and longitude of a point referred to the axes of reference by θ and ϕ: cos θ and sin θ by μ and ν respectively. We denote the LEGENDRE's coefficient or zonal harmonic of the i^{th} degree by the symbol P_i so that P_i is defined by the equation

$$i!\ \mathrm{P}_i=(-1)^i r^{i+1}\frac{d^i}{dz}\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

According to CLERK MAXWELL'S theory of poles the general expression for the harmonic of the i^{th} degree is given by the equation

$$i!\ \mathrm{Y}_i=(-1)^i r^{i+1}\frac{d^i}{dh_1 dh_2 \ldots dh_i}\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where $\frac{d}{dh}$ means differentiation with regard to an axis h.

In the harmonic (7) the i poles are the points where the i axes of differentiation cut the sphere of reference. In the harmonic (6) the i poles all coincide with the point where the axis of z cuts the sphere. If the i poles coincide at any other point the harmonic will be denoted by Q_i.

In the tesseral and sectorial system we have ventured to depart from the usual notations by denoting them as follows :—

$$i!\ (i,\sigma)=(-1)^i r^{i+1}\left(\frac{d^\sigma}{d\xi}+\frac{d^\sigma}{d\eta}\right)\frac{d^{i-\sigma}}{dz}\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$i!\ [i,\sigma]=(-1)^i r^{i+1} j\left(\frac{d^\sigma}{d\xi}-\frac{d^\sigma}{d\eta}\right)\frac{d^{i-\sigma}}{dz}\frac{1}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where $j=\sqrt{-1}$, $\xi=x+jy$, $\eta=x-jy$. It will be observed that (8) is not in accordance with the general definition (7), for if we put $\sigma=0$ we get $(i, o)=2P_i$.

As regards the name of Spherical Harmonic introduced by THOMSON and TAIT, it seems desirable that the names of LEGENDRE and LAPLACE should be in some way retained in connexion with the different kinds of harmonics with which they are associated. From the point of view of the former writers there is perhaps some difficulty in doing this. We shall however continue to give the name of LAPLACE's coefficient to what THOMSON and TAIT call the Biaxal Harmonic or the formula which expresses Q_i in terms of the harmonics referred to fixed axes of reference. With the definitions given above, this is

$$Q_i=P_iP_i'+, \&c., +\frac{2^{2\sigma-1}i!\,i!}{\overline{i+\sigma}!\,\overline{i-\sigma}!}\{(i, \sigma)(i, \sigma)'+[i, \sigma][i, \sigma]'\}+ \quad . \quad . \quad (10)$$

in which any accented function is derived from the corresponding unaccented function by the substitution of the coordinates of the pole of Q_i in place of the running coordinates.

The following are, in the first instance, the integrals to which the method of this paper will be applied :—

$$(1) \quad \iint \mu^m P_n dS$$

$$(2) \quad \iint (1-\mu^2)^m P_{2n} dS$$

$$(3) \quad \iint (1-\mu^2)^{m-\frac{1}{2}} P_{2n} dS$$

$$(4) \quad \iint P_p P_q P_r dS$$

$$(5) \quad \iint (p, \alpha)(q, \beta)(r, \alpha+\beta) dS$$

$$\iint (p, \alpha)[q, \beta][r, \alpha+\beta] dS, \&c.$$

$$(6) \quad \iint P_p P_q P_r P_s dS$$

Of these cases the first three require no special comment. The fourth case was for the first time solved by Professor J. C. ADAMS, in the Proceedings of the Royal Society, vol. xxvii., pp. 63–71, although the result had been independently found and published by Mr. FERRERS in his 'Treatise on Spherical Harmonics.'

Professor ADAMS, as well as Mr. FERRERS, made the discovery of the value of (4) by an inductive process, and Mr. TODHUNTER showed in a subsequent number of the Proceedings how the proof could be thrown into a compact form. The present writer was, however, convinced that the integral could be found deductively according to the method described above, and the simple character of the result led him to hope that the same method might be applied in cases of greater complexity. Accordingly cases (5) and (6) have been dealt with, but they have not been solved with the completeness of case (4), the results being practically expressed in the form of series.

The most interesting as well as the most important applications of the present method are connected with ellipsoids and ellipses. We have thereby obtained in series of harmonics the potentials of

(1) An ellipsoidal shell.
(2) A solid ellipsoid.
(3) An elliptic plate of uniform density.
(4) An electric current in an elliptic circuit.

Of these cases (2) leads to the approximate determination of the forces expressing the mutual action between two solid ellipsoids or uniformly magnetized ellipsoids, and (4) in certain cases to the forces between two currents in elliptic circuits, the particular case of circular circuits being completely determinate.

Similar results also hold for rectangular solids and circuits.

The case (2) derives additional interest from the fact that it engaged the attention of LAGRANGE, who obtained the expansion as far as the first four terms. The same four terms have also been worked out in a very interesting paper on the potential of an ellipsoid at an external point, by Colonel A. R. CLARKE ('Philosophical Magazine,' 1877, vol. ii., pp. 458–461).

In regard to case (3), Professor CAYLEY, in the 'Proceedings of the Mathematical Society' for 1875, has obtained the solution in the form of an integral, from which he derives interesting properties of the potential depending on certain particular positions of the attracted point. The expansion in harmonic series would seem, however, to be practically more useful in determining the mutual forces between two electrical circuits.

Throughout the following investigations, the method of treating Spherical Harmonics introduced by THOMSON and TAIT will be almost exclusively adopted.

$\iint \mu^m \mathrm{P}_n d\mathrm{S}$

---

§ 2. Taking a point A on the axis of z produced negatively at a distance a from the origin, let us consider the result of integrating

$$\iint \frac{z^m}{r} d\mathrm{S}, \text{ or } \iint z^m u\, d\mathrm{S}$$

over the surface of the sphere, where r is the distance of any point P from A, and u is the reciprocal of r.

If we denote by the suffix 1 the fact of the operator being upon z^m, and by the suffix 2 of its being upon u, then the operator

$$\left(\frac{d^2}{dx}+\frac{d^2}{dy}+\frac{d^2}{dz}\right)^i$$

upon the product, $z^m u$, may be written,

$$\left\{\frac{d^2}{dx_2}+\frac{d^2}{dy_2}+\left(\frac{d}{dz_2}+\frac{d}{dz_1}\right)^2\right\}^i$$

that is, in virtue of u satisfying LAPLACE'S equation,

$$\left(\frac{d^2}{dz_1}+2\frac{d}{dz_1}\frac{d}{dz_2}\right)^i$$

Referring now to the general result (3), and picking out the general term, we find

$$\frac{4\pi R^{2i+2}}{2i+1!}\,\frac{i!}{n!\,i-n!}2^n\frac{d^{2i-n}}{dz_1}\,\frac{d^n}{dz_2}$$

In order that this may lead to a value not zero, we must have

$$2i-n=m$$

or

$$2i=m+n$$

Substituting then this value of i, and differentiating $z^m u$, putting x, y, $z=0$ after the differentiations, we get

$$(-1)^n\frac{4\pi R^{m+n+2}}{m+n+1!}\,\frac{\frac{m+n}{2}!}{n!\frac{m-n}{2}!}\,2^n m!\;n!\frac{1}{a^{n+1}}$$

But u can be expanded in a series of zonal harmonics, viz. : it is

$$\frac{1}{a}-\frac{R}{a^2}P_1+\ldots+(-1)^n\frac{R^n}{a^{n+1}}P_n+\ldots$$

Substituting this expansion in $\iint z^m u dS$, and equating the coefficients of the different powers of the reciprocals of a to the values already found for them, we obtain finally, in the case where m and n are integers,

$$\iint \mu^m P_n dS=4\pi R^2\frac{2^n m!\frac{m+n}{2}!}{m+n+1!\frac{m-n}{2}!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

It is obvious from the above proof that $m+n$ must be an even number, and that n must not be greater than m. In all other cases in which m and n are integers, the integral must be zero.

It is obvious the same method of solution as the foregoing will apply generally towards determining

$$\iint \frac{f(z)}{r}\, d\mathrm{S}$$

where $f(z)$ is any function whose differential coefficients are finite at the origin. In fact, we find

$$\iint f(z)\mathrm{P}_n d\mathrm{S} = \pi(2\mathrm{R})^{n+2}\frac{n!}{2n+1!}\left[\frac{d^n}{dz}+\frac{\mathrm{R}^2}{2(2n+3)^4}\frac{d^{n+2}}{dz}+\frac{\mathrm{R}^4}{2.4(2n+3)(2n+5)}\frac{d^{n+4}}{dz}+\&\mathrm{c.}\right]f(z). \quad (12)$$

where, after the differentiations are performed, z is to be put equal to zero.

A result practically equivalent to the above is given by HEINE ("Kugelfunctionen," second edition, p. 76), and applied to various cases where the functions can be expanded in simple series of zonal harmonics. The functions are

$$(1+k\mu^2)^{-(n+\frac{3}{2})},\ (1-\mu^2)^{\frac{1}{2}},\ \sin^{-1}\mu,\ (1-\mu^2)^{-\frac{1}{2}}$$

Similar functions are also to be found expanded in TODHUNTER'S "Functions of LAPLACE, LAME, and BESSEL," §§ 48, 145–147. We shall now discuss more general cases of some of these functions.

$$\iint (1-\mu^2)^m \mathrm{P}_{2n} d\mathrm{S}$$

---

§ 3. The case now indicated will be discussed only for integral values of m and n. Let us consider the integral

$$\iint \frac{(x^2+y^2)^m}{r} d\mathrm{S}, \text{ or } \iint (x^2+y^2)^m u\, d\mathrm{S}$$

If we use the suffix 1 in any operator when it is upon $(x^2+y^2)^m$, and 2 when it is upon u, we have as the general term operating,

$$\frac{4\pi\mathrm{R}^{2i+2}}{2i+1!}\left(\left(\frac{d}{dx_1}+\frac{d}{dx_2}\right)^2+\left(\frac{d}{dy_1}+\frac{d}{dy_2}\right)^2+\frac{d^2}{dz_2}\right)^i$$

or, since

$$\frac{d}{dx_1}+\frac{d}{dx_2}=\frac{d}{d\xi_1}+\frac{d}{d\eta_1}+\frac{d}{d\xi_2}+\frac{d}{d\eta_2}$$

$$\frac{d}{dy_1}+\frac{d}{dy_2}=j\left(\frac{d}{d\xi_1}-\frac{d}{d\eta_1}+\frac{d}{d\xi_2}-\frac{d}{d\eta_2}\right)$$

the general term above is

$$\frac{4\pi\mathrm{R}^{2i+2}}{2i+1!}\left(4\left(\frac{d}{d\xi_1}+\frac{d}{d\xi_2}\right)\left(\frac{d}{d\eta_1}+\frac{d}{d\eta_2}\right)+\frac{d^2}{dz_2}\right)^i$$

that is, in virtue of u satisfying LAPLACE's equation,

$$\frac{4\pi R^{2i+2}}{2i+1!}4^i\left(\frac{d}{d\xi_1}\frac{d}{d\eta_1}+\frac{d}{d\xi_1}\frac{d}{d\eta_2}+\frac{d}{d\xi_2}\frac{d}{d\eta_1}\right)^i$$

The general term of the operator just found is

$$\frac{4\pi R^{2i+2}}{2i+1!}4^i\frac{i!}{u!\,i-u!}\left(\frac{d^2}{d\xi_1 d\eta_1}\right)^u\left(\frac{d}{d\xi_1}\frac{d}{d\eta_2}+\frac{d}{d\xi_2}\frac{d}{d\eta_1}\right)^{i-u}$$

It is obvious that $i-u$ must be even, otherwise the differentiated expressions could not but vanish when $x=0$, $y=0$, $z=0$. We must thus have

$$i-u=2n$$

$$i+u=2m$$

Substituting these values in the general operator, and retaining only the middle term of the last factor, which alone in that factor will produce a result differing from zero, we get, as the expression of the effective operator,

$$\frac{4\pi R^{2i+2}}{2m+2n+1!}\,4^i\,\frac{m+n!}{m-n!\,n!\,n!}\left(\frac{d}{d\xi_1}\frac{d}{d\eta_1}\right)^m\left(\frac{d}{d\xi_2}\frac{d}{d\eta_2}\right)^n$$

of which the last factor may be replaced by

$$\left(-\frac{1}{4}\frac{d^2}{dz_2}\right)^n$$

And now performing the operations indicated, we find as the result

$$(-1)^n\frac{4\pi R^{2i+2}}{2m+2n+1!}\,4^m\,\frac{m+n!\,m!\,m!\,2n!}{m-n!\,n!\,n!}\,\frac{1}{a^{2n+1}}$$

Since $x^2+y^2=R^2(1-\mu^2)=R^2\nu^2$, and the term involving P_{2n} in the expansion of u in zonal harmonics is

$$\frac{R^{2n}}{a^{2n+1}}P_{2n}$$

we have, finally,

$$\iint(1-\mu^2)^m P_{2n}dS=4\pi R^2\frac{(-1)^n 4^m}{2m+2n+1!}\,\frac{m+n!\,m!\,m!\,2n!}{m-n!\,n!\,n!}\quad\ldots\ldots\quad(13)$$

If $\phi(\nu^2)$, any function of ν^2, be expanded in the form

$$A_0+A_1\nu^2+\ldots+A_{n+p}\nu^{2n+2p}+\ldots$$

then it may be shown that

$$\int_0^1 \phi(\nu^2)\mathrm{P}_{2n}d\mu$$

$$=(-1)^n\frac{2n!}{1.3.5\ldots(4n+1)}\left(\mathrm{A}_n+\frac{(n+1)^2}{4n+3}\,2\mathrm{A}_{n+1}+\frac{(n+1)^2(n+2)^2}{(4n+3)(4n+5)1.2}\,2^2\mathrm{A}_{n+1}+\&\mathrm{c.}\right). \quad (14)$$

In certain cases this series admits of summation. For example, let us examine under what circumstances $\iint\mu^{-k}\mathrm{P}_{2n}d\mathrm{S}$ admits of being evaluated

$$\mu^{-k}=(1-\nu^2)^{-\frac{k}{2}}=\Sigma\frac{\frac{k}{2}\left(\frac{k}{2}+1\right)\ldots\left(\frac{k}{2}+n-1\right)}{n!}\nu^{2n}$$

Hence

$$\mathrm{A}_{n+1}=\frac{\frac{k}{2}+n}{n+1}\,\mathrm{A}_n$$

$$\mathrm{A}_{n+2}=\frac{\left(\frac{k}{2}+n\right)\left(\frac{k}{2}+n+1\right)}{(n+1)(n+2)}\mathrm{A}_n,\ \&\mathrm{c.}$$

If we now write

$$\alpha=\frac{k}{2}+n$$

$$\beta=n+1$$

$$\gamma=2n+\frac{3}{2}$$

we find for the integral $\iint\mu^{-k}\mathrm{P}_{2n}d\mathrm{S}$, the expression

$$(-1)^n\frac{k(k+2)(k+4)\ldots(k+2n-2)}{(2n+1)(2n+3)(4n+1)}\Sigma$$

where

$$\Sigma=1+\frac{\alpha\beta}{\gamma}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)1.2}+\ldots$$

$$=\frac{\Gamma(\gamma)\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\beta)\Gamma(\gamma-\alpha)}\text{*}$$

$$=\frac{\Gamma\left(2n+\frac{3}{2}\right)\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+1+\frac{1-k}{2}\right)}$$

$$=\frac{\left(n+\frac{1}{2}\right)\ldots\left(2n+\frac{1}{2}\right)}{\frac{1-k}{2}\ldots\left(\frac{1-k}{2}+n\right)}$$

---

* BERTRAND, 'Calcul. Intégral,' 1870, pp. 495–6.

The value of the integral under discussion is accordingly

$$(-1)^n\frac{k(k+2)\ldots(k+2n-2)}{(1-k)(3-k)\ldots(2n+1-k)} \quad \ldots\ldots\ldots \quad (15)$$

and the above reasoning shows that k must be less than 1 : it may, however, have any negative value.

$\iint(1-\mu^2)^{m-\frac{1}{2}}\mathrm{P}_{2n}d\mathrm{S}$

§ 4. To evaluate the integral now indicated, m and n being integers, we must modify our previous solution, for otherwise we should be led to infinite values of the differential coefficients at the origin.

Let us consider the integral

$$\iint(\xi\eta)^{m-\frac{1}{2}}\mathrm{V}d\mathrm{S}$$

If V is a function which is symmetrical round the axis of z we may write this

$$2\pi\int\rho^{2m}\mathrm{V}ds$$

where ds is an element of arc of any section of the sphere of reference containing the axis of z, and $\rho^2+z^2=\mathrm{R}^2$.

Now the expression (5) gives the form of solution in this case. If we put

$$\mathrm{V}=\frac{1}{\pi}\int_0^\pi(z+j\rho\cos\psi)^{2n}d\psi$$

we have, in fact,

$$\iint(1-\mu^2)^{m-\frac{1}{2}}\mathrm{P}_{2n}d\mathrm{S}=\frac{4\pi\mathrm{R}^2}{2^{2m+2n}m+n!m+n!}\times\int_0^\pi\left(\frac{d^2}{d\rho}+\frac{d^2}{dz}\right)^{m+n}\rho^{2m}(z+j\rho\cos\psi)^{2n}d\psi$$

Expanding the quantities under the integral sign, and performing the differentiations and integrations, we obtain

$$\frac{2\pi^2\mathrm{R}^2 2m!2n!}{2^{2m+2n}m+n!m!n!}\mathrm{S}$$

where

$$\mathrm{S}=1-\frac{n(m+\frac{1}{2})}{1.2}+\frac{n(n-1)(m+\frac{1}{2})(m+\frac{1}{2}+1)}{1.2 \qquad\qquad 1.2}-\&\mathrm{c}.$$

$$=\frac{(n-m-\frac{1}{2})(n-m-\frac{1}{2}-1)\ldots(-m-\frac{1}{2}+1)}{1.2\ldots n}$$

First let $n>m$, then in S we have $n-m$ positive and m negative factors, and in that case S becomes

$$(-1)^m \frac{2n-2m!\,2m!}{2^{2n}n-m!\,m!\,n!}.$$

Next, let $n<m$, then S becomes

$$(-1)^n \frac{2m!\,m-n!}{2^{2n}2m-2n!\,m!\,n!}.$$

Substituting these values of S in the expression for the integral we find

(1) if $n>m$

$$\frac{(-1)^m 2\pi^2 R^2 2m!\ 2n!\ 2n!\ 2n-2m!}{2^{2m+4n}n+m!\ n-m!\ m!\ m!\ n!\ n!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

(2) If $n<m$

$$\frac{(-1)^n 2\pi^2 R^2 2m!\ 2m!\ 2n!\ m-n!}{2^{2m+4n}m+n!\ 2m-2n!\ m!\ m!\ n!\ n!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$\iint Q_i \frac{dP_j}{d\mu} dS, \iint Q_i \frac{d^2P_j}{d\mu^2} dS$

§ 5. The results for the two integrals now stated are, if α be the colatitude of the pole of Q_i,

$$4\pi R^2 P_i(\cos\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$2\pi R^2 (j-i)(j+i+1) P_i(\cos\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where, in the second integral, $j-i$ must be a positive even integer.

GENERAL THEOREMS IN DIFFERENTIATION.

§ 6. Before we proceed with the remaining cases, it will be convenient to state and prove various theorems in differentiation.*

Theorem i. The general operator (7) upon the reciprocal of r may be made to operate, instead, upon a homogeneous function in x, y, z of the i^{th} degree; this effect being expressed by the relation

$$(-1)^i \frac{d^i}{dh_1 dh_2 \ldots dh_i} \frac{1}{r} = \frac{1}{r^{i+1}} \frac{d^i}{dh_1 dh_2 \ldots dh_i}(r^i Q_i) \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where the pole of Q_i is in the direction of r.

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* Theorems i. and ii. were given in a paper by the author in the 'Messenger of Mathematics,' No. 73, 1877. On account of the brevity of the proofs, and in order to secure completeness, it is thought best to reproduce them here.

Theorem ii. relates to the differentiation of the reciprocal of $r^{2\sigma+1}$ and may be stated thus :

$$(-1)^{i-\sigma}\frac{d^{i-\sigma}}{dh_1dh_2\ldots dh_{i-\sigma}}\frac{1}{r^{2\sigma+1}}=\frac{2^\sigma\sigma!}{2\sigma!}\frac{d^{i-\sigma}}{dh_1dh_2\ldots dh_{i-\sigma}}\left(r^{i-\sigma}\frac{d^\sigma Q_i}{d\mu^\sigma}\right)\quad . \quad . \quad . \quad (21)$$

Theorem iii. If $f\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)$ be any homogeneous operator of the i^{th} degree operating on a homogeneous expression $\phi\ (x, y, z)$ of the i^{th} degree, then

$$f\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)\phi(x, y, z)=\phi\left(\frac{d}{dx},\frac{d}{dy},\frac{d}{dz}\right)f\ (x, y, z)\quad . \quad . \quad . \quad . \quad . \quad (22)$$

This theorem is obviously true, and though stated for only three quantities, x, y, z, is true for any number.

Theorems i. and ii. are proved as follows: Let $\mathrm{OP}=r$, $\mathrm{QP}=\rho$ and $\mathrm{O\hat{P}Q}=\pi-\theta$, where Q is any point near to P. Then

$$\frac{1}{\mathrm{OQ}}=\frac{1}{r}-\frac{\rho}{r^2}\mathrm{Q}_1+\ \ldots\ +(-1)^i\frac{\rho^i}{r^{i+1}}\mathrm{Q}_i+\&c.$$

Keeping r fixed let us perform the operation (7) on the two sides of this equation and then put $\rho=0$, in which case OQ becomes r. It is to be observed that with a homogeneous operator like (7), it is immaterial what point is taken for origin of coordinates. If then we take P for origin when we are dealing with the right hand side of the above expansion, we see that the first i terms will disappear by repeated differentiations, and the terms beyond the $i+1^{th}$ disappear in consequence of the zero value of ρ. The theorem stated therefore follows, the substitution of r for ρ making no difference in a result from which x, y, z are finally made to disappear.

Theorem ii. may be proved in the same way, if instead of the expansion for the reciprocal of OQ, as above, we take the expansion for the reciprocal $\mathrm{OQ}^{2\sigma+1}$, found by differentiating the above series σ times with regard to μ.

DIFFERENT FORMS OF TESSERAL AND SECTORIAL HARMONICS.

§ 7. In several of the following investigations frequent use will be made of different forms of the zonal and tesseral harmonics. With the view of classifying these various forms, and of bringing the various expressions for the tesseral and sectorial system into harmony with the corresponding expressions for the zonal, a proof is here given which will be a direct illustration of the foregoing theorems.

The most important forms of the zonal harmonic P_i are

$$\frac{1}{2^i i!}\frac{d^i}{d\mu}(\mu^2-1)^i\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

$$\frac{2i}{2^i i!\, i!}\left(\mu^i - \frac{i(i-1)}{2(2i-1)}\mu^{i-2} + \ldots\right) \quad \ldots\ldots\ldots \quad \text{(B)}$$

$$\frac{1}{\pi}\int_0^\pi (\mu + j\nu\cos\psi)^i d\psi,\ \frac{1}{\pi}\int_0^\pi \frac{d\psi}{(\mu + j\nu\cos\psi)^{i+1}} \quad \ldots\ldots\ldots \quad \text{(C)}$$

$$\mu^i - \frac{i(i-1)}{4}\mu^{i-2}\nu^2 + \quad \ldots\ldots\ldots \quad \text{(D)}$$

Now by definition

$$(i,\ \sigma) = (-1)^i \frac{r^{i+1}}{i!}\frac{d^{i-\sigma}}{dz}\left(\frac{d^\sigma}{d\xi} + \frac{d^\sigma}{d\eta}\right)\frac{1}{\sqrt{z^2 + \xi\eta}}$$

$$= (-1)^{i+\sigma}\frac{2\sigma!}{2^{2\sigma} i!\, \sigma!} r^{i+1}(\xi^\sigma + \eta^\sigma)\frac{d^{i-\sigma}}{dz}\ \frac{1}{r^{2\sigma+1}}$$

By Theorem ii. this is

$$\frac{1}{2^\sigma i!} 2\cos\sigma\phi\nu^\sigma \frac{d^{i-\sigma}}{dz}\left(r^{i-\sigma}\frac{d^\sigma Q_i}{d\mu^\sigma}\right)$$

We observe that, in this result, the axis of the harmonic Q_i is the line OP, as explained in the proof of Theorem i. We must therefore change the axes of coordinates, making the new axis of z coincide with OP.

If $r^{i-\sigma}\frac{d^\sigma P_i}{d\mu^\sigma}$, when expanded, becomes

$$Az^{i-\sigma} - Bz^{i-\sigma-2}\rho^2 + \ldots$$

we have to consider the effect of operating upon this with

$$\left(\cos\alpha\frac{d}{dz} - \sin\alpha\frac{d}{d\rho}\right)^{i-\sigma}$$

There will result

$$i-\sigma\,!\ (A\cos^{i-\sigma}\alpha - B\cos^{i-\sigma-2}\alpha\sin^2\alpha + \ldots)$$

that is

$$i-\sigma\,!\ \frac{d^\sigma P_i}{d\mu^\sigma}$$

where μ is now the cosine of the angle between OP and the axis of z. Hence we obtain the first and second forms of the harmonic corresponding to the forms (A) and (B) of the zonal harmonic, viz. :—

$$(i,\ \sigma) = \frac{i-\sigma!}{2^\sigma i!} 2\cos\sigma\phi\nu^\sigma\frac{d^\sigma P_i}{d\mu^\sigma} \quad \ldots\ldots\ldots \quad (a)$$

$$= \frac{2\,i!}{2^{i+\sigma} i!\, i!} 2\cos\sigma\phi\nu^\sigma\left(\mu^{i-\sigma} - \frac{(i-\sigma)(i-\sigma-1)}{2(2i-1)}\mu^{i-\sigma-2} + \ldots\right) \quad \ldots\ldots \quad (b)$$

Let us now consider the definite integral

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{p\cos^2\phi+q\sin^2\phi}=\frac{\pi}{2}\frac{1}{\sqrt{pq}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

This result will still be true if for p, q we substitute the unreal quantities $c+re^{\theta j}$, $c+re^{-\theta j}$, as may indeed be shown by direct integration.

Let us differentiate both sides of the equation (α), σ times with regard to p and σ times with regard to q. We find

$$2\sigma!\int_0^{\frac{\pi}{2}} \frac{\sin^{2\sigma}\phi\cos^{2\sigma}\phi}{(p\cos^2\phi+q\sin^2\phi)^{2\sigma+1}}d\phi=\frac{\pi}{2}\left\{\frac{1.3\ldots(2\sigma-1)}{2^\sigma}\right\}^2\frac{1}{(pq)^{\sigma+\frac{1}{2}}}$$

Making the above substitutions for p, q we get

$$\frac{2^\sigma\sigma!}{1.3\ldots 2\sigma-1}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin^{2\sigma}2\phi}{\{c+r(e^{\theta j}\cos^2\phi+c^{-\theta j}\sin^2\phi)\}^{2\sigma+1}}d\phi=\left(\frac{1}{c^2+2cr\mu+r^2}\right)^{\frac{2\sigma+1}{2}}$$

$$=\frac{1}{1.3\ldots(2\sigma-1)}\frac{1}{c^{2\sigma+1}}\left(\frac{d^\sigma \mathrm{P}_\sigma}{d\mu^\sigma}+\ldots+(-1)^{i-\sigma}\left(\frac{r}{c}\right)^{i-\sigma}\frac{d^\sigma \mathrm{P}_i}{d\mu^\sigma}+\ldots\right)$$

the last line being derived from its predecessor by σ differentiations with regard to μ.

And now expanding the expression under the integral sign in powers of r, and equating the coefficients of $r^{i-\sigma}$, we find

$$\frac{d^\sigma \mathrm{P}_i}{d\mu^\sigma}=\frac{i+\sigma!\,\sigma!}{2\sigma!\,i-\sigma!}2^\sigma\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(e^{\theta j}\cos^2\phi+e^{-\theta j}\sin^2\phi)^{i-\sigma}\sin^{2\sigma}2\phi d\phi$$

On putting $2\phi=\psi$, and substituting the expression just found in equation (a), we finally obtain for (i, σ) one of the forms corresponding to C, viz. :—

$$\frac{i+\sigma!\,\sigma!}{2\sigma!\,i!}2\cos\sigma\phi\nu^\sigma\frac{1}{\pi}\int_0^\pi(\mu+j\nu\cos\psi)^{i-\sigma}\sin^{2\sigma}\psi d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

There is obviously another expression, corresponding to the second form of C, which is easily written. It is to be found by expanding, as above, in powers of the reciprocal of r.

The expansion of the binomial inside of the integral (c) and subsequent integration with regard to ψ give the form corresponding to D, viz. :—

$$\frac{i+\sigma!}{2^{2\sigma}i!\,\sigma!}2\cos\sigma\phi\nu^\sigma\left\{\mu^{i-\sigma}-\frac{(i-\sigma)(i-\sigma-1)}{4(\sigma+1)}\mu^{i-\sigma-2}\nu^2+\ldots\right\} \quad . \quad . \quad . \quad . \quad (d)$$

Integration of the Product of Two Harmonics and Proof of LAPLACE'S *Coefficient.*

§ 8. Let us take two points A and B at distances a and b from the origin, A being on the axis of z and B at an angular distance α from it.

Let P be any point on the sphere of reference, and let us consider the integral

$$\iint \frac{d\mathrm{S}}{\mathrm{AP.BP}}$$

By (3) we may write as the result of this

$$4\pi\mathrm{R}^2\sum_0^\infty \frac{\mathrm{R}^{2i}}{2i+1!}\left(\frac{d^2}{dx}+\frac{d^2}{dy}+\frac{d^2}{dz}\right)^i \frac{1}{\mathrm{AO.BO}}$$

If we use the suffix 1 to denote differentiation upon AP only and 2 upon BP only, the above becomes by virtue of LAPLACE's equation

$$4\pi\mathrm{R}^2\sum_0^\infty \frac{\mathrm{R}^{2i}}{2i+1!}2^i\left(\frac{d}{dx_1}\frac{d}{dx_2}+\frac{d}{dy_1}\frac{d}{dy_2}+\frac{d}{dz_1}\frac{d}{dz_2}\right)^i \frac{1}{\mathrm{AO.BO}}$$

or, what is the same thing,

$$4\pi\mathrm{R}^2\sum_0^\infty \frac{\mathrm{R}^{2i}}{2i+1!}2^i\left(2\frac{d}{d\xi_1}\frac{d}{d\eta_2}+2\frac{d}{d\xi_2}\frac{d}{d\eta_1}+\frac{d}{dz_1}\frac{d}{dz_2}\right)^i \frac{1}{\mathrm{AO.BO}}$$

Now on consideration of Theorem i. we will see that in expanding the operator just found, the only terms which will lead to results not zero will be those which, so far as the operator with suffix 1 is concerned, are of the form

$$\frac{d^m}{d\xi_1}\frac{d^m}{d\eta_1}\frac{d^n}{dz_1}$$

and this by virtue of LAPLACE's operator

$$\frac{d^2}{dz_1}+4\frac{d}{d\xi_1}\frac{d}{d\eta_1}=0$$

is expressible in terms of $\frac{d}{dz_1}$ only. Hence we conclude that the only effective terms may be found directly as those not containing a power of v in the expansion of

$$4\pi\mathrm{R}^2\sum_0^\infty \frac{\mathrm{R}^{2i}}{2i+1!}\left(-v^2\frac{d^2}{dz_1}-\frac{1}{v^2}\frac{d^2}{dz_2}+2\frac{d}{dz_1}\frac{d}{dz_2}\right)^i \frac{1}{\mathrm{AO.BO}}$$

i.e., in

$$4\pi\mathrm{R}^2\sum_0^\infty (-1)^i \frac{\mathrm{R}^{2i}}{2i+1!}\left(v\frac{d}{dz_1}-\frac{1}{v}\frac{d}{dz_2}\right)^{2i} \frac{1}{\mathrm{AO.BO}}$$

i.e.,

$$4\pi\mathrm{R}^2\Sigma\frac{\mathrm{R}^{2i}}{2i+1!}\frac{2i!}{i!\,i!}\frac{d^i}{dz_1}\frac{d^i}{dz_2}\frac{1}{\mathrm{AO.BO}}$$

Hence

$$\iint \frac{d\mathrm{S}}{\mathrm{AP.BP}}=4\pi\mathrm{R}^2\Sigma\frac{\mathrm{R}^{2i}}{2i+1}\frac{\mathrm{P}_i(\cos\alpha)}{a^{i+1}b^{i+1}}$$

If now we expand the reciprocals of AP and BP in harmonics we arrive at the well-known theorems

$$\iint \mathrm{P}_i \mathrm{Q}_j d\mathrm{S}=0$$

$$\iint \mathrm{P}_i \mathrm{Q}_i d\mathrm{S}=\frac{4\pi\mathrm{R}^2}{2i+1}\mathrm{P}_i(\cos\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The last result, when properly considered, can be made to yield the integral of any two harmonics. For it can be easily seen from the foregoing work that if ρ_1, ρ_2 are any two vectors drawn from the origin, then we have this very remarkable formula for Q_i

$$2i!\ \mathrm{Q}_i=\rho_1{}^{i+1}\rho_2{}^{i+1}2^i\left(\frac{d}{dx_1}\frac{d}{dx_2}+\frac{d}{dy_1}\frac{d}{dy_2}+\frac{d}{dz_1}\frac{d}{dz_2}\right)^i\frac{1}{\rho_1\rho_2}$$

On account of the operator being an invariant we may suppose the axes to be any rectangular axes whatever.

By what has just been shown we may throw this into the form

$$2i!\ \mathrm{Q}_i=\rho_1{}^{i+1}\rho_2{}^{i+1}(-1)^i\left(v\frac{d}{dz_1}-\frac{1}{v}\frac{d}{dz_2}\right)^{2i}\frac{1}{\rho_1\rho_2}$$

where v^2 stands for

$$\frac{d}{d\eta_2}\div\frac{d}{d\eta_1}$$

The general term of the value of $2i!\ \mathrm{Q}_i$ just found is

$$\frac{\rho_1{}^{i+1}\rho_2{}^{i+1}(-1)^\sigma 2i!}{i-\sigma!\ i+\sigma!}\frac{v^{i-\sigma}}{v^{i+\sigma}}\quad\frac{d^{i-\sigma}}{dz_1}\frac{d^{i+\sigma}}{dz_2}\frac{1}{\rho_1\rho_2}$$

i.e.,

$$\frac{\rho_1{}^{i+1}\rho_2{}^{i+1}(-1)^\sigma 2i!}{i-\sigma!\ i+\sigma!}\frac{d^\sigma}{d\eta_1}\frac{d^{-\sigma}}{d\eta_2}\frac{d^{i-\sigma}}{dz_1}\frac{d^{i+\sigma}}{dz_2}\frac{1}{\rho_1\rho_2}$$

or

$$\frac{\rho_1{}^{i+1}\rho_2{}^{i+1}2i!}{i-\sigma!\ i+\sigma!}2^{2\sigma}\frac{d^\sigma}{d\eta_1}\frac{d^\sigma}{d\xi_2}\frac{d^{i-\sigma}}{dz_1}\frac{d^{i-\sigma}}{dz_2}\frac{1}{\rho_1\rho_2}$$

It is obvious a term similar to this can be obtained with only ξ_1 put for ξ_2 and η_2 for η_1, by merely changing the sign of σ in the first of the last three expressions. Adding the two terms and recollecting the definitions of the harmonics we may write the result

$$\frac{2i!\ i!\ i!}{i-\sigma!\ i+\sigma!}2^{2\sigma-1}\{(i,\ \sigma)(i,\ \sigma)'+[i,\ \sigma][i,\ \sigma]'\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Hence LAPLACE's Coefficient as given in § 1.

Reverting now to equation (23), let us take two fixed poles, and let Q_i, Q_i' be expanded according to LAPLACE's Formula; we shall then be at once led to the well-known surface integrals of two tesseral harmonics.

The discussion entered into in this article is necessary for the treatment of the product of three harmonics to which we now proceed.

It may be remarked that the expansion of the operator

$$\left(\frac{d}{dx_1}\frac{d}{dx_2}+\frac{d}{dy_1}\frac{d}{dy_2}+\frac{d}{dz_1}\frac{d}{dz_2}\right)^i\frac{1}{\rho_1\rho_2}$$

is, according to the above investigation,

$$\left\{2^{-i}\frac{2i!}{i!\,i!}\frac{d^i}{dz_1}\frac{d^i}{dz_2}+\ldots+2^{2\sigma-i}\frac{2i!}{i+\sigma!\,i-\sigma!}\left(\frac{d^\sigma}{d\xi_1}\frac{d^\sigma}{d\eta_2}+\frac{d^\sigma}{d\xi_2}\frac{d^\sigma}{d\eta_1}\right)\left(\frac{d}{dz_1}\frac{d}{dz_2}\right)^{i-\sigma}+\ldots\right\}\frac{1}{\rho_1\rho_2}$$

(See also THOMSON and TAIT, p. 157).

General Method of Dealing with the Integral $\iint Q_pQ_qQ_r\ldots dS$.

§ 9. In discussing the general method of evaluating this integral it will be convenient to confine ourselves to the case of three harmonics, though the first steps of the reasoning will apply to any number.

Let A, B, C be any three points whose distances from the origin are a, b, c, and let P be any point on the sphere of reference. Then by § 1

$$\iint\frac{dS}{\text{AP.BP.CP}}=4\pi\text{R}^2\Sigma\frac{\text{R}^{2i}}{2i+1!}\left(\frac{d^2}{dx}+\frac{d^2}{dy}+\frac{d^2}{dz}\right)^i\frac{1}{\text{AO.BO.CO}}$$

$$=4\pi\text{R}^2\Sigma\frac{\text{R}^{2i}}{2i+1!}2^i(v_{23}+v_{31}+v_{12})^i\frac{1}{\text{AO.BO.CO}}$$

where v_{23} stands for

$$\frac{d}{dx_2}\frac{d}{dx_3}+\frac{d}{dy_2}\frac{d}{dy_3}+\frac{d}{dz_2}\frac{d}{dz_3},\ \&\text{c.}$$

the suffixes having the same signification as in the previous article.

The general term of $2i+2$ dimensions in R may therefore be written

$$\frac{4\pi\text{R}^{2i+2}}{2i+1!}2^i\frac{i!}{\lambda!\,\mu!\,\nu!}v^\lambda{}_{23}v^\mu{}_{31}v^\nu{}_{12}\frac{1}{\text{AO.BO.CO}}$$

Now the general expansion of each of the operators in this expression was found in the previous article.

If therefore we adopt a somewhat more convenient notation we may expand $v_{23}{}^\lambda$, $v_{31}{}^\mu$, $v_{12}{}^\nu$ in a series of terms of which the type is

$$2^{2(l+m+n)-i}\frac{2\lambda!\,2\mu!\,2\nu!}{\lambda+l!\,\lambda-l!\,\mu+m!\,\mu-m!\,\nu+n!\,\nu-n!}$$

$$\times\left(\frac{d^l}{d\xi_2}\frac{d^l}{d\eta_3}+\frac{d^l}{d\xi_3}\frac{d^l}{d\eta_2}\right)\left(\frac{d^m}{d\xi_3}\frac{d^m}{d\eta_1}+\frac{d^m}{d\xi_1}\frac{d^m}{d\eta_3}\right)\left(\frac{d^n}{d\xi_1}\frac{d^n}{d\eta_2}+\frac{d^n}{d\xi_2}\frac{d^n}{d\eta_1}\right)$$

$$\times\frac{d^{\mu+\nu-m-n}}{dz_1}\frac{d^{\nu+\lambda-n-l}}{dz_2}\frac{d^{\lambda+\mu-l-m}}{dz_3}$$

where the values of l, m, n range between 0 and λ, μ, ν.

The general operator just found we shall denote by the symbol $\pi\ (l, m, n)$.

§ 10. If we expand each of the reciprocals of AP, BP, CP in harmonics we shall arrive at the general term of $2i+2$ dimensions in R in a different form, viz. : it will be

$$\frac{\mathrm{R}^{2i}}{a^{p+1}b^{q+1}c^{r+1}}\iint \mathrm{Q}_p\mathrm{Q}_q\mathrm{Q}_r d\mathrm{S}$$

In order therefore that the general term as found by the work of the last article may correspond with this we must have

$$\mu+\nu=p$$
$$\nu+\lambda=q$$
$$\lambda+\mu=r$$

The quantities λ, μ, ν, as depending upon p, q, r, are therefore perfectly determinate, and the equation expressing the identity of the general terms may by Theorem i. be written

$$\iint \mathrm{Q}_p\mathrm{Q}_q\mathrm{Q}_r d\mathrm{S}=\frac{4\pi\mathrm{R}^2}{2i+1!}2^i\frac{i!}{\lambda!\,\mu!\,\nu!}\Sigma\Sigma\Sigma\pi(l,\ m,\ n)(\mathrm{Q}_p r_1{}^p\mathrm{Q}_q r_2{}^q\mathrm{Q}_r r_3{}^r)\quad .\quad .\quad .\quad .\quad (25)$$

On examining the equations determining λ, μ, ν, we see that in order to their being positive, $p+q+r$ must be an even number, and no one of the three quantities p, q, r, must be greater than the sum of the other two.

If these conditions are not satisfied, then $\iint \mathrm{Q}_p\mathrm{Q}_q\mathrm{Q}_r d\mathrm{S}=0$ for integral values of p, q, r, wherever the poles of the harmonics may be.

$\iint \mathrm{P}_p\mathrm{P}_q\mathrm{P}_r d\mathrm{S}$

§ 11. Let us now suppose that the three points A, B, C are in the axis of z. Then the harmonic Q_p becomes P_p, which we will suppose expressed by the series D. In like manner P_q and P_r may be similarly expressed. It follows that, in selecting from the general operator what terms are practically effective, and discarding those which are not, we need retain only such terms as are, for example, of the form

$$\frac{d^\sigma}{d\xi_1}\frac{d^\sigma}{d\eta_1}\frac{d^{p-2\sigma}}{dz_1}$$

Now the terms of this form are obviously got by putting $l=m=n, =\sigma$ say. In that case, when we omit inoperative terms, π becomes

$$2^{6\sigma-i+1}\frac{2\lambda!\,2\mu!\,2\nu!}{\lambda+\sigma!\,\lambda-\sigma!\,\mu+\sigma!\,\mu-\sigma!\,\nu+\sigma!\,\nu-\sigma!}$$

$$\times\left(\frac{d}{d\xi_1}\frac{d}{d\eta_1}\frac{d}{d\xi_2}\frac{d}{d\eta_2}\frac{d}{d\xi_3}\frac{d}{d\eta_3}\right)^\sigma\frac{d^{p-2\sigma}}{dz_1}\frac{d^{q-2\sigma}}{dz_2}\frac{d^{r-2\sigma}}{dz_3}$$

that is, in virtue of LAPLACE'S equation being satisfied,

$$(-1)^\sigma\frac{2}{2^i}\frac{2\lambda!\,2\mu!\,2\nu!}{\lambda+\sigma!\,\lambda-\sigma!\,\mu+\sigma!\,\mu-\sigma!\,\nu+\sigma!\,\nu-\sigma!}\frac{d^p}{dz_1}\frac{d^q}{dz_2}\frac{d^r}{dz_3}$$

Taking all possible terms of π, such as that just found, and performing the operation directed by them, we have for $\iint P_pP_qP_r dS$, the expression

$$\frac{4\pi R^2}{2i+1!}\frac{2\lambda!\,2\mu!\,2\nu!\,i!}{\lambda!\,\mu!\,\nu!}\left(\frac{p!\,q!\,r!}{(\lambda!\,\mu!\,\nu!)^2}+2\Sigma(-1)^\sigma\frac{p!\,q!\,r!}{\lambda+\sigma!\,\lambda-\sigma!\,\mu+\sigma!\,\mu-\sigma!\,\nu+\sigma!\,\nu-\sigma!}\right)$$

where σ ranges from 0 to the smallest of the quantities λ, μ, ν.

Now the series within brackets is clearly the coefficient of $x^{2\lambda}$, $y^{2\mu}$, $z^{2\nu}$, or of $x^{\mu+\nu}$, $y^{\nu+\lambda}$, $z^{\lambda+\mu}$, in

$$(-1)^i(y-z)^{\mu+\nu}(z-x)^{\nu+\lambda}(x-y)^{\lambda+\mu}$$

or, again, the same series is also obviously equal to

$$\frac{p!\,q!\,r!}{2\lambda!\,2\mu!\,2\nu!}K$$

where K is the coefficient of $x^{\mu+\nu}\,y^{\nu+\lambda}\,z^{\lambda+\mu}$ in

$$(-1)^i(y-z)^{2\lambda}(z-x)^{2\mu}(x-y)^{2\nu}$$

that is, in

$$(-1)^i x^{\mu+\nu}y^{\nu+\lambda}z^{\lambda+\mu}\left(\sqrt{\frac{y}{z}}-\sqrt{\frac{z}{y}}\right)^{2\lambda}\left(\sqrt{\frac{z}{x}}-\sqrt{\frac{x}{z}}\right)^{2\mu}\left(\sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}\right)^{2\nu}$$

If we now write herein

$$z=xe^{2\phi j}=ye^{2\theta j}$$

we see that K is the term not involving the cosine of an angle of the value $m\theta+n\phi$ in the expansion in such cosines of W, where

$$W=(-1)^i\, 2^{2i} \sin^{2\lambda}\theta \sin^{2\mu}\phi \sin^{2\nu}(\theta-\phi)$$

so that

$$K=\frac{1}{\pi^2}\int_0^\pi\int_0^\pi W d\theta d\phi$$

Now expanding $\sin^{2\nu}(\theta-\phi)$ in powers of sines and cosines by the binomial theorem, we observe that the odd terms in the expansion disappear on integration, so that the value of K is given by

$$\Sigma\frac{1}{\pi^2}\int_0^\pi\int_0^\pi(-1)^i\, 2^{2i}\frac{2\nu!}{2n!\; 2\nu-2n!}\sin^{2\lambda+2n}\theta \cos^{2\nu-2n}\theta \sin^{2\mu+2\nu-2n}\phi \cos^{2n}\phi\; d\theta\; d\phi$$

On integration and reduction this becomes

$$(-1)^i\, 2^{-2\nu}\frac{2\nu!}{\lambda+\nu!\;\mu+\nu!}\,\Sigma\,\frac{2\lambda+2n!\;2\mu+2\nu-2n!}{\lambda+n!\;\mu+\nu-n!\;\nu-n!\;n!}$$

that is,

$$(-1)^i\, 2^{-2\nu}\frac{2\nu!}{\lambda+\nu!\;\mu+\nu!}\,\frac{2\lambda!}{\lambda!}\,\frac{2\mu+2\nu!}{\mu+\nu!}\,\frac{1}{\nu!}S$$

where

$$S=1+\frac{(2\lambda+1)\nu}{2\mu+2\nu-1}+\frac{(2\lambda+1)(2\lambda+3)\nu(\nu-1)}{(2\mu+2\nu-1)(2\mu+2\nu-3)1.2}+\&c.$$

This hypergeometric series is capable of summation by a method similar to that given by BERTRAND, 'Calcul. Intégral,' 1870, pp. 495–496. If we put

$$\lambda+\tfrac{1}{2}=\alpha$$
$$\mu+\nu-\tfrac{1}{2}=\beta$$

S becomes

$$1+\frac{\alpha\nu}{\beta}+\frac{\alpha(\alpha+1)}{\beta(\beta-1)}\,\frac{\nu(\nu-1)}{1.2}+\ \&c.$$

$$=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\,\frac{\Gamma(\beta-n+1)}{\Gamma(\beta+1)}\,\frac{\nu(\nu-1)\ldots(\nu-n+1)}{n!}$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\Sigma\int_0^\infty\frac{\nu(\nu-1)\ldots\nu-n+1}{n!}\,\frac{y^{\alpha+n-1}}{(1+y)^{\alpha+\beta+1}}dy$$

$$=\frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha)\Gamma\beta+1)}\int_0^\infty\frac{y^{\alpha-1}dy}{(1+y)^{\lambda+\mu+1}}$$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\,\frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(\lambda+\mu+1)}$$

$$=2^{2\nu}\frac{i!}{\lambda+\mu!}\,\frac{2\mu!}{\mu!}\,\frac{\mu+\nu!}{2\mu+2\nu!}$$

On substituting this value for S in the value of K we find

$$\mathrm{K}=(-1)^i\frac{i!\,2\lambda!\,2\mu!\,2\nu!}{p!\,q!\,r!\,\lambda!\,\mu!\,\nu!}$$

Finally, performing the operations with regard to the z's, we find

$$\iint \mathrm{P}_p\mathrm{P}_q\mathrm{P}_r d\mathrm{S}=\frac{4\pi\mathrm{R}^2}{2i+1!}\,\frac{i!\,i!}{2i!}\,\frac{2\lambda!\,2\mu!\,2\nu!}{\lambda!\,\lambda!\,\mu!\,\mu!\,\nu!\,\nu!}. \quad \ldots\ldots \quad (26)$$

This agrees with the result obtained by Professor ADAMS.

§ 12. Before leaving this case we may remark that, so far as the foregoing proof is concerned, the poles of all the three harmonics are not necessarily coincident. The proof will hold when two are coincident and the third is at an angular distance, say α, from them. In those circumstances the value of $\iint \mathrm{P}_p\mathrm{P}_q\mathrm{Q}_r d\mathrm{S}$ will be the result found above multiplied by $\mathrm{P}_r(\cos\alpha)$. The same is, of course, true of the integral $\iint \mathrm{Q}_p\mathrm{Q}_q\mathrm{P}_r d\mathrm{S}$, provided the two Q's have the same poles. Now let Q_p, Q_q be each expanded according to LAPLACE's Formula. Then we will have the curious result that any zonal harmonic of degree r can be expanded in products of harmonics, zonal and tesseral, of degrees p and q, the quantities p, q, r being, of course, subject to the foregoing restrictions. If q is less than p the number of terms in such expansion will be $2q+1$

$$\iint(p,\ \alpha)(q,\ \beta)(r,\ \alpha+\beta)d\mathrm{S}$$

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§ 13. Reverting to the general expression for the operator $\pi\ (l,\ m,\ n)$ given in § 8, let us leave out for an instant the numerical multipliers and the operators with regard to the z's and let us multiply out those with regard to ξ and η. Let us suppose l, m, n in descending order of magnitude, and for the sake of brevity, instead of writing the differential operators, let us write only the characteristic letter, *e.g.*, instead of $\frac{d}{d\xi_1}$ write ξ_1. We find

$$\begin{aligned}&\xi_1{}^n\eta_1{}^n\xi_2{}^n\eta_2{}^n\xi_3{}^m\eta_3{}^m(\xi_1{}^{m-n}\eta_2{}^{l-n}\xi_3{}^{l-m}+\eta_1{}^{m-n}\xi_2{}^{l-n}\eta_3{}^{l-m})\\ &+\xi_1{}^n\eta_1{}^n(\xi_1{}^{m-n}\xi_2{}^{l+n}\eta_3{}^{l+m}+\eta_1{}^{m-n}\eta_2{}^{l+n}\xi_3{}^{l+m})\\ &+\xi_2{}^n\eta_2{}^n(\xi_1{}^{m+n}\xi_2{}^{l-n}\eta_3{}^{l+m}+\eta_1{}^{m+n}\eta_2{}^{l-n}\xi_3{}^{l+m})\\ &+\xi_3{}^m\eta_3{}^m(\xi_1{}^{m+n}\eta_2{}^{l+n}\xi_3{}^{l-m}+\eta_1{}^{m+n}\xi_2{}^{l+n}\eta_3{}^{l-m})\end{aligned}$$

Now since any product such as $\frac{d}{d\xi_1}\,\frac{d}{d\eta_1}$ can be replaced by $-\frac{1}{4}\,\frac{d^2}{dz_1}$, we see that each of the lines just written corresponds to the sum of tesseral harmonics, in such manner that if we suppose the differentiations performed by the whole of the operator, and if one of the resulting terms is

$$\frac{\mathrm{L}}{a^{p+1}b^{q+1}c^{r+1}}(p,\ \alpha)_a(q,\ \beta)_b(r,\ \gamma)_c$$

where the suffixes indicate that the angular coordinates of A, B, C are substituted in the harmonics to which they are respectively attached, then we have in all cases one of the quantities α, β, γ equal to the sum of the other two. This we see from the expanded form of the operator just given, and it is a result which was to be expected, for supposing Q_p, Q_q, Q_r severally expanded according to LAPLACE's Formula we should have in the surface integral a series of terms of which the type involves the integral

$$\int_0^{2\pi}\cos\alpha\phi\cos\beta\phi\cos\gamma\phi d\phi$$

and this integral vanishes unless the above conditions be complied with, the case where $\alpha+\beta+\gamma=0$ being of course out of the reckoning.

In accordance with the explanation just given we may now put

$$\left.\begin{array}{ccc} m-n=\alpha & \text{or} & m+n=\alpha \\ l+n=\beta & & l-n=\beta \\ \multicolumn{3}{c}{\text{and } \therefore\ l+m=\alpha+\beta} \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

If then we substitute in π $(l,\ m,\ n)$ for l and m the values $\beta+n$ and $\alpha-n$, we see that n may range in numerical value between 0 and the least of the integers $\lambda-\beta$, $\mu-\alpha$, ν, and may be positive or negative. The operator π, in fact, becomes

$$(-1)^n 2^{2(\alpha+\beta)-i}\frac{2\lambda!\ 2\mu!\ 2\nu!}{\lambda+\beta+n!\ \lambda-\beta-n!\cdot\mu+\alpha-n!\ \mu-\alpha+n!\ \nu+n!\ \nu-n!}$$

$$\times\left(\frac{d^\alpha}{d\xi_1}\frac{d^\beta}{d\xi_2}\frac{d^{\alpha+\beta}}{d\eta_3}+\frac{d^\alpha}{d\eta_1}\frac{d^\beta}{d\eta_2}\frac{d^{\alpha+\beta}}{d\xi_2}\right)\times\frac{d^{\mu+\nu-\alpha}}{dz_1}\frac{d^{\nu+\lambda-\beta}}{dz_2}\frac{d^{\lambda+\mu-\gamma}}{dz_3}$$

We have therefore now to consider a series in which any term is the product of (1) an invariable power of 2, (2) an invariable operator, (3) a variable coefficient whose parameter is n. If we denote the first two of these factors by the symbol ϖ, then the sum of all such terms is $\varpi\mathrm{K}$, where

$$\mathrm{K}=\frac{2\lambda!\ 2\mu!\ 2\nu!}{\mu+\nu-\alpha!\ \nu+\lambda-\beta!\ \lambda+\mu+\alpha+\beta!}\mathrm{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and W is the coefficient of $x^{2\lambda}y^{2\mu}z^{2\nu}$ in

$$(-1)^i(y-z)^{\mu+\nu-\alpha}(z-x)^{\nu+\lambda-\beta}(x-y)^{\lambda+\mu+\alpha+\beta}$$

that is, in

$$(-1)^i(y-z)^{p-\alpha}(z-x)^{q-\beta}(x-y)^{r+\alpha+\beta}$$

The expression upon which the operator ϖ has to operate may be found by expressing Q_p, &c., by LAPLACE's Formula, and using the series D: it is as follows :—

$$\frac{2^{-1}p!}{\alpha!\,p-\alpha!}\{(\xi_1^{\,a}+\eta_1^{\,a})(p,\,\alpha)_a-j(\xi_1^{\,a}-\eta_1^{\,a})[p,\,\alpha]_a\}z_1^{\,p-\alpha}$$

$$\times\frac{2^{-1}q!}{\beta!\,q-\beta!}\{(\xi_2^{\,\beta}+\eta_2^{\,\beta})(q,\,\beta)_b-j(\xi_2^{\,\beta}-\eta_2^{\,\beta})[q,\,\beta]_b\}z_2^{\,q-\beta}$$

$$\times\frac{2^{-1}r!}{\alpha+\beta!\,r-\alpha-\beta!}\{(\xi_3^{\,\alpha+\beta}+\eta_3^{\,\alpha+\beta})(r,\,\alpha+\beta)_c-j(\xi_3^{\,\alpha+\beta}-\eta_3^{\,\alpha+\beta})[r,\,\alpha+\beta]_c\}z_3^{\,r-\alpha-\beta}$$

If we omit irrelevant terms this becomes

$$\frac{2^{-3}p!\,q!\,r!}{\alpha!\,\beta!\,\alpha+\beta!\,p-\alpha!\,q-\beta!\,r-\alpha-\beta}\times\{(p,\,\alpha)_a(q,\,\beta)_b(r,\,\alpha+\beta)_c$$
$$-(p,\,\alpha)[q,\,\beta][r,\,\alpha+\beta]_c-[p,\,\alpha]_a(q,\,\beta)_b\,[r,\,\alpha+\beta]_c-[p,\,\alpha]_a[q,\,\beta]_b(r,\,\alpha+\beta)_c\}$$
$$\times(\xi_1^{\,\alpha}\xi_2^{\,\beta}\eta_3^{\,\alpha+\beta}+\eta_1^{\,\alpha}\eta_2^{\,\beta}\xi_3^{\,\alpha+\beta})z_1^{\,p-\alpha}z_2^{\,q-\beta}z_3^{\,r-\alpha-\beta}$$

The result of the operation is accordingly

$$2^{2(\alpha+\beta)-i-2}p!\;q!\;r!\;\mathrm{K}\{(p,\,\alpha)_a(q,\,\beta)_b(r,\alpha+\beta)_c-\&c.\;.\;.\;.\;\}$$

Turning now to the integral $\iint Q_pQ_qQ_rd\mathrm{S}$, let us expand each of the Q's according to LAPLACE's Formula. We thus find for the general term

$$\frac{2^{4(\alpha+\beta)-3}p!\,p!\,q!\,q!\,r!\,r!}{p+\alpha!\,p-\alpha!\,q+\beta!\,q-\beta!\,r+\alpha+\beta!\,r-\alpha-\beta!}$$
$$\times\iint\{(p,\,\alpha)_a(p,\,\alpha)+[p,\,\alpha]_a\,[p,\,\alpha]\}\{(q,\,\beta)_b(q,\,\beta)+q,\,\beta]_b[q,\,\beta]\}\{(r,\,\alpha+\beta)_c(r,\,\alpha+\beta)$$
$$+[r,\,\alpha+\beta]_c\,[r,\,\alpha+\beta]\}d\mathrm{S}$$

Now, by the formula of § 10, when we come to equate the two results just found, we shall get

$$\frac{2^{2(\alpha+\beta)-1}p!\,q!\,r!}{p+\alpha!\,q+\beta!\,r-\alpha-\beta!}\iint(p,\,\alpha)(q,\,\beta)(r,\,\alpha+\beta)d\mathrm{S}=\frac{4\pi\mathrm{R}^2}{2i+1!}\,\frac{i!\,2\lambda!\,2\mu!\,2\nu!}{\lambda!\,\mu!\,\nu!}\mathrm{W}$$

or finally

$$\iint(p,\,\alpha)(q,\,\beta)(r,\,\alpha+\beta)d\mathrm{S}=\frac{4\pi\mathrm{R}^2}{2i+1!}\,\frac{i!\,2\lambda!\,2\mu!\,2\nu!\,p+\alpha!\,q+\beta!\,r-\alpha-\beta!}{\lambda!\,\mu!\,\nu!\,p!\,q!\,r!}\,\mathrm{W}.\quad.\quad(29)$$

where W is defined above (28) as the coefficient of a term in a certain product, or as a series whose terms depend upon a single parameter.

If we denote by U the value of the expression on the right hand of the equation just found, we have further

$$\iint(p,\ \alpha)(q,\ \beta)[r,\ \alpha+\beta]d\mathrm{S}=0$$

$$\iint(p,\ \alpha)[q,\ \beta][r,\ \alpha+\beta]d\mathrm{S}=-\mathrm{U}$$

&c.

$\iint \mathrm{P}_p\mathrm{P}_q\mathrm{P}_r\mathrm{P}_s d\mathrm{S}$

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§ 14. We may remark in regard to the case which we have now to examine, that since $\mathrm{P}_r\mathrm{P}_s$ may by the foregoing results be expanded in a series of the form

$$\mathrm{A}_0\mathrm{P}_{r-s}+\mathrm{A}_1\mathrm{P}_{r-s+2}+\ \ldots\ +\mathrm{A}_m\mathrm{P}_{r-s+2m}+\ \ldots\ +\mathrm{A}_s\mathrm{P}_{r+s}$$

where the value of A_m determined by (26) is

$$\frac{2r-2s+4m+1}{2r+2m+1}\ \frac{r+m\,!\ r+m\,!\ 2m\,!\ 2r-2s+2m\,!\ 2s-2m\,!}{2r+2m\,!\ m\,!\ m\,!\ r-s+m\,!\ r-s+m\,!\ s-m\,!\ s-m\,!}$$

the above integral is reducible to the computation of a series of s terms of the form

$$\mathrm{A}_m\iint \mathrm{P}_p\mathrm{P}_q\mathrm{P}_{r-s+2m}d\mathrm{S}$$

The last integral is to be found by writing in (26)

$$2\lambda=-p+q+r-s+2m$$
$$2\mu=p-q+r-s+2m$$
$$2\nu=p+q-r+s-2m$$

We are thus led to a complete though practically laborious evaluation of $\iint\mathrm{P}_p\mathrm{P}_q\mathrm{P}_r\mathrm{P}_s d\mathrm{S}$. It will be observed that the result is zero unless $p+q+r+s$ be an even number.

With the view of finding out how far the method of this paper is applicable in this case, and what difficulties stand in the way of its general application, we will now briefly apply it. It will be convenient to expand the operator in a somewhat different manner to that pursued in the case of the product of three harmonics. The operator was then expanded in a form which would render it useful for application in the case of the tesseral and sectorial system. If, however, the poles of the harmonic are all in the axis of z, a much simpler mode of expansion may be adopted.

When the product of four harmonics is under consideration we have to discuss an operator of the form

$$\left\{\left(\frac{d}{dx_1}+\frac{d}{dx_2}+\frac{d}{dx_3}+\frac{d}{dx_4}\right)^2+\left(\frac{d}{dy_1}+\frac{d}{dy_2}+\frac{d}{dy_3}+\frac{d}{dy_4}\right)^2+\left(\frac{d}{dz_1}+\frac{d}{dz_2}+\frac{d}{dz_3}+\frac{d}{dz_4}\right)^2\right\}^i$$

or, what is the same thing,

$$\left\{\left(\frac{d}{dz_1}+\frac{d}{dz_2}+\frac{d}{dz_3}+\frac{d}{dz_4}\right)^2+4\left(\frac{d}{d\xi_1}+\frac{d}{d\xi_2}+\frac{d}{d\xi_3}+\frac{d}{d\xi_4}\right)\left(\frac{d}{d\eta_1}+\frac{d}{d\eta_2}+\frac{d}{d\eta_3}+\frac{d}{d\eta_4}\right)\right\}^i$$

If this operate upon the reciprocal of $\rho_1\rho_2\rho_3\rho_4$, where these distances are measured from four points A, B, C, D on the axis of z, we have four relations of the form

$$\frac{d^2}{dz}+4\frac{d^2}{d\xi d\eta}=0$$

Further, on picking out the general term, we observe that, to be effectually operative, it must consist of products of the form

$$\frac{d^l}{dz}\frac{d^m}{d\xi}\frac{d^m}{d\eta}$$

and this form of operator may, by the foregoing relation, be expressed in terms of $\frac{d}{dz}$ only.

It is obvious, however, that the sum of the terms so modified can be found directly from the part of the expansion not containing k's of

$$\left\{\left(\frac{d}{dz_1}+\frac{d}{dz_2}+\frac{d}{dz_3}+\frac{d}{dz_4}\right)^2-(k_1+k_2+k_3+k_4)\left(\frac{1}{k_1}\frac{d^2}{dz_1}+\frac{1}{k_2}\frac{d^2}{dz_2}+\frac{1}{k_3}\frac{d^2}{dz_3}+\frac{1}{k_4}\frac{d^2}{dz_4}\right)\right\}^i$$

or, what is the same thing, from the part of the expansion not containing k's of

$$\left\{\left(\frac{d}{dz_1}+\frac{d}{dz_2}+\frac{d}{dz_3}+\frac{d}{dz_4}\right)^2-\left(k_1\frac{d}{dz_1}+k_2\frac{d}{dz_2}+k_3\frac{d}{dz_3}+k_4\frac{d}{dz_4}\right)\left(\frac{1}{k_1}\frac{d}{dz_1}+\frac{1}{k_2}\frac{d}{dz_2}+\frac{1}{k_3}\frac{d}{dz_3}+\frac{1}{k_4}\frac{d}{dz_4}\right)\right\}^i$$

that is, of

$$(-1)^i\left\{\left(\sqrt{\frac{k_1}{k_2}}-\sqrt{\frac{k_2}{k_1}}\right)^2\frac{d}{dz_1}\frac{d}{dz_2}+5\text{ similar terms}\right\}^i$$

If we expand this the general term will be

$$(-1)^i\frac{i!}{\lambda!\,\mu!\,\nu!\,l!\,m!\,n!}\times(k_2-k_3)^{2\lambda}(k_3-k_1)^{2\mu}(k_1-k_2)^{2\nu}(k_4-k_1)^{2l}(k_4-k_2)^{2m}(k_4-k_3)^{2n}$$
$$\times\left(\frac{1}{k_1}\frac{d}{dz_1}\right)^p\left(\frac{1}{k_2}\frac{d}{dz_2}\right)^q\left(\frac{1}{k_3}\frac{d}{dz_3}\right)^r\left(\frac{1}{k_3}\frac{d}{dz_4}\right)^s$$

where

$$\left.\begin{aligned}\mu+\nu+l&=p\\ \nu+\lambda+m&=q\\ \lambda+\mu+n&=r\\ l+m+n&=s\end{aligned}\right\}\quad\text{. . . . .}\quad(30)$$

The quantities λ, μ, ν, l, m, n are therefore indeterminate, and there will be a series of terms having the required indices.

The coefficient of the operator given above may be put into another form. Let us consider the expansion of

$$\{a(k_2-k_3)+b(k_3-k_1)+c(k_1-k_2)+d(k_4-k_1)+e(k_4-k_2)+f(k_4-k_3)\}^{2i}$$

first in powers of a, b, c, . . . and then in powers of k_1, k_2, k_3, . . . We then see that the coefficient of $k_1^p\ k_2^q\ k_3^r\ k_4^s$ in $(k_2-k_3)^{2\lambda}$ . . . $(k_4-k_3)^{2n}$ is equal to

$$\frac{2\lambda!\ 2\mu!\ 2\nu!\ 2l!\ 2m!\ 2n!}{p!\ q!\ r!\ s!}\mathrm{W}$$

where W is the coefficient of $a^{2\lambda}\ b^{2\mu}\ c^{2\nu}\ d^{2l}\ e^{2m}\ f^{2n}$ in

$$(-b+c-d)^p(-c+a-e)^q(-a+b-f)^r(d+e+f)^s$$

We have, finally,

$$\iint \mathrm{P}_p\mathrm{P}_q\mathrm{P}_r\mathrm{P}_s d\mathrm{S}=(-1)^i\frac{4\pi\mathrm{R}^2 i!}{2i+1!}\ \Sigma\ \frac{2\lambda!\ 2\mu!\ 2\nu!\ 2l!\ 2m!\ 2n!}{\lambda!\ \mu!\ \nu!\ l!\ m!\ n!}\mathrm{W}$$

where Σ denotes the sum of all such values of W multiplied by their respective coefficients corresponding to values of λ, μ, ν, l, m, n determined by equations (30).

This investigation serves to exhibit the peculiar practical difficulties which beset the problem of integrating products of harmonics over the sphere of reference, if more than three harmonics be considered.

POTENTIALS OF ELLIPSOIDS.

§ 15. Let p be the perpendicular upon the tangent plane at any point of an ellipsoid, and let us consider the integral

$$\iint e^{\alpha x+\beta y+\gamma z} p\, d\mathrm{S}$$

taken over the surface.

By the theory of corresponding points this integral may clearly be thrown into the form of an integral taken over the surface of a sphere of radius R, viz.: it is

$$\frac{abc}{\mathrm{R}^2}\iint e^{\frac{a\alpha x'+b\beta y'+c\gamma z'}{\mathrm{R}}} d\mathrm{S}',$$

the value of which, by what we have already shown in (3) is

$$2\pi abc\frac{e^{\nabla}-e^{-\nabla}}{\nabla}$$

where

$$\nabla^2=a^2\alpha^2+b^2\beta^2+c^2\gamma^2.$$

Now let V be any function of x, y, z, which can be expanded by TAYLOR'S Theorem; then

$$\iint \mathrm{V}p\,d\mathrm{S}$$

taken over the surface of the ellipsoid is

$$2\pi abc\frac{e^{\nabla}-e^{-\nabla}}{\nabla}\mathrm{V}$$

or, in series,

$$4\pi abc\left(1+\frac{\nabla^2}{3!}+\ .\ .\ .\ +\frac{\nabla^{2i}}{2i+1!}+\ .\ .\ .\ \right)\mathrm{V}$$

where

$$\nabla^2=a^2\frac{d^2}{dx}+b^2\frac{d}{dy^2}+c^2\frac{d^2}{dz},$$

and after the differentiations are performed, x, y, z are to be put equal to zero.

§ 16. If V satisfies LAPLACE'S equation, then the operator ∇^2 will remain unaltered if for a^2, b^2, c^2 we substitute $a^2+\epsilon$, $b^2+\epsilon$, $c^2+\epsilon$. It follows, therefore, that the average value of V over any ellipsoid as measured by

$$\frac{\iint\frac{\mathrm{V}p\,d\mathrm{S}}{3}}{\text{volume of ellipsoid}}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \qquad (31)$$

is the same for that ellipsoid as for any ellipsoid confocal with it.

This theorem,* which may be regarded as the anologue for ellipsoids of the corresponding theorem given by GAUSS for spheres, is due to Professor CHARLES NIVEN, who also showed that, if V be due to attracting matter E inside the surface of the ellipsoid, then the expression (31) becomes

$$\frac{\mathrm{E}}{2}\int_0^{\infty}\frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}$$

§ 17. As a particular case, let $\mathrm{V}=\frac{1}{\mathrm{QP}}$ where Q is any point outside of the ellipsoid. Then

$$\iint\frac{p\,d\mathrm{S}}{\mathrm{QP}}$$

is the potential at Q due to matter of density p coated over the ellipsoid. The quantity of matter is $4\pi abc$ and the potential due to it at Q, according to a well known result is

$$2\pi abc\int_{\epsilon}^{\infty}\frac{d\psi}{\sqrt{(a^2+\psi)(b^2+\psi)(c^2+\psi)}}$$

or

$$2\pi abc\alpha$$

where α is the ellipsoidal coordinate of Q.

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* Mathematical Tripos Solutions for 1878.

If P′ be any point inside the ellipsoid we then have ultimately when P′ coincides with the origin

$$\alpha=\frac{e^{\nabla}-e^{-\nabla}}{\nabla}\frac{1}{\mathrm{QP}'}$$

But, if f, g, h be the coordinates of Q,

$$\mathrm{QP}'^2=(f-x)^2+(g-y)^2+(h-z)^2$$

Hence

$$\frac{d}{dx}\frac{1}{\mathrm{QP}'}=-\frac{d}{df}\frac{1}{\mathrm{QP}'}$$

$$\frac{d^2}{dx}\frac{1}{\mathrm{QP}'}=\frac{d^2}{df}\frac{1}{\mathrm{QP}'} \qquad \&c.$$

We may therefore take ∇^2 to stand for

$$a^2\frac{d^2}{df}+b^2\frac{d^2}{dg}+c^2\frac{d^2}{dh}$$

and in that case we may put x, y, z zero before differentiating. If then OQ be denoted by r, we have

$$\alpha=\frac{e^{\nabla}-e^{-\nabla}}{\nabla}\frac{1}{r} \quad \ldots \ldots \quad (32)$$

or, in series,

$$2\left(1+\frac{1}{3!}\nabla^2+\ldots+\frac{1}{2i+1!}\nabla^{2i}+\ldots\right)\frac{1}{r} \quad \ldots \ldots \quad (33)$$

If we multiply α by $2\pi\rho abc\delta\theta$ we have the potential of an infinitely thin shell bounded by two similar and similarly situated ellipsoids. When a shell is hereafter referred to, this kind of shell will be meant, unless otherwise specified.

§ 18. Since the whole ellipsoid may be divided up into shells, the potential of the ellipsoid at Q is

$$\iiint\rho\frac{pd\mathrm{S}d\theta}{\mathrm{QP}\theta} \quad \ldots \ldots \quad (34)$$

where ρ is the density at any point in the shell whose semi-axes are $a\theta$, $b\theta$, $c\theta$. It will be observed that, since $p\delta\theta=\theta\delta p$, $\frac{p\delta\mathrm{S}\delta\theta}{\theta}$ is an element of volume, so that (34) may be written

$$\iiint\frac{\rho dxdydz}{\mathrm{QP}}$$

If we take the series (33) we find for the potential of the ellipsoid at Q the series

$$\int_0^1 4\pi\rho abc\theta^2\left(1+\frac{\theta^2}{3!}\nabla^2+\ldots+\frac{\theta^{2i}}{2i+1!}\nabla^{2i}+\&c.\right)\frac{1}{r}d\theta \quad \ldots \ldots \quad (35)$$

This can be integrated if ρ be a function of θ.

In the particular case where ρ is constant and the mass is denoted by M, we find

$$\mathrm{M}\left(1+\frac{3}{5}\frac{1}{3!}\nabla^2+\ldots\ldots+\frac{3}{2i+3}\frac{\nabla^{2i}}{2i+1!}+\ldots\right)\frac{1}{r}\quad\ldots\ldots\quad(36)$$

§ 19. The result just formed exhibits in a very simple manner MACLAURIN'S Theorem, that the attractions of confocal ellipsoids at external points are proportional to their masses. For, as has been already pointed out, the operator ∇^2 is unaltered by the addition of $k\left(\frac{d^2}{df}+\frac{d^2}{dg}+\frac{d^2}{dh}\right)$.

The series (35) also shows in what direction MACLAURIN'S Theorem may be generalised.

(i.) Let $\rho=f(\theta)$. We then see that if there be two ellipsoids which would be confocal if they were coaxal, and if the matter in them were arranged in layers similar to their bounding surfaces according to any specified law depending on the parameter of the layer, then the attraction of such ellipsoids at external points whose coordinates referred to the axes of the ellipsoids are equal, are proportional to the masses of the ellipsoids.

(ii.) If we multiply the value of α given in (33) by

$$2\pi\sqrt{(a^2-\phi)(b^2-\phi)(c^2-\phi)}\mathrm{F}(\phi)d\phi$$

and integrate between the values ϕ_1 and ϕ_2, we find on the right hand side

$$4\pi\int_{\phi_2}^{\phi_1}\sqrt{(a^2-\phi)(b^2-\phi)(c^2-\phi)}\mathrm{F}(\phi)d\phi\times\left(1+\frac{\nabla^2}{3!}+\ldots+\frac{\nabla^{2i}}{2i+1!}+\ldots\right)\frac{1}{r}\quad.\quad(37)$$

in which the operator is independent of ϕ.

On the left hand side we have

$$\int_{\phi_2}^{\phi_1}2\pi\sqrt{(a^2-\phi)(b^2-\phi)(c^2-\phi)}\alpha\mathrm{F}(\phi)d\phi$$

$$=\int_{\phi_2}^{\phi_1}\iint\frac{\mathrm{F}(\phi)pd\mathrm{S}}{\mathrm{QP}}d\phi$$

$$=\iiint\frac{2\mathrm{F}(\phi)p^2}{\mathrm{QP}}dxdydz$$

where p is the perpendicular from the centre on the tangent plane at the point x, y, z to the confocal, passing through the point of parameter ϕ.

Hence, if the matter between two confocal ellipsoids be affected with a density varying as $\mathrm{F}(\phi)p^2$, the equipotential surfaces will be confocal ellipsoids, as may be seen from the expression (37), which also gives the ratios of the attractions due to different shells.

(iii.) The theorem is true for any value of V satisfying LAPLACE'S equation.

§ 20. If we put $a=b$ then ∇^2 becomes $-(a^2-c^2)\frac{d^2}{dz}$, and the potential of a solid ellipsoid of revolution of uniform density becomes

$$\frac{\mathrm{M}}{r}\left(1-\frac{3\mathrm{P}_2}{3\cdot5}\,\frac{a^2-c^2}{r^2}+\frac{3\mathrm{P}_4}{5\cdot7}\left(\frac{a^2-c^2}{r^2}\right)^2-\frac{3\mathrm{P}_6}{7\cdot9}\left(\frac{a^2-c^2}{r^2}\right)^3+\ldots\right)\quad .\ \ .\ \ .\ \ .\quad (38)$$

§ 21. The expansion involving surface spherical harmonics of the potential of a solid ellipsoid of uniform density is easily derived from (36).

In virtue of LAPLACE's equation, the operator ∇^2 becomes

$$(a^2-b^2)\frac{d^2}{dx}-(b^2-c^2)\frac{d^2}{dz}$$

which we will abbreviate to

$$f^2\frac{d^2}{dx}-g^2\frac{d^2}{dz}$$

Now by Theorem i.

$$\left(f^2\frac{d^2}{dx}-g^2\frac{d^2}{dz}\right)^i\frac{1}{r}=(-1)^i\frac{1}{r^{2i+1}}\left(g^2\frac{d^2}{dz}-f^2\frac{d^2}{dx}\right)^i r^{2i}\mathrm{Q}_{2i}$$

Since there is here no operator in regard to y we may put $y=0$ in the expression for $\mathrm{Q}_{2i}r^{2i}$ before differentiating. In that case $\xi=\eta=x$, and the expansion will take a comparatively simple form.

We take, according to custom, the axis of z for the axes of the zonal and tesseral-sectorial system. The expansion of $\mathrm{Q}_{2i}r^{2i}$ then becomes, by LAPLACE's Formula

$$\mathrm{P'}_{2i}r^{2i}\mathrm{P}_{2i}+\ .\ .\ .\ \frac{2^{4\sigma-1}\ 2i!\ 2i!}{2i+2\sigma!\ 2i-2\sigma!}\,(2i,\ 2\sigma)'r^{2i}(2i,\ 2\sigma)+\ .\ .\ .$$

$$=\mathrm{P'}_{2i}\left(z^{2i}-\frac{2i(2i-1)}{4}z^{2i-2}x^2+\ .\ .\ .\right)+\ .\ .\ .\ +\ .\ .\ .$$

$$+\frac{2i!}{2\sigma!\ 2i-2\sigma!}(2i,\ 2\sigma)'\left(z^{2i-2\sigma}x^{2\sigma}-\frac{(2i-2\sigma)(2i-2\sigma-1)}{4(2\sigma+1)}z^{2i-2\sigma-2}x^{2\sigma+2}+\ .\ .\ .\right)+\ \&\text{c.}$$

The general term of the expansion of the potential of an ellipsoid may therefore be expressed thus :—

$$(-1)^i\frac{3}{(2i+3)(2i+1)}\,\frac{1}{r^{2i+1}}(\mathrm{AP'}_{2i}+\ .\ .\ .\ +\mathrm{B}_{2\sigma}(2i,\ 2\sigma)'+\ .\ .\ .\ )$$

where

$$\mathrm{A}=g^{2i}-\frac{i}{2}g^{2i-2}f^2+\frac{i(i-1)}{1.2}\,\frac{3}{2}\,\frac{1}{2^2}g^{2i-4}f^4-\ \&\text{c.}$$

$$\mathrm{B}_{2\sigma}=(-1)^\sigma\frac{i!}{\sigma!\ i-\sigma!}\left(g^{2i-2\sigma}f^{2\sigma}-\frac{i-\sigma}{2}g^{2i-2\sigma-2}f^{2\sigma+2}\right.$$

$$\left.+\frac{(i-\sigma)(i-\sigma-1)}{1.2}\,\frac{2\sigma+3}{2\sigma+2}\,\frac{1}{2^2}g^{2i-2\sigma-4}f^{2\sigma+4}-\ \&\text{c.}\right)\quad .\ \ .\ \ .\ \ .\quad (39)$$

§ 22. It may be observed that the symbolical form of the operator in the case of the potential of a shell leads to a symbolical form for the potential of the ellipsoid, viz. :—

$$\frac{3M}{2}\frac{\nabla(e^{\nabla}+e^{-\nabla})-e^{\nabla}+e^{-\nabla}}{\nabla^3}\frac{1}{r}$$

Now let there be a second solid ellipsoid of mass M′ and of semi-axes a', b', c', and let it be placed in a position where the coordinates of its centre referred to the axes of the first ellipsoid are f, g, h, and its axes are inclined at direction cosines $(l_1\ m_1\ n_1)$, $(l_2\ m_2\ n_2)$, $(l_3\ m_3\ n_3)$. Then if we put

$$\nabla_1^2=a'^2\frac{d^2}{dx}+b'^2\frac{d^2}{dy}+c'^2\frac{d^2}{dz}$$

where

$$\frac{d}{dx}=l_1\frac{d}{df}+m_1\frac{d}{dg}+n_1\frac{d}{dh}$$

$$\frac{d}{dy}=l_2\frac{d}{df}+m_2\frac{d}{dg}+n_2\frac{d}{dh}$$

$$\frac{d}{dz}=l_3\frac{d}{df}+m_3\frac{d}{dg}+n_3\frac{d}{dh}$$

a double application of the reasoning of §§ 16–18 leads us to the conclusion that the exhaustion of the potential energy of the two ellipsoids due to mutual action is

$$\frac{9}{4}MM'\frac{\nabla(e^{\nabla}+e^{-\nabla})-e^{\nabla}+e^{-\nabla}}{\nabla^3}\ \frac{\nabla_1(e^{\nabla_1}+e^{-\nabla_1})-e^{\nabla_1}+e^{-\nabla_1}}{\nabla_1^3}\ \frac{1}{r}\quad .\ .\ .\ .\quad (40)$$

where r is the distance between their centres.

If we turn the second ellipsoid through an angle $\delta\theta$ about the line (λ, μ, ν) passing through its centre, we have

$$\left.\begin{array}{l}\delta l_1\ =(\nu m_1-\mu n_1)\delta\theta\\ \delta m_1=(\lambda n_1\ -\nu l_1\ )\delta\theta\\ \delta n_1\ =(\mu l_1\ -\lambda m_1)\delta\theta\end{array}\right\}\quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad (41)$$

with similar increments for the other direction cosines. Hence the expansion of (40) in harmonics leads to an approximate determination of the forces and couples representing the mutual action between the two ellipsoids.

It is obvious that a similar investigation will apply to the determination of the forces and couples between two magnets in the form of ellipsoids uniformly magnetised.

It may be interesting to notice that the foregoing method of expansion (36) shows that for points at a considerable distance the potential due to a solid ellipsoid is the same as if its mass were distributed as follows :—Two-fifths at the centre and one-tenth at the extremities of each of the axes.

General Remarks on Ellipsoidal Surface and Volume Integrals.

§ 23. We have found for the ellipsoid that

$$\iint \mathrm{V}p\,d\mathrm{S}=4\pi abc\left(1+\frac{\nabla^2}{3!}+\ldots\right)\mathrm{V}$$

$$\iiint \mathrm{V}\,dx\,dy\,dz=4\pi abc\left(\frac{1}{3}+\frac{1}{5}\,\frac{\nabla^2}{3!}+\ldots\right)\mathrm{V}$$

where V is any function of x, y, z whose differential coefficients are finite at the origin. It is obvious, therefore, that these two results lead to an infinite variety of definite integrals. For example, let us use the second result in finding the value of $\iiint x^{2l}y^{2m}z^{2n}dx\,dy\,dz$, which is an integral evaluated by LAGRANGE in determining the potential due to an ellipsoid (TODHUNTER'S 'History of the Theory of Attractions,' vol. ii, p. 158).

Putting $i=l+m+n$, we find at once

$$\iiint x^{2l}y^{2m}z^{2n}dx\,dy\,dz=\frac{4\pi abc\;i!\;a^{2l}b^{2m}c^{2n}}{(2i+3)(2i+1)!\;l!\;m!\;n!}\,\frac{d^{2l}}{dx}\,\frac{d^{2m}}{dy}\,\frac{d^{2n}}{dz}\,x^{2l}y^{2m}z^{2n}$$

$$=4\pi a^{2l+1}b^{2m+1}c^{2n+1}\frac{1.3.5\ldots(2l-1)1.3.5\ldots(2m-1)1.3.5\ldots(2n-1)}{1.3.5\ldots(2l+2m+2n+3)}$$

If we suppose the function V such that it satisfies the equation

$$a^2\frac{d^2\mathrm{V}}{dx^2}+b^2\frac{d^2\mathrm{V}}{dy^2}+c^2\frac{d^2\mathrm{V}}{dz^2}=0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (42)$$

then the surface volume integrals are the simplest possible in regard to results, but it will depend upon some other condition attached to V what class of function we shall have succeeded in integrating. For example, if V also satisfies the condition

$$x\frac{d\mathrm{V}}{dx}+y\frac{d\mathrm{V}}{dy}+z\frac{d\mathrm{V}}{dz}=n\mathrm{V}$$

we should be led to a class of results similar to those obtained in the case of spherical harmonics for the sphere, and, in fact, derivable from these by the theory of corresponding points. As another example, let us suppose that V, besides satisfying equation (42), also satisfies LAPLACE'S equation; then one solution for V will be of the form

$$\mathrm{F}(\sqrt{b^2-c^2}\,x+\sqrt{c^2-a^2}\,y+\sqrt{a^2-b^2}\,z)$$

We thus get

$$\left.\begin{aligned}\iint \mathrm{F}p\,d\mathrm{S}\quad&=4\pi abc\mathrm{F}(0)\\ \iiint \mathrm{F}\,dx\,dy\,dz&=\frac{4\pi abc}{3}\,\mathrm{F}(0)\end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (43)$$

As a limiting case, let the ellipsoid wear down to the focal ellipse, and for the sake of brevity let the semi-axes of that ellipse be denoted by f and g. We find

$$\left.\begin{aligned}&\iint\frac{\mathrm{F}(gx+jfy)}{\sqrt{1-\frac{x^2}{f^2}-\frac{y^2}{g^2}}}dxdy=2\pi fg\mathrm{F}(0)\\&\iint\sqrt{1-\frac{x^2}{f^2}-\frac{y^2}{g^2}}\,\mathrm{F}(gx+jfy)dxdy=\frac{2\pi fg}{3}\mathrm{F}(0)\end{aligned}\right\}\quad\ldots\ldots\quad(44)$$

These results may be further simplified if the ellipse becomes a circle.

The Potential due to an Ellipse of uniform density.

§ 24. The integral

$$\int e^{\alpha x+\beta y}pds$$

taken round the perimeter of an ellipse of semi-axes a, b, is derivable from the corresponding case of the circle of radius R by the theory of corresponding points, and the result is

$$2\pi ab\sum_0^\infty\frac{1}{2^{2i}\;i!\;i!}\,(a^2\alpha^2+b^2\beta^2)^i\quad\ldots\ldots\ldots\ldots\quad(45)$$

If we write herein $a\theta$, $b\theta$, and recollect that $p\delta\theta=\theta\delta p$, then

$$\iint e^{\alpha x+\beta y}dxdy$$

taken over the area of the ellipse is equal to

$$\iint e^{\alpha x+\beta y}\frac{p}{\theta}dsd\theta$$

the limits of θ being from 0 to 1. The result of integrating this is

$$2\pi ab\sum_0^\infty\frac{(a^2\alpha^2+b^2\beta^2)^i}{2^{2i+1}\;i!\;\overline{i+1}!}\quad\ldots\ldots\ldots\ldots\quad(46)$$

Similar reasoning would give us

$$\iint f\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)e^{\alpha x+\beta y}dxdy$$

Confining our attention, however, to the result just found (46), we see that the integration over the area of the ellipse of any function V of x, y, having finite differential coefficients at the origin, leads to

$$\iint \mathrm{V}dxdy = 2\pi ab\sum_0^\infty \frac{\left(a^2\frac{d^2}{dx}+b^2\frac{d^2}{dy}\right)^i}{2^{2i+1}\ i!\ i+1!}\mathrm{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

where, after the differentiations are performed, x, y, z are to be put equal to zero.

If we suppose the ellipse coated with matter of unit density we thus get for the potential at any outside point (f, g, h), sufficiently distant,

$$\iint \frac{dxdy}{\sqrt{(f-x)^2+(g-y)^2+h^2}} = 2\pi ab\sum_0^\infty \frac{\left(a^2\frac{d^2}{dx}+b^2\frac{d^2}{dy}\right)^i}{2^{2i+1}\ i!\ i+1!}\ \frac{1}{r}$$

$$= 2\pi ab\sum_0^\infty \frac{\left(a^2\frac{d^2}{df}+b^2\frac{d^2}{dg}\right)^i}{2^{2i+1}\ i!\ i+1!}\ \frac{1}{\sqrt{f^2+g^2+h^2}}$$

$$= 2\pi ab\sum_0^\infty \frac{(-1)^i 2i!}{2^{2i+1}\ i!\ i+1!}\ \frac{1}{r^{i+1}}\left(\mathrm{AP}'_{2i}+\ .\ .\ .\ +\mathrm{B}_{2\sigma}(2i,\ 2\sigma)+\ .\ .\ .\ \right) \quad . \quad . \quad (48)$$

Where A, . . . $\mathrm{B}_{2\sigma}$ . . . are the same as in § 21, provided we write $f^2=a^2-b^2$ and $g^2=b^2$ in the expansion of that article. We must have $r>b$ and $>\sqrt{a^2-b^2}$, if the above expansion is convergent.

If $a=b$ or $f=0$ the expansion just found reduces to the well known expansion for a circular plate of uniform thickness, given in THOMSON and TAIT, p. 406.

(Added September 23, 1879.)

§ 25. The series given above for the potential of an ellipse may be thrown into the form of an integral. Writing the result for the elliptic ring in the form

$$2\pi ab\sum_0^\infty \frac{\left\{(a^2-b^2)\frac{d^2}{df}-b^2\frac{d^2}{dh}\right\}^i}{2^{2i}\ i!\ i!}\ \frac{1}{r}$$

we see that this series is the part of the expansion not involving powers of k or its reciprocal of

$$2\pi abe^{\frac{kj}{2}\left(b\frac{d}{dh}+c\frac{d}{df}\right)}\ e^{\frac{j}{2k}\left(b\frac{d}{dh}-c\frac{d}{df}\right)}\frac{1}{r}$$

where $c^2=a^2-b^2$. That is, the series is equal to

$$ab\int_0^{2\pi} e^{jb\cos\psi\frac{d}{dh}-c\sin\psi\frac{d}{df}}\frac{1}{r}d\psi.$$

Hence writing herein $a\theta$, $b\theta$, multiplying by $\frac{d\theta}{\theta}$ and integrating, we find for the potential at (f, g, h) the expression

$$ab\int_0^{2\pi}\int_0^1 e^{\left(jb\cos\psi\frac{d}{dh}-c\sin\psi\frac{d}{df}\right)\theta}\frac{1}{r}\theta d\theta d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(a)}$$

The expansion of the exponential leads of course to the series (48) given above, and it is easy to find thereby the expressions for the action of an elliptic current on a magnet placed at any distance from a coil outside a certain boundary. In like manner, subject to a similar restriction, we might find, as in the case of two ellipsoids, the action of one elliptic current on another. Within the boundary referred to, the expressions we have obtained are no longer convergent, and in the case of the elliptic current the potential in the neighbourhood of the centre of the circuit must be found by an independent process.

The case of the circle, however, admits of a simple and complete determination.

Potential of Electric Currents in Circular Circuits.

§ 26. Reverting to the integral (a) let us put $c=0$; then the potential at (f, g, h) due to a circle of unit density is given by

$$\mathrm{V}=a^2\int_0^{2\pi}\int_0^1 e^{ja\theta\cos\psi\frac{d}{dh}}\frac{1}{r}\theta d\theta d\psi$$

To integrate this in regard to θ we observe first that

$$\int_0^1 \theta e^{u\theta}d\theta=\frac{e^u}{u}-\frac{e^u}{u^2}+\frac{1}{u^2}\;.$$

Let us next consider the integral

$$\int_0^{2\pi} e^{k\cos\psi}d\psi, \text{ or } \int_0^{2\pi}\cos\psi\frac{e^{k\cos\psi}}{\cos\psi}d\psi$$

Integrating by parts we find

$$\Big[e^{k\cos\psi}\tan\psi\Big]_0^{2\pi}+\int_0^{2\pi}\frac{k\sin^2\psi}{\cos\psi}e^{k\cos\psi}d\psi-\int_0^{2\pi}\frac{\sin^2\psi}{\cos^2\psi}e^{k\cos\psi}d\psi$$

The first term becomes infinite when $\psi=\frac{\pi}{2}$ and when $\psi=\frac{3\pi}{2}$; we may replace it by $\int_0^{2\pi}\sec^2\psi d\psi$. We thus find

$$\int_0^{2\pi}\left(\frac{e^{k\cos\psi}}{k\cos\psi}-\frac{e^{k\cos\psi}}{k^2\cos^2\psi}+\frac{1}{k^2\cos^2\psi}\right)d\psi=\frac{1}{k}\int_0^{2\pi}\cos\psi e^{k\cos\psi}d\psi=\int_0^{2\pi}\sin^2\psi e^{k\cos\psi}d\psi$$

Combining these results, we find for the potential at (f, g, h) due to a circular plate of unit density the expression

$$a^2\int_0^{2\pi}\sin^2\psi e^{ja\cos\psi\frac{d}{dh}}\frac{1}{r}d\psi \quad \ldots\ldots\ldots\ldots \quad \text{(b)}$$

If we move the origin to a point at distance v on the negative side of the axis of z, we find

$$2a^2\int_0^{\pi}e^{-(ja\cos\psi+v)\frac{d}{dh}}\frac{1}{r}d\psi$$

in which form the expansion of $-\frac{dV}{dh}$ readily gives CLERK MAXWELL'S expansion for the potential of unit current in the circle ('Electricity and Magnetism,' vol. ii., p. 301 (6′)).

The result expressed by (b) may be written

$$V=a^2\int_0^{2\pi}\frac{\sin^2\psi}{\sqrt{f^2+g^2+(h+aj\cos\psi)^2}}d\psi$$

Hence the potential due to unit current in the circle is

$$-a^2\int_0^{2\pi}\sin^2\psi\frac{d}{dh}\frac{1}{\sqrt{f^2+g^2+(h+aj\cos\psi)^2}}d\psi$$

$$=aj\int_0^{2\pi}\sin\psi\frac{d}{d\psi}\frac{1}{\sqrt{f^2+g^2+(h+aj\cos\psi)^2}}d\psi$$

$$=\int_0^{2\pi}\frac{aj\cos\psi}{\sqrt{f^2+g^2+(h+aj\cos\psi)^2}}d\psi$$

$$=2\int_0^{\pi}\frac{aj\cos\psi}{\sqrt{f^2+g^2+(h+aj\cos\psi)^2}}d\psi \quad \ldots\ldots\ldots\ldots \quad \text{(c)}$$

This result shows that the potential of a circular current is the same as that of an imaginary bar in the axis of z joining the points whose distances from the origin are aj and $-aj$, the density at a point ψ being $-2\cot\psi$. Now the integral (c) is true at all points whose distances from the origin are greater than a. We can, however, determine the corresponding result for points within the radius a by the ordinary theory of inversion. We have seen that for points outside the radius a the potential is the same as that of a bar joining two imaginary points. If z be the distance of any point of this bar from the origin and z' the corresponding inverted distance, we have

$$z'=\frac{a^2}{z}=-aj\sec\psi$$

$$\therefore\ dz'=-aj\sec\psi\tan\psi d\psi$$

and the corresponding density is $-2\cot\psi\dfrac{a}{aj\cos\psi}$

Hence the potential at all points inside the radius a is

$$2\int_0^\pi\frac{a\sec^2\psi}{\sqrt{f^2+g^2+(h-aj\sec\psi)^2}}d\psi\quad.\ .\ .\ .\ .\ .\ .\ .\ .\ .\quad\text{(d)}$$

This result may also be proved from the consideration that the expression for the magnetic force is continuous as we pass from one form of potential to the other.

§ 27. The results (c) and (d) give the mutual potential energy between two circular currents placed in any positions whatever, in which the smaller circle is either wholly outside or wholly inside the radius of the other. For the sake of simplicity we shall suppose that the axes of the two currents intersect. Let the radius of the larger be A and of the smaller a, and let u, v be the distances of their centres from the point of intersection, the axes being inclined at an angle $\cos^{-1}\mu$. Then, if unit current circulates in each, and if the smaller circle lies wholly inside the sphere of radius A containing the larger, the mutual potential energy is given by

$$\mathrm{M}=4a\mathrm{A}\int_0^\pi\int_0^\pi\frac{j\cos\phi\sec{}^2\psi}{\sqrt{(u+\mathrm{A}j\sec\psi)+(v+aj\cos\phi)^2-2(u+\mathrm{A}j\sec\psi)(v+aj\cos\phi)\mu}}d\phi d\psi$$

If this series be expanded in harmonics

$$\mathrm{B}_0\mathrm{P}_0+\mathrm{B}_1\mathrm{P}_1+\ .\ .\ .\ +\mathrm{B}_n\mathrm{P}_n+\ .\ .\ .$$

we have

$$\mathrm{B}_n=4a\mathrm{A}\int_0^\pi\int_0^\pi\frac{j\cos\phi\sec^2\psi(v+aj\cos\phi)^n}{(u+\mathrm{A}j\sec\psi)^{n+1}}d\phi d\psi$$

The series in the form of zonal harmonics is given by CLERK MAXWELL (‘Electricity and Magnetism,’ vol. ii., p. 303).

On comparison with his series, since

$$\int_0^\pi j\cos\phi(\mu+j\nu\cos\phi)^n d\phi=-\frac{\pi}{n+1}\nu\frac{d\mathrm{P}_n}{d\mu}$$

when we take account of the above value of B_n we deduce

$$\int_0^\pi\frac{\sec^2\psi d\psi}{(\mu+\nu j\sec\psi)^{n+1}}=-\frac{\pi}{n}\nu\frac{d\mathrm{P}_n}{d\mu}$$

true for integral values of n, from 1 upwards.

If the smaller circle is wholly outside the sphere of radius A, the value of M given above is inapplicable, and we ought to take a double application of the formula (c).

§ 28. In the important case of two coaxal circular currents, if b be the distance between them, the mutual potential energy is given by

$$\mathrm{M}=4a\mathrm{A}\int_0^\pi\int_0^\pi\frac{j\cos\phi\sec^2\psi}{b+j(\mathrm{A}\sec\psi-a\cos\phi)}d\phi d\psi\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (e)$$

provided the sphere of radius A encloses the smaller circle.

Performing the integration in regard to ψ we find

$$\mathrm{M}=4\pi a\int_0^\pi\frac{\cos\phi(a\cos\phi+bj)}{\sqrt{\mathrm{A}^2-(a\cos\phi+bj)^2}}d\phi$$

With the view of simplifying this let us put $a\cos\phi+bj=\mathrm{A}\cos(x+yj)$, so that $a\cos\phi=\mathrm{A}\cos x\cos yj$ and $-bj=\mathrm{A}\sin x\sin yj$. Thus x and y are not independent, and by the substitution proposed it can readily be shown that M takes the form

$$8\pi\mathrm{A}^2\int\frac{\cos^2 x\sin x}{\sqrt{\mathrm{A}^2\cos^4 x-(a^2+\mathrm{A}^2+b^2)\cos^2 x+a^2}}dx$$

the limits of integration being determined as follows:

When $\phi=\frac{\pi}{2}$, $x=\frac{\pi}{2}$, and when $\phi=0$, $\mathrm{A}^2\cos^4 x-(a^2+\mathrm{A}^2+b^2)\cos^2 x+a^2=0$.

The roots of the last equation are

$$p^2=\cos^2\theta_1=\frac{\mathrm{A}^2+a^2+b^2+\sqrt{(\mathrm{A}^2+a^2+b^2)^2-4a^2\mathrm{A}^2}}{2\mathrm{A}^2}$$

$$q^2=\cos^2\theta_2=\frac{\mathrm{A}^2+a^2+b^2-\sqrt{(\mathrm{A}^2+a^2+b^2)-4a^2\mathrm{A}^2}}{2\mathrm{A}^2}$$

Since p is greater than unity we must take q as the value of $\cos\theta$ when $\phi=0$. Hence the value of M is

$$8\pi\mathrm{A}\int_{\cos^{-1}q}^{\frac{\pi}{2}}\frac{\cos^2 x\sin x}{\sqrt{(p^2-\cos^2 x)(q^2-\cos^2 x)}}dx$$

$$=8\pi\mathrm{A}q^2\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta}{\sqrt{p^2-q^2\cos^2\theta}}d\theta$$

$$=8\pi\mathrm{A}p\left\{\mathrm{F}\left(\frac{q}{p}\right)-\mathrm{E}\left(\frac{q}{p}\right)\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (f)$$

where F and E are complete elliptic integrals whose modulus is $\frac{q}{p}$.

If b be small and if $A-a=c$ be also small, then approximately

$$p=1+\sqrt{\frac{b^2+c^2}{A^2}} \text{ and } q=1-\sqrt{\frac{b^2+c^2}{A^2}}$$

and M becomes

$$4\pi A\left\{\log 8\sqrt{\frac{A^2}{b^2+c^2}}-2\right\}$$

This result is given by CLERK MAXWELL.

§ 29. If the circle of radius a be outside the sphere of radius A passing through the larger circuit, the value of M is

$$4aA\int_0^\pi\int_0^\pi\frac{\cos\phi\cos\psi}{b+j(A\cos\phi+a\cos\psi)}d\phi d\psi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (g)$$

The forms (e) and (g) obtained for M are interesting from their simplicity, and would seem to be useful in calculations connected with induction coils.

XII. *Measurements of Electrical Constants.*—No. II.
*On the Specific Inductive Capacities of Certain Dielectrics.**—Part I.

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Communicated by Professor J. Clerk Maxwell, *F.R.S.*

Received October 21,—Read December 12, 1878.

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Introductory.

In the autumn of 1876 it was suggested in the course of a conversation with Professor Clerk Maxwell that I should, under his direction, undertake a series of measurements of specific inductive capacities—particularly those of transparent dielectrics—with a view of testing Professor Maxwell's electro-magnetic theory of light, which requires that the square of the refractive index should be equal to the dielectric capacity, multiplied by the magnetic permeability.

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* A paper of mine with the above title was communicated to the Royal Society by Professor J. Clerk Maxwell, F.R.S., on March 9, 1878. It was read on March 28, and an abstract of it was published in the 'Proceedings of the Royal Society,' vol. xxvii., p. 270.

In the course of the summer it was pointed out to me that, owing to a mistake in the formula of calculation, all the results were wrong.

I therefore requested permission to withdraw my paper, in order to recalculate the results.

The new values of K arrived at led me to make some determinations of refractive indices, and to re-write the theoretical deductions at the end of the paper.

I now beg, through Professor Maxwell, to present the paper in an amended form, with the hope that it may be found not entirely unworthy the attention of the Royal Society.

By March, 1877, the method of work had been sketched out, and most of the instruments ordered.

The greater portion of the expense of the investigation has been met by grants from the Government Fund of £4,000.

The general outline of the method, which is, with the exception of the first idea of the 5-plate balance,* entirely due to Professor MAXWELL, is as follows :—

Metal plates, some connected with the electrometer and some with the source of electrification, are so placed that, when there is only air between them, the potentials of the two pairs of quadrants are equal and the needle remains at zero, however the strength of the electrification may vary.

On placing one of the dielectrics under examination between two of the plates a stronger action takes place between these two plates, and the needle is deflected. The plates are now moved farther apart by a screw so as to reduce the action until the needle returns to zero.

From the distance which the plates have to be separated, and the thickness of the dielectric, the specific inductive capacity can be calculated.

The great difficulty in all experiments in specific inductive capacity is to make the experiments quickly enough. If the electric stress is continued in the same direction for any appreciable time, a permanent strain is produced, and the apparent specific inductive capacity differs from the true one.

To overcome this difficulty Professor MAXWELL arranged that the electrification should be constantly reversed. It was found possible to reverse it 12,000 times in a second, and so to eliminate all phenomena of "residual charge."

How, in spite of the constant reversal, it was arranged that the deflection, when the balance was not established, should be steady, and depend only on the position of the plates, and not on the direction of the electrification, will be seen later on in the paper.

Description of instruments employed.

For producing a rapidly reversed electrification of high potential, a coil, wheel-break, secondary reversing engine, and batteries were used.

The coil was one of APPS' induction coils, on which is wound 22 miles of secondary wire, and which, with a suitable battery and break, is capable of giving a 17-inch spark in air.

The wheel-break consists of a small magnetic engine the fly-wheel of which is of brass, and about 2½ inches in diameter. In the circumference of the wheel 60 slots are cut and filled with ebonite. A light spring faced with platinum presses on the edge of the wheel with a force regulated by a capstan screw.

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* Due, I believe, to Sir WILLIAM THOMSON.

A current passing from the spring to the wheel would thus be made 60 times, and broken 60 times, in each revolution of the engine.

The engine, which is by APPS, is of particularly good construction for high speed work. Two horse-shoe electro-magnets are placed with their poles facing each other. One is fixed and the other revolves round an axis which is parallel to the cores and half way between them. The current in the fixed magnet has a constant direction, that in the revolving magnet is reversed every half revolution. The commutator is so arranged that the force between poles which are approaching each other is always attractive, and that between receding poles repulsive. Two screws regulate the pressure of the commutator springs.

With a little care as to the adjustment, it was found possible to drive this engine at a speed of 100 revolutions a second.

The secondary reversing engine consists of a square ebonite plate, near the corners of which are four holes filled with mercury. Upon it is supported an oscillating frame, which by dipping into the right and left hand pair of cups alternately, reverses at each oscillation any current passing through the instrument. The oscillations of the frame were produced by a rod going from it to a crank in the axis of a magnetic engine. The engine was similar to that of the wheel-break, except that as it was not driven at so high a speed, the screw adjustments of the springs were dispensed with.

To regulate the speed, a friction-brake was used, consisting of a loop of silk which passed round a pulley on the axis, and was connected by an india-rubber band to a cord. This cord could be tightened by turning a handle round which it was wound.

About 30 reversals per second was a sufficient speed for the purpose for which the engine was employed, and about the maximum which could be used without splashing the mercury from the cups.

The batteries.—Ten small LECLANCHÉ cells, in series, worked the coil, four pint GROVE's the wheel-break, and two pint GROVE's the reversing engine.

When the wheel-break was driven at full speed, the coil gave a spark about $\frac{1}{25}$ inch in length, the current being reversed 120 times in each revolution, or 12,000 times per second.

Comparison of make and break currents.

To test the equality of the make and break currents, the discharge was passed through a small vacuum tube. When the wheel-break was turned slowly the usual difference in the illuminations of the ends of the tube was observed, which difference was reversed on reversing the primary current of the coil. On working the break at high speed this difference disappeared, and no effect was produced by reversing the primary.

This shows that at high speed the sums of the make and break currents in opposite directions were approximately equal.

The secondary reversing engine was, however, used to guard against any difference in the above sums which might be produced by difference of leakage, &c., by reversing the secondary current about 30 times per second.

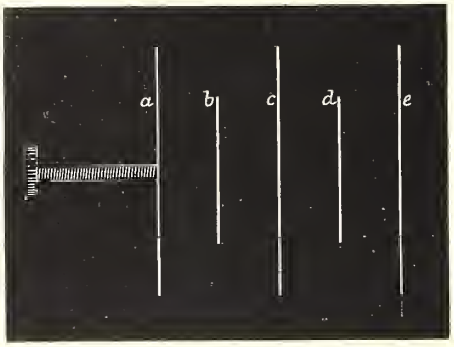
Description of instruments (*continued*).

The 5-plate induction balance.—This instrument was invented some years ago by Sir WILLIAM THOMSON.

Only a rough lecture-model was constructed at the time of the invention. The present instrument is the first which has been made for purposes of measurement. The working drawings were made by me from sketches and verbal explanations supplied by Professor MAXWELL.

The instrument was constructed with great skill by Mr. KIESER, of the firm of ELLIOTT Brothers.

Fig. 1.



It consists of three circular metal plates, *a c e*, each 6 inches diameter, and two, *b d*, each 4 inches diameter, arranged as shown in fig. 1. There is a space of rather more than an inch between each plate and the one next it. The plate *a* is movable parallel to itself, so that it can be placed either in contact with *b* or nearly three inches from it. The other four plates are fixed. The plates *b c d e* are supported from above by steel rods. The lower end of each rod is screwed into the upper edge of a plate, the upper into an ebonite plug fixed into a small triangular horizontal brass plate, at the corners of which are levelling screws. The screws rest on a flat brass stage, a slit in which allows the rod to pass through. As there is not room for all four triangles close together there are two stages one above the other.

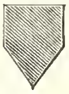
The plates *c e* hang from the lower stage, and *b d*, which are furnished with longer rods, from the upper one. The triangles when adjusted are clamped by screws which for *c e* are fixed in the second stage, while for *b d* the clamps are carried on a third stage made especially to hold them. Four stout brass pillars support the three stages. The feet of these pillars are screwed into a large brass plate let into the wooden base of the instrument. On this plate also stand the pillars carrying plate *a*. Plate *a* is fixed to the end of a brass rod of section  from which it is insulated by a block of ebonite (E, fig. 3). This slides on two pillars (A A, figs. 2 and 3), which it touches only on its inclined surfaces (fig. 2). It is pressed downward by stout springs (*α α*, figs. 2 and 3). It is moved backwards and forwards by a screw B, pressed by a spring against a hardened steel plate at C (fig. 3). The screw inside D turns in two collars, one fixed to D and the other only kept from revolving, and forced away from the first by a stout spiral spring. This prevents what is called "back lash," *i.e.*, it ensures that the longitudinal motion shall be reversed at the same time as the motion of the screw.

Fig. 2.

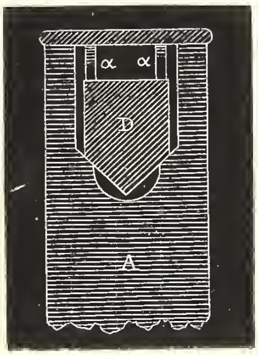
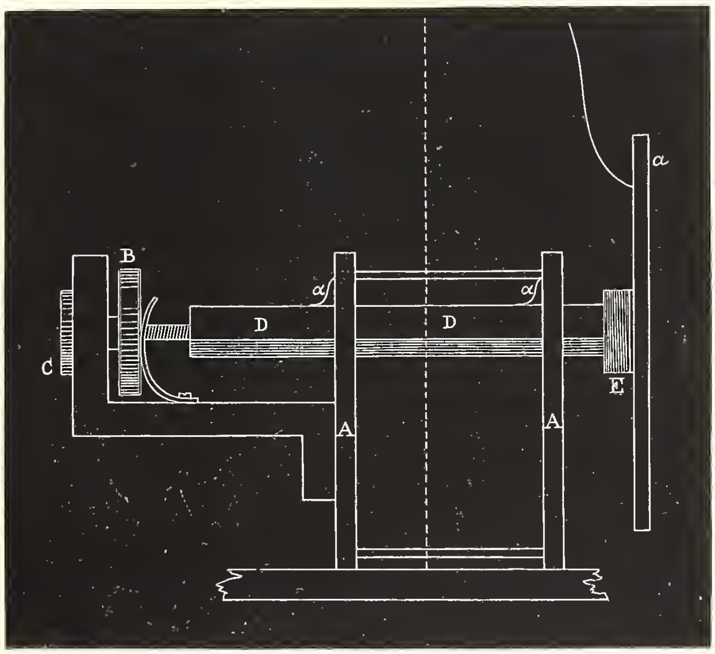


Fig. 3.



A scale divided to $\frac{1}{50}$ inch is engraved upon D, and a vernier fixed to one of the uprights reads the position of the plate to $\frac{1}{1000}$ inch. The scale is read by a microscope, fixed some three inches distant on the case of the instrument.

The five plates are enclosed in a glass case, like a balance case, about 15 inches long. It passes below the stages, and holes in the top glass admit the four steel rods. A hole in the side admits the rod D, so that *a* is inside the case, while the screw-head is outside. The dotted line in fig. 3 shows the position of the glass side of the case.

The five plates are placed close to one end of the case (see fig. 5, *Induction Balance*). On the base, inside the case, are slides, in which stages for carrying the dielectrics move. The stages can be moved by rods projecting beyond the case. Thus a dielectric under examination can be placed between two of the plates, or removed from them, without opening the glass case.

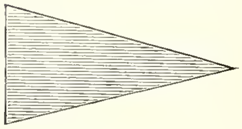
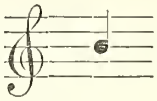
On the upper fixed stages are ebonite pillars with double-binding screws on them; to the lower nut of each is attached a flexible spiral wire, leading to one of the plates by way of the steel rod. By means of the upper nuts the plates can be connected to other instruments.

The mechanical slide.—One of the sliding stages—viz.: that used to place a dielectric between *a* and *b*[*]—has "mechanical motions." In addition to the rod by which it is drawn in and out, there are three other parallel rods, with milled heads. Turning, one gives a lateral motion—viz.: moves any dielectric placed on the slide nearer *a* or *b*. By turning the second the dielectric plate can be placed either exactly vertical or inclined a little in either direction; and turning the third gives the dielectric a small angular motion round a vertical axis.

The THOMSON *electrometer* is a quadrant of the simple form by ELLIOTT; one of WHITE'S which I have was found to be unsuitable for use with the reversing gear. When, as in this case, the instrument is only used as an electroscope, the superior sensitiveness of ELLIOTT'S pattern gives it great advantages.

The callipers were a pair with especially long jaws, made for use in adjusting the plates of the balance. When laid upon a bracket fixed to the outside of the balance case, the jaws projected right in between the plates. This bracket could be inclined either up or down, for measuring at the upper or lower parts of the plates. The "outsides" scale was used in the ordinary way for measuring the thickness of the dielectric plates.

Determination of speed of wheel-break.

The fly-wheel of the engine had five spokes. A piece of card of this shape  being held with its pointed end against the spokes, was struck five times in each revolution, and gave the note B=  which corresponds to 495 vibrations per second.[†] Dividing this by 5, we find that the wheel revolves 99 times per second—say 100, as the card would check the speed slightly. This determination was made with a battery which had been charged 24 hours, and had done a good deal of work.

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[*] In practice, the dielectrics are only put between *a* and *b*.

[†] 'Sound and Music,' SEDLEY TAYLOR, p. 200.

Estimation of electromotive force of secondary current.

The spark given by the coil with the wheel-break and LECLANCHÉ battery is about ·04 inch. Let N be the number of cells this corresponds to. Now Messrs. DE LA RUE and MÜLLER have[*] enunciated the following formula for the spark produced by any number of their cells :—

$$\text{Length of spark} = \frac{\cdot 0033 N^2}{600^2}$$

$$\text{Here length} = \cdot 04.$$

Hence we have

$$N^2 = \cdot 04 \frac{600^2}{\cdot 0033} = 4{,}300{,}000$$

$$N = \text{about } 2050.$$

The electromotive force is then equal to that of about 2050 cells of a chloride of silver battery.

This determination was made with the coil disconnected from the induction balance. When it is connected to it the spark obtainable is much shorter, owing, possibly, to the leakage, or more probably to the capacity of the induction balance and electrometer, a certain proportion of the electricity being required to charge these at every make and break. I had no means of accurately measuring the lengths of very short sparks.

Connections and general method of working.

The connections are shown in figs. 4 and 5. The wheel-break and LECLANCHÉ battery were connected to the coil primary in the usual way. The secondary current after passing the secondary reversing engine came to the induction balance.

One of the wires carrying the secondary current was connected to the centre plate *c* (fig. 4), the other to the two outer ones, *a* and *e* ; the smaller plates *b* and *d* were connected to the quadrants of the electrometer respectively. Now, for a moment, let us suppose the electrification to be produced by a battery acting constantly in the same direction, and the needle of the electrometer to have a permanent charge ; then when the plates are all placed symetrically and with only air between them, however strong the battery may be, the spot of light will remain at zero. This may be seen by noting in fig. 4 the actions of the large plates on the small ones respectively. If, however, *a* is moved away so as to make the distance *a b* greater than *d e*, the needle will be deflected in one direction. If instead of moving *a* away we had placed a dielectric plate in the space *a b*, there would have been a greater action from *a* to *b*, and the deflection would have been in the opposite direction. It is clear that if we insert a dielectric, and also move a away from b, we can, by properly adjusting the motion, keep the electrometer at zero.

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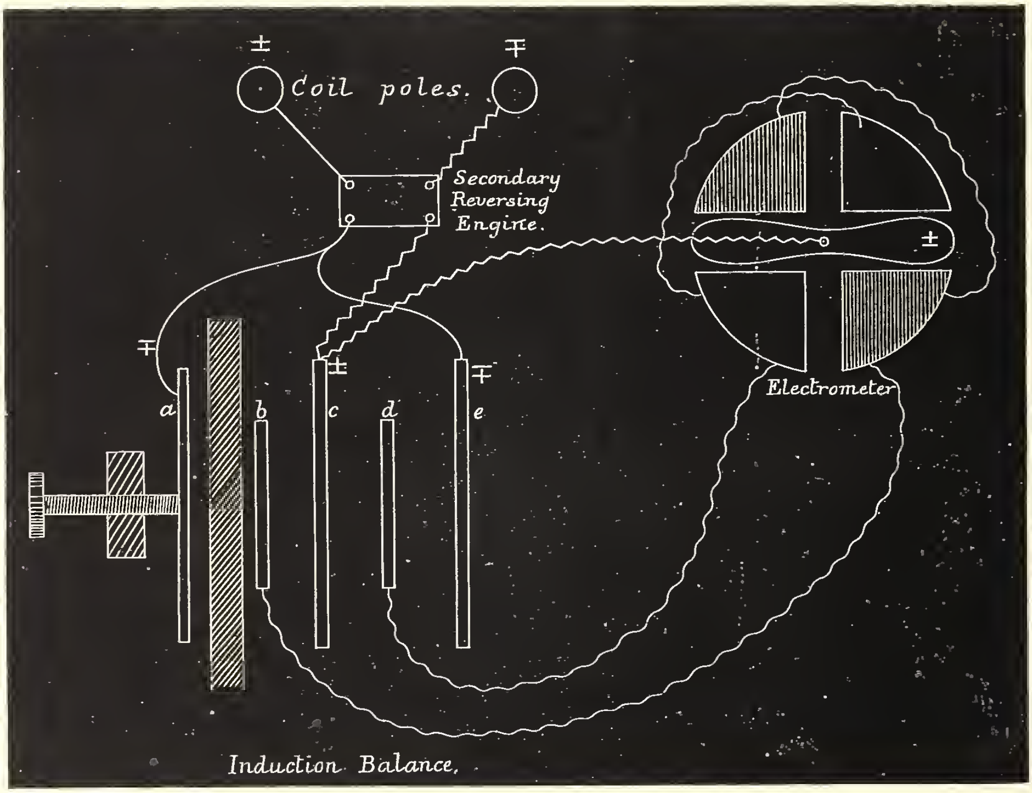
[*] 'Proc. Roy. Soc.,' 1875-76, p. 170.

Fig. 4.

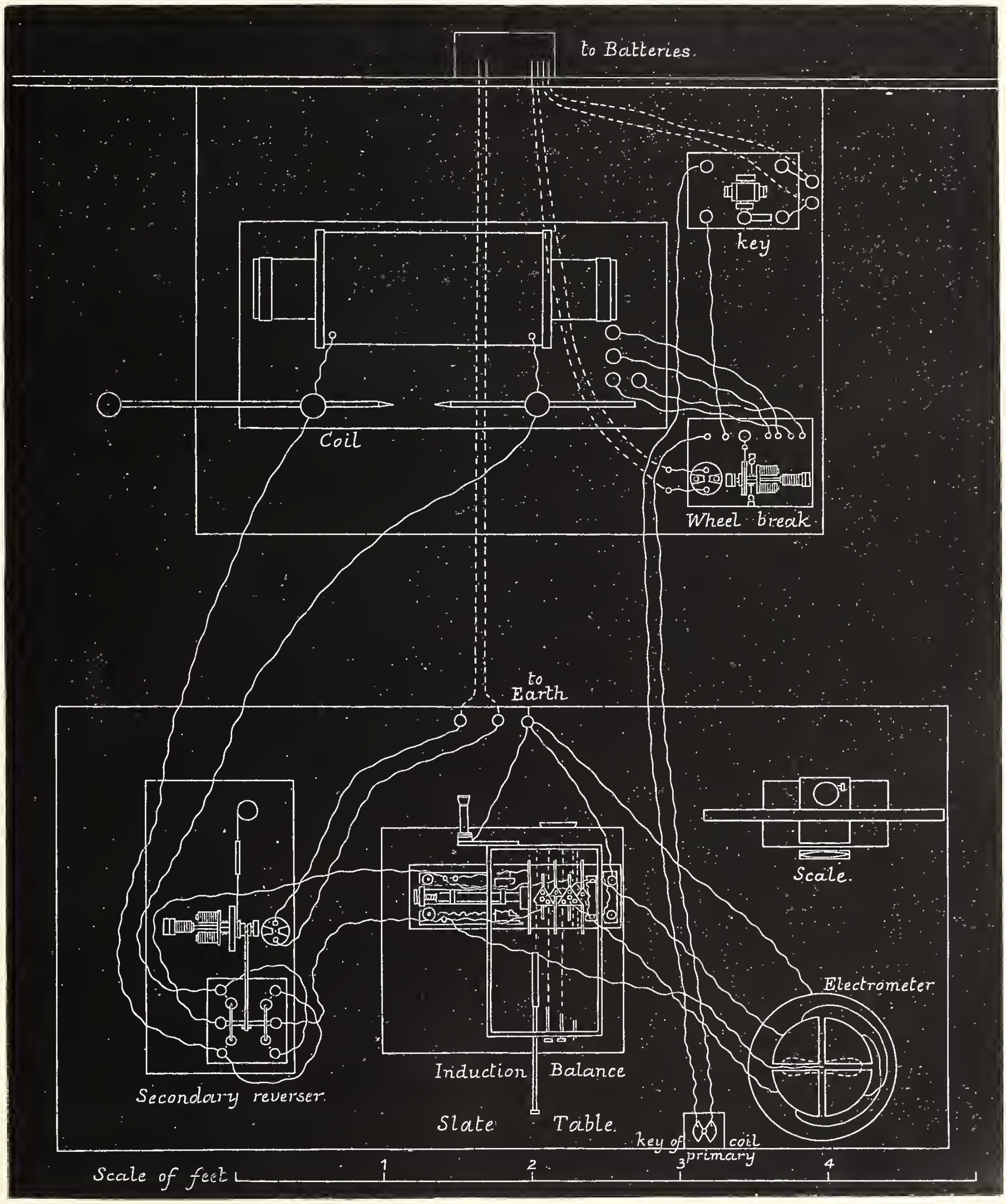


The amount of motion of a required in each case, will depend on the thickness of the dielectric, *and on its specific inductive capacity.*

Let us suppose the balance not perfectly established, and a deflection, say, to the right—now reverse the poles (we are still supposed to be working with a battery); the deflection will be to the left, and with the electrifications which are actually used, and which are reversed 12,000 times in each second, no disturbance of equilibrium would produce any deflection if the needle were charged in the ordinary way.

To escape from this difficulty Professor MAXWELL devised the following arrangement:—The needle, instead of being permanently charged, was connected with the plate c, and every time the electrification of the plates was reversed, that of the needle was reversed also. As the sign of the force between two electrified bodies depends only on whether the electrifications are like or unlike, and not on whether they are positive or negative, this arrangement caused the force in the needle to depend only on the numerical difference between the potentials of the quadrants, and not on the sign of the electrification.

Fig. 5.


to Batteries.
key
Coil
Wheel break
to Earth
Scale.
Electrometer
Secondary reverser.
Induction Balance
Slate Table.
key of primary coil
Scale of feet
1
2
3
4

Thus the electrifications, of the needle, the five plates, the dielectric, and the four quadrants were all reversed 12,000 times in each second, and yet the light spot was found to be absolutely steady, and completely under the control of the screw of *a*.

A motion of *a* of ·001 inch usually caused a deflection of from 1 to 2 millims. on the scale. I so arranged the connections that, to bring the light spot to zero, *a* had to be moved so as to follow the deflection. This was a very convenient way of remembering in which direction to turn the screw.

The actual determinations were made as follows:—The dielectric was removed and the slide on which it stood being pushed in under the plates, the position of *a* was determined which brought the light spot to zero with only air in the balance. The slide was then drawn out, the dielectric placed upon it, and it was than pushed in so as to place the dielectric between *a* and *b*. The dielectric plates were all 7 inches high and 7 inches or 8 inches long, and varied from $\frac{1}{4}$ inch to 1 inch in thickness. A second reading was taken when the dielectric was in position.

Before each set of measurements, preliminary experiments were made by means of which *a* was placed very nearly in its right position in each case before the primary circuit was closed. This prevented large deflections of the needle, which were found injurious, as after a very large deflection the light spot seldom returned exactly to the same zero.

Before commencing a measurement the engines were started: the observer standing where he could see the scale, was able with his right hand to make and break the coil primary circuit, and with his left to work the screw of *a* (see fig. 5). As soon as equilibrium was established, contact was broken, the engines stopped, and the reading of *a* taken with the microscope.

The formula used for calculating the results assumes that the lines of force through the dielectric are straight and perpendicular to its sides. Now the lines of force are very nearly straight over most of the space *a b*, but not quite straight close to either *a* or *b*. When the lines of force through the dielectric are perpendicular to its sides they will be at their shortest in it, and it will produce its minimum effect. By means of the mechanical slide the dielectric was in each case adjusted to give its minimum effect. The field of force was, however, found to be very nearly uniform except close to the plates. A cardboard box covered with tinfoil, and connected to earth, protected the second stage which carried the electrometer connections.

All wires leading from the quadrants to the balance were protected by wide cardboard* tubes covered with tinfoil connected to earth. The metal portions of the cases of balance and electrometer were also connected to earth.

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* May, 1879.—Brass tubes and a brass box over the upper stage have since been substituted with advantage.

The formula of calculation.

Let the reading of a when there is only air in the balance be a_1, and that when the dielectric is inserted a_2.

A dielectric plate of thickness b being inserted displaces a thickness b of air. A dielectric plate of thickness b and specific inductive capacity K acts like a plate of air of thickness $\frac{b}{\mathrm{K}}$.

Therefore the effect of inserting the plate has been the same as if we had decreased the distance $a\,b$ by a quantity $b-\frac{b}{\mathrm{K}}$.

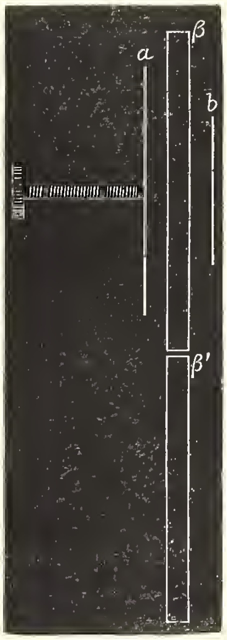
But when we have again brought the electrometer to zero by the screw we have increased $a\,b$ by a quantity a_2-a_1, and as this increase exactly compensates the apparent decrease produced by the dielectric, we must have

$$b-\frac{b}{\mathrm{K}}=a_2-a_2, \qquad \text{or} \qquad \mathrm{K}=\frac{b}{b-(a_2-a_1)}$$

and this formula was used to calculate the results of the following experiments.

The experiments on ebonite show that the results given by the formula agree with each other over a very wide range of values of b.

Fig. 6.



This formula assumes that those lines of force between a and b which pass through the dielectric are approximately straight lines, also that all the lines pass through it, and that no portion, or at least no sensible portion, of them pass round its edge. To test the truth of this assumption the following experiment was made:—A plate of ebonite, β, fig. 6, being placed in the usual way between a and b, a was adjusted so as to bring the light spot to zero. The circuit being kept closed, a second similar piece of ebonite was placed at β'. Now, if any of the lines of force had curved round the edge of β, the introduction of β' would have caused a deflection in the direction showing an increase of specific inductive capacity in $a\ b$. No such deflection however took place, which shows that no lines of force strong enough to affect the electrometer pass round the edge of a 7-inch dielectric.

The experiments.

After about five months spent in preliminary experiments and in perfecting the adjustments of the various instruments, the following measurements were made :—

Optical glass.—Four beautiful slabs, each seven inches square, of perfectly homogeneous glass, were cast in Birmingham by Messrs. CHANCE, and polished in London under the directions of Mr. KIESER.

The glasses were—

Double extra-dense flint.
Extra-dense flint.
Light flint.
Hard crown.

Small prisms of each glass were also prepared in order that the refractive indices might be compared with the specific inductive capacities.*

The following are the details of the experiments and values of K deduced from them :—†

Optical glass.

DOUBLE extra-dense flint.

$b = {\cdot}847$.

a_2	(1) 1·613	(3) 1·615	(5) 1·610
a_1	(2) 1·034	(4) 1·034	1·032
$a_2 - a_1$	·579	·581	·578

Mean ·5793.

K=3·164.

---

[*] See p. 440.

[†] a_1, a_2, and b are all given in inches.

Extra-dense flint.

b=·717.

a_2	(2) 1·514	(4) 1·515	(6) 1·511
a_1	(1) 1·032	(3) 1·031	(5) 1·030
a_2-a_1	·482	·484	·481

Mean ·48223.

K=3·054.

Light flint.

b=·699.

a_2	(1) 1·503	(3) 1·498	(5) 1·495	(7) 1·500
a_1	(2) 1·035	(4) 1·034	(6) 1·032	(8) 1·032
a_2-a_1	·468	·464	·467	·468

Mean ·46675.

K=3·013.

Hard crown.

b=·864.

a_2	(2) 1·621	(4) 1·617	(6) 1·616	(8) 1·619
a_1	(1) 1·032	(3) 1·032	(5) 1·031	(7) 1·034
a_2-a_1	·589	·585	·585	·585

Mean ·586.

K=3·108.

Ordinary plate glass.

b=·996.

a_2	1·733	1·734	1·731
a_1	1·043	1·042	1·042
a_2-a_1	·690	·692	·689

Mean ·6903.

K=3·258.

ANOTHER piece of the same.

b=1·003.

a_2	1·739	1·737	1·730
a_1	1·044	1·043	1·042
a_2-a_1	·695	·694	·688

Mean ·6923.
K=3·228.

Ebonite.

Four slabs of different thickness.

No. 1.

b=·754 (mean).

a_2	(1) 1·460	(3) 1·453	(5) 1·453	(7) 1·452
a_1	(2) 1·031	(4) 1·032	(6) 1·036	(8) 1·032
a_2-a_1	·429	·421	·417	·420

Mean ·42175.
K=2·2697.

No. 2.

b=·509 (mean).

a_2	(2) 1·313	(4) 1·318	(6) 1·315
a_1	(1) 1·032	(3) 1·033	(5) 1·033
a_2-a_1	·281	·285	·282

Mean ·2826.
K=2·2482.

No. 3.*

b=·516.

a_2	(1) 1·325	(3) 1·323	(5) 1·321
a_1	(2) 1·030	(4) 1·030	1·031
a_2-a_1	·295	·293	·290

Mean ·2926.
K=2·3097.

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* Slabs Nos. 3 and 4 were much warmer than Nos. 1 and 2 when experimented on. But on the experiment with No. 4 being repeated after it was cool no difference was perceptible.

No. 4.

$b = \cdot264$ (mean).

a_2	1·184	1·181	1·180
a_1	1·031	1·032	1·033
$a_2 - a_1$	·153	·149	·147

Mean ·1496.

K=2·3077.

SUMMARY of experiments on ebonite.

Slab.	Thickness.	Spec. ind. capacity.
	inch.	
No. 1	·754	2·2697
,, 2	·509	2·2482
,, 3	·516	2·3097
,, 4	·264	2·3077
Mean		2·284

Extreme difference from the mean= −1·5 per cent.

Vulcanized india-rubber.

$b = \cdot789.$

a_2	(2) 1·506	(4) 1·523	(6) 1·505	1·505
a_1	(1) 1·035	(3) 1·035	(5) 1·041	1·035
$a_2 - a_1$	·471	·488	·464	·470

Mean ·47325.

K=2·497.

Plain black india-rubber.

$$b\left\{\begin{matrix}\cdot992\\1\cdot006\end{matrix}\right\}\text{mean }\cdot999$$

a_2	1·586	1·590	1·590
a_1	1·042	1·038	1·037
	·544	·552	·553

Mean ·5496

K=2·220.

Best quality gutta percha.

$$b\left\{\begin{array}{l}\cdot786\\ \cdot789\\ \cdot787\\ \cdot789\end{array}\right\}\text{mean }\cdot788$$

a_2	(2)	1·517	1·515	1·515
a_1	(1)	1·048	1·048	1·047
a_2-a_1		·469	·467	·468

Mean ·468.
$K=2\cdot462$.

CHATTERTON'S *compound.*

$$b\left\{\begin{array}{ll}\left.\begin{array}{l}\cdot842\\ \cdot834\end{array}\right\} & \cdot834^{*}\\ \left.\begin{array}{l}\cdot826\\ \cdot830\end{array}\right\} & \cdot832\end{array}\right\}\text{mean }\cdot833$$

a_2	1·555	1·555	1·555
a_1	1·048	1·049	1·048
a_2-a_1	·507	·506	·507

Mean ·5066.
$K=2\cdot547$.

Sulphur.

This plate was not a very good one, and was cracked across one corner.[†]

$$b\left\{\begin{array}{l}\cdot745\\ \cdot751\end{array}\right\}\text{mean }\cdot748.$$

a_2	1·496	1·492	1·507
a_1	1·035	1·037	1·038
a_2-a_1	·461	·455	·469

Mean ·4583.
$K=2\cdot58$.

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[*] The four measures being taken at the four sides in order, the agreement of these two means shows that the slope is uniform.

[†] It was prepared by casting in a mould with glass sides. It was semi-transparent for some hours after it had set, but became opaque before the experiments were made.

Shellac.

b measured from centre of each of the four sides.

$$\left.\begin{array}{r}·791\\ ·779\\ ·743\\ ·772\end{array}\right\}\text{ mean }·769$$

a_2	(1)	1·536	1·536	1·536
a_1	(2)	1·047	1·047	1·046
a_2-a_1		·489	·489	·490

Mean ·4893.
K=2·74.

Solid paraffin (Messrs. CLARK and MUIRHEAD'S).

Six slabs cut in planing machine from larger blocks.

No. 1.

b=·730.

a_2	(1)	1·382	1·378	1·379	1·379
a_1	(2)	1·030	1·029	1·027	1·028
a_2-a_1		·352	·349	·352	·351

Mean ·351.
K=1·9261.

No. 2.

b=·750.

a_2	(2)	1·392	1·392	1·393
a_1	(1)	1·037	1·034	1·035
a_2-a_1		·355	·358	·358

Mean ·357.
K=1·9084.

No. 3.

$b=\cdot748$.

a_2	(1)	1·397	1·398	1·397
a_1	(2)	1·035	1·035	1·037
a_2-a_1		·362	·363	·360

Mean ·3616.

K=1·9358.

No. 4.

$b=\cdot782$.

a_2	(2)	1·425	1·418	1·421
a_1	(1)	1·037	1·035	1·037
a_2-a_1		·388	·383	·384

Mean ·385.

K=1·9697.

No. 5.

$b=\cdot755$.

a_2	(1)	1·402	1·402	1·404
a_1	(2)	1·037	1·036	1·037
a_2-a_1		·365	·366	·367

Mean ·366.

K=1·9408.

No. 6.

$b=\cdot754$.

a_2	(2)	1·411	1·411	1·411
a_1	(1)	1·037	1·034	1·034
a_2-a_1		·374	·377	·377

Mean ·376.

K=1·9947.

Correction for cavities in paraffin.—All the plates contain cavities more or less numerous. The following approximate method of allowing for the effect of these cavities was employed:—

In the calculations we substitute for the plate of thickness b, an imaginary plate whose thickness is that which the plate would have had if it had been of the same length and breadth, contained the same quantity of paraffin, and had no cavities. Let us call this thickness b', and we shall have

$$\frac{b}{b'}=\frac{\text{density of solid paraffin}}{\text{density of plate}}$$

The above will not be a complete correction, owing to the unequal distribution of the cavities, but when the formula is applied to each of the six determinations, their mean should not be far from the truth.

To determine the density of the plates, two methods were used. One method was to carefully plane their edges, and then weigh and measure each plate, and so determine the weight of a cubic inch of each of them. The other was to weigh the plates in water with a lead sinker. The objection to the second method was, that some of the plates had cavities open to the edges into which the water could run. On trying both methods, however, the uncertainty introduced by this was found to be less than that caused by errors of measuring.

The specific gravity of a small flat piece of paraffin, free from cavities, was determined in the usual way by means of a fine balance of ŒRTLING's.

	grammes.
Weight of paraffin in air	2·164
Weight of paraffin and lead sinker in water at 11° C. . .	2·090
Weight of lead sinker in water	2·290
Thence weight of paraffin in water	— ·200
Weight of suspending hair	·005
Specific gravity of paraffin at 11° C.	=·9109

Dr. MUIRHEAD kindly made another determination for me, and found specific gravity =·912, but he did not state the temperature.

To determine the density of the paraffin plates a much larger balance was used, which was provided only with English weights. One end of the beam projected over the edge of the table, and the plates were slung horizontally in a double stirrup of fine iron wire, which was attached to the balance by a wire about three feet long. The weight of the wire and stirrup was compensated in air. The difference in its weight caused by immersing the lower portion of it in water was neglected. The plate hung in a large foot-pan full of water placed on the floor. The lead sinker was simply laid on the plate. When the balance was loaded with 18 ounces, it turned with about 2 grains. Weight of lead sinker in water, 1 lb. 4 oz. 4 dr. 1 scr. 2 gr.

WEIGHTS of plates in water equal weights of plates with sinker, minus weight of sinker in water.

Temperature of water 11° C.

Plate.	Weight in air in grains.	Weight in water in grains.	Specific gravity at 11° C.
No. 1	9600	−1330	·8783
,, 2	9282	−1330	·8771
,, 3	9904	−1160	·8951
,, 4	10,797	−1322	·8909
,, 5	10,027	−1201	·8933
,, 6	10,585	−1149	·9021

Let us write

$$\phi=\frac{\text{Sp. gr. of plate}}{\text{Sp. gr. of paraffin}}=\frac{\text{Sp. gr. of plate}}{\cdot 9109}$$

We shall then have

$$b'=b\phi.$$

Let us call K′ the corrected value of K, that is, the value of K calculated from b' instead of from b, and we shall have from the experiments, and the above determinations the following:—

SUMMARY of experiments on paraffin.

Plate.	ϕ.	b.	b'.	K.	K′.
No. 1	·9642	·730	·7038	1·9261	1·9940
,, 2	·9628	·750	·7221	1·9084	1·9784
,, 3	·9826	·748	·7249	1·9358	1·9969
,, 4	·9780	·782	·7648	1·9697	2·0126
,, 5	·9868	·755	·7450	1·9408	1·9654
,, 6	·9903	·754	·7467	1·9947	2·0143

$$\text{Mean value of}\begin{cases}K=1\cdot 9459\\ K'=1\cdot 9936^{*}\end{cases}$$

Among the corrected values the extreme difference from mean is −1·4 per cent.

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* N.B.—In future this number will be called K.

Among the uncorrected values the extreme difference from the mean is +2·5 per cent., which shows that though the correction is a rough one it gives numbers more nearly true than the uncorrected experiments.

The melting point of the paraffin was 68° C.

Bisulphide of carbon.—The liquid was contained in a glass trough. In this case a_1 was the reading when the empty trough was placed in $a\,b$; a_2 the reading when it was full. The trough was 7 inches wide and 9 inches high; b is its internal thickness.

We have from direct measurement $b=$ ·637. The external thickness was ·779. The thicknesses of the sides ·076 and ·066, respectively, and ·779−(·076+·066)=·637; showing that the inside and outside scales of the callipers agree very well.

The experiments had to be made somewhat hastily owing to leakage of the bisulphide.

$b=$ ·637

a_1	1·192		1·189		1·192
a_2		1·471		1·467	
a_1	1·176		1·173		

Mean $a_2=$ 1·469
Mean $a_1=$ 1·184

a_2-a_1 ·285

K=1·81.

The following table gives a general summary of the results of the experiments, and compares the values of K obtained with those given by various former experimenters:*—

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* Some of the numbers given in the third column are from 'Units and Physical Constants,' by Professor EVERETT, p. 134 (Macmillan, 1879). Where no reference is given, the number is from a list sent to me by Professor MAXWELL.

Name of dielectric.	K as determined by present experiments.	K as determined by experimenters mentioned in next column.	Previous experimenters.	References.
Glass—				
Double extra-dense flint	3·164*	10·1	Hopkinson	'Proc. Roy. Soc.,' 1877, p. 298, vol. xxvii.
Extra-dense flint	3·054.*			
Dense flint		7·4	Hopkinson	,, ,,
Light flint	3·013*	6·85	Hopkinson	,, ,,
Hard crown	3·108.*			
Common plate, two specimens—				
No. 1	3·258 } mean 3·243	6·10	Wüllner	'Sitzb. Königl. bayer. Akad.,' March, 1877.
,, 2	3·228			
Halbweisses glass		2·96 3·66 } mean 3·31	Schiller.	
White murren		5·83	Schiller	Pogg. Ann., vol. clii., p. 535, 1874.
Common flint		Greater than 1·76	Faraday	'Exp. Res.'
Ebonite, four slabs—				
No. 1	2·2697 } mean 2·284	2·76	Siemens.	
,, 2	2·2482	3·15	Boltzmann	'Weiner Sitzb.,' Jan. 9, 1873.
,, 3	2·3097	2·21	Schiller.	
,, 4	2·3077	2·56	Wüllner	'Sitzb. Königl. bayer. Akad.,' March, 1877.
Best quality gutta percha	2·462.			
Chatterton's compound	2·547.			
India-rubber—				
Black	2·220	2·12 to 2·34	Schiller	Pogg. Ann., vol. clii., p. 535, 1874.
Grey vulcanized	2·497	2·69 to 2·94	Schiller	
Solid paraffin, specific gravity at 11° C. = ·9109, melting point 68° C. Six slabs cut in planing machine; results corrected for cavities—		2·32	Boltzmann	Carl's 'Repertorium,' vol. x., pp. 92, 165.
		1·96	Wüllner	'Sitzb. Königl. bayer. Akad.,' March, 1877.
		1·977	Gibson and Barclay	Phil. Trans., 1871, p. 573.
No. 1	1·9940 } 1·9936	Specific gravity at 11° C., calculated from data in Messrs. Gibson and Barclay's paper, = ·9080. Multiplying Messrs. Gibson and Barclay's value of K by ratio of specific gravities, we have 1·9833 for my paraffin by Gibson and Barclay's method.		
,, 2	1·9784			
,, 3	1·9969			
,, 4	2·0126			
,, 5	1·9654			
,, 6	2·0143			
Paraffin, clear, quickly cooled		1·68	Schiller.	
Slow cooled, white		1·81 1·89 } 1·85	Schiller.	
Shellac	2·74	2·00	Faraday	'Exp. Res.,' 1270.
		2·95 to 3·73	Wüllner	'Sitzb. Königl. bayer. Akad.,' March, 1877.
Sulphur	2·58	2·88 to 3·21	Wüllner	,, ,,
		2·24	Faraday	'Exp. Res.,' 1266.
		3·84	Boltzmann	Carl's 'Repertorium,' vol. x., pp. 92, 165.
Bisulphide of carbon	1·81.			

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* Note added September 25, 1879.—At the time (Christmas, 1877) when these experiments were made the glasses were newly cast. On repeating the experiments in August, 1879, it was found that the specific inductive capacities of all the optical glasses had considerably increased in the course of eighteen months, the new values being: D.E.D.F., 3·838; E.D.F., 3·621; L.F., 3·443; H.C., 3·310. For full details of these experiments see 'Report of the British Association,' 1879, p. 250.

It has been suggested to me that I should give some evidence that—

(1.) The insulation of the apparatus was perfect.
(2.) That the plates were good insulators.
(3.) That their surfaces were not damp.

With regard to the first point, an examination of the method will show that an absolutely perfect insulation is not necessary.

As long as no electricity leaked to the small plates *b* and *d*, a leakage from the large plates *a c e* would only produce the same effect as a diminution of the electromotive force of the coil current. It would diminish the sensitiveness of the instrument, but would not affect the results obtained by it.

To prevent the electricity which leaked from the large plates leaking to, or acting by induction on, the small ones, the wires and supports belonging to the latter were completely protected by metal screens well connected to earth.

The efficiency of this protection was shown by the perfect steadiness of the light spot when the earth connection was good, and its perfect obedience to the screw of *a*.

The instant, however, that from any cause the earth connection became imperfect, the instrument became entirely unmanageable, and the light spot was in continual irregular and violent motion. I have, for the continuation of the investigation, lately introduced some further improvements, both in the insulation and in the arrangement of the metal screens.

No error is to be anticipated from defective insulation, as with the apparatus in a leaky state, instead of incorrect results being obtained, measurement becomes impossible.

With regard to any possibility of leakage occurring to one of the small plates, I may add that, after a few months' experience of the balance, it would be impossible to mistake a deflection due to this cause for the regular deflection caused by induction.

As to the second point, the only proof that the plates are good insulators is that the substances of which they are composed, glass, paraffin, &c., are generally considered to be so.

With respect to the possible existence of a film of moisture, the following is the state of the case :—

Before putting any plate into the balance it was cleaned and rubbed dry with a wash-leather. It was then rapidly passed over the flame of a spirit lamp to discharge any electrification that it might have acquired from the friction.

The spirit lamp, of course, deposited a dew on the surface. This, however, was very slight, and could be seen to evaporate rapidly in the course of a few seconds. In one or two cases the plates were carefully dried by a fire, but no particular difference was noticed.

The exact agreement with each other of the experiments on those substances, of which I possessed more than one plate, such as paraffin, ebonite, and particularly common plate glass, shows, I think, that this film of moisture can never have produced any serious effect, as the experiments were made on many different days during which the temperature and dew-point varied very considerably.

The effect of the film of moisture, if any, would have been to decrease $a_2 - a_1$. Now as each set of determinations took about 20 minutes, the film, if it existed, would have dried gradually during this time, and in the last experiments of a set, $a_2 - a_1$ would always have been greater than in the earlier ones. An examination of the details of the experiments does not show that this was the case.*

The close agreements of the values of K for paraffin found by me and by Messrs. GIBSON and BARCLAY, who worked by an entirely different method, shows that there cannot be any serious defect either in their method or in mine.

Measurement of the refractive indices of the transparent dielectrics.

In order to see how far the results of these experiments bore out Professor MAXWELL's electromagnetic theory of light, the square roots of the specific inductive capacities of the various transparent dielectrics were compared with the refractive indices given in works on physics. Wherever an approximate agreement was found the refractive index was carefully re-determined. It was not, however, thought necessary to make experiments on the refractive indices of the dielectrics when either there was an obvious and wide difference between μ and $\sqrt{K}$, or when, as in the case of paraffin, a previous determination, probably more trustworthy than any I could make, was available.

Optical glass.—In order to see whether Dr. HOPKINSON's† determinations of the refractive indices of CHANCE's optical glass could be compared with those obtained by me, the specific gravities of the specimens used by me were taken and compared with the specific gravities of those used by Dr. HOPKINSON.

The following were the results:—

The name given to the glass by Messrs. CHANCE.	Specific gravity of Dr. HOPKINSON's specimens.	Specific gravity of GORDON's specimens.
Double extra-dense flint .	4·42162	4·421
Extra-dense flint. . . .	3·88947	3·884
Light flint	3·20609	3·198
Hard crown	2·48664	2·484

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* September 25, 1879.—Some later experiments on the effect of wetting the plates have shown that no error could have been introduced accidentally by the presence of a film of moisture.—Rep. Brit. Assoc., 1879, p. 250.

† 'Proc. Roy. Soc.,' 1877, p. 290.

Experimental determinations of μ.—A large 4-prism spectroscope by Browning was used.

It has a silver circle about $9\frac{1}{2}$ inches in diameter or nearly 30 inches in circumference.

The vernier reads to 10″.

The prisms were removed and replaced by one prism of the glass under examination.

μ was determined in the ordinary way by first measuring the angle of each prism and then mapping a spectrum observed through it at minimum deviation.

An attempt was made to observe the solar spectrum, but owing to the uncertainty of the weather in October it was abandoned, and a spark between magnesium poles in air was used as the source of light instead.

The spark was produced by means of my 17-inch coil working with its own vibrating break. A Leyden jar was connected to the secondary circuit in the usual way, and a convenient discharging frame caused the spark to take place in front of the slit.

The wires of the primary circuit were brought round to a plug-key fixed to the table close to the spectroscope. This enabled the observer to start and stop the coil without leaving his place.

To determine the refractive index for waves of infinite length, we proceed as follows :—

We have the general equation

$$\mu=\mathrm{A}+\frac{\mathrm{B}}{\lambda^2}$$

To determine A, a determination of the values of μ for two rays of different wave lengths λ and λ' are necessary and sufficient, for we have

$$\mu\lambda^2=\mathrm{A}\lambda^2+\mathrm{B}$$
$$\mu'\lambda'^2=\mathrm{A}\lambda'^2+\mathrm{B}$$

Subtracting one equation from the other we eliminate B and obtain

$$\mathrm{A}=\frac{\mu\lambda^2-\mu'\lambda'^2}{\lambda^2-\lambda'^2}$$

But when $\lambda=\infty$, $\mu=\mathrm{A}$. Hence

$$\mu_{\lambda=\infty}=\frac{\mu\lambda^2-\mu'\lambda'^2}{\lambda^2-\lambda'^2}$$

The following are the results of the experiments on glass :—

REFRACTIVE index of double extra-dense flint glass.

$$\text{Specific gravity}\begin{cases}4{\cdot}421 & \text{GORDON.}\\ 4{\cdot}42162 & \text{HOPKINSON.}\end{cases}$$

$$\sqrt{K}=1{\cdot}7783.$$

Ray.	Wave length.	μ.		Ray.	Wave length.
		GORDON.	HOPKINSON.		
Bright line in extreme red . .	..	1·7032	1·7011	B	6·867
D.	5·892	1·7099	1·7102	D	
b_1.	5·174	1·7201	1·7209	b	
b_2.		1·7234			
Bright line in central blue . .	..	1·7374	1·7272	F	
Band in extreme violet . . .	..	1·7460	1·7432	G	
			1·7577	H_1	3·968

$$\text{Which give }\mu_{\lambda=\infty}=\begin{cases}1{\cdot}655\ \text{GORDON from D and } b.\\ 1{\cdot}672\ \text{HOPKINSON from B and } H_1.\end{cases}$$

REFRACTIVE index of extra-dense flint glass.

$$\text{Specific gravity}\begin{cases}3{\cdot}884 & \text{GORDON.}\\ 3{\cdot}88947 & \text{HOPKINSON.}\end{cases}$$

$$\sqrt{K}=1{\cdot}7474.$$

Ray.	Wave length.	μ.		Ray.	Wave length.
		GORDON.	HOPKINSON.		
Bright line in extreme red . .	..	1·6423	1·6429	B	6·867
D.	5·892	1·6478	1·6505	D	
b_1.	5·174	1·6561	1·6591	b	
b_2.		1·6586			
Bright line in central blue . .	..	1·6693	1·6642	F	
Band in extreme violet . . .	..	1·6757	1·6770	G	
			1·6885	H_1	3·968

$$\text{Which give }\mu_{\lambda=\infty}=\begin{cases}1{\cdot}615\ \text{GORDON from D and } b.\\ 1{\cdot}620\ \text{HOPKINSON from B and } H_1.\end{cases}$$

REFRACTIVE index of light flint glass.

$$\text{Specific gravity} \begin{cases} 3{\cdot}198 & \text{GORDON.} \\ 3{\cdot}20609 & \text{HOPKINSON.} \end{cases}$$

$$\sqrt{K} = 1{\cdot}7342.$$

Ray.	Wave length.	μ.		Ray.	Wave length.
		GORDON.	HOPKINSON.		
Bright line in extreme red	..	1·5685	1·5685	B	6·867
D.	5·892	1·5722	1·5740	D	
b_1.	5·174	1·5787	1·5803	b	
b_2.		1·5805			
Bright line in central blue	..	1·5879	1·5839	F	
Band in extreme violet	..	1·5920	1·5928	G	
			1·6007	H_1	3·968

$$\text{Which give } \mu_{\lambda=\infty} = \begin{cases} 1{\cdot}547 \text{ GORDON from D and } b. \\ 1{\cdot}555 \text{ HOPKINSON from B and } H_1. \end{cases}$$

REFRACTIVE index of hard crown glass.

$$\text{Specific gravity} \begin{cases} 2{\cdot}484 & \text{GORDON.} \\ 2{\cdot}48664 & \text{HOPKINSON.} \end{cases}$$

$$\sqrt{K} = 1{\cdot}7629.$$

Ray.	Wave length.	μ.		Ray.	Wave length.
		GORDON.	HOPKINSON.		
Bright line in extreme red	..	1·5022	1·5141	B	6·867
D.	5·892	1·5045	1·5176	D	
b_1.	5·174	1·5083	1·5215	b	
b_2.		1·5093			
Bright line in central blue	..	1·5017	1·5236	F	
Band in extreme violet	..	1·5132	1·5288	G	
			1·5333	H_1	3·968

$$\text{Which give } \mu_{\lambda=\infty} = \begin{cases} 1{\cdot}491 \text{ GORDON from D and } b. \\ 1{\cdot}504 \text{ HOPKINSON from B and } H_1. \end{cases}$$

I here give for what it may be worth an attempt to determine for two glasses for what wave length of light μ would equal $\sqrt{K}$.

To determine the wave length of the ray for which $\mu = \sqrt{K}$, we have

$$\mu = A + \frac{B}{\lambda^2}$$

For double extra-dense flint we have, using HOPKINSON's numbers,

$$A=1{\cdot}672.$$

For the D line the equation becomes

$$1{\cdot}710=1{\cdot}672+\frac{B}{3{\cdot}471}$$

or

$$B=3{\cdot}471(1{\cdot}710-1{\cdot}672)$$
$$={\cdot}1319$$

Putting $\mu=\sqrt{K}=1{\cdot}778$, we have

$$\lambda=3{\cdot}527$$

In DRAPER's photograph line N is $3{\cdot}528$

EXTRA-dense flint.

For the D line we have

$$1{\cdot}650=1{\cdot}620+\frac{B}{3{\cdot}471}$$

or

$$B=3{\cdot}471({\cdot}030)={\cdot}1041$$

Putting $\mu=\sqrt{K}=1{\cdot}747$, we have

$$1{\cdot}747=1{\cdot}620+\frac{{\cdot}1041}{\lambda^2}$$
$$\lambda=2{\cdot}862$$

The following table compares the various refractive indices of each dielectric with the square root of the specific inductive capacity:—

TABLE comparing μ and $\sqrt{K}$.

Dielectric.	$\sqrt{K}$.	$\mu_{\lambda=\infty}$.	μ_D.	μ_{H_1}.	$\lambda_{\mu=\sqrt{K}}$.
Double extra-dense flint glass	1·778	1·672	1·710	1·757	3·527, the wave length for N in ultra violet.
Extra-dense flint glass .	1·747	1·620	1·650	1·688	2·862
Light flint glass* . . .	1·734	1·555	1·574	1·601	
Hard crown glass . . .	1·763	1·504	1·517	1·533	
Paraffin	1·4119	1·4220†			
Sulphur	1·606				
Bisulphide of carbon . .	1·345	. .	1·611		
Common plate glass . .	1·801	. .	1·543		

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* In the abstract of the paper published in 'Proc. Roy. Soc.,' No. 191, the refractive indices for hard crown and light flint were interchanged. The mistake, which has only just been discovered, was caused by the polisher to whom the prisms were entrusted having interchanged the "H.C." and "L.F." labels.—June 17, 1879.

† GLADSTONE and CLERK MAXWELL, 'Maxwell's Electricity,' § 789, vol. ii., p. 389. The melting point of my paraffin was 68° C., that of Dr. GLADSTONE's was less than 57° C.

Conclusion.

It will be seen from the foregoing table that the results of the experiments on certain dielectrics, notably two out of the four specimens of optical glass,[*] and the paraffin agree tolerably closely with the formula deduced from Professor CLERK MAXWELL's theory. For other dielectrics the formula does not even approximately hold good. I think we may fairly conclude that the square of the refractive index is a term in the expression of the specific inductive capacity, and that in some dielectrics it is by far the most important term. In other dielectrics it is overshadowed by some other term whose nature is at present unknown.

Pending the discovery of this unknown term, I propose the following method of eliminating it in a particular case :—

Let two plates of Iceland spar be constructed with their faces, perpendicular and parallel, to the axis of the crystal respectively. Let the specific inductive capacities be determined, and then if Professor CLERK MAXWELL's views of the nature of an electromagnetic disturbance in the ether are correct, I think we ought to have

$$\left\{\begin{matrix}\text{Ratio of specific inductive capacities}\\ \text{along and across axis.}\end{matrix}\right\}^{\frac{1}{2}} = \left\{\begin{matrix}\text{Ratio of refractive indices}\\ \text{along and across axis.}\end{matrix}\right\}$$

To me it seems possible that this equation may be unaffected by the existence of the unknown term, whatever it may be.[†]

I am proposing to continue this research, using 1000 LECLANCHÉ cells instead of secondary current ; 500 of these are already erected, and a rapid reverser and chronograph have been constructed.

Since the appearance of Messrs. AYRTON and PERRY's paper on the viscosity of dielectrics,[‡] it has become of interest to determine whether there is any change in the specific inductive capacity when the duration of the electrification is varied from $\frac{1}{500}$th second to several days.

It is impossible to use the coil for this work, as any variation in the speed of the break (or reverser) produces large changes in the electromotive force of secondary currents.

It is also important to determine whether difference of electromotive force produces any difference in specific inductive capacity. The agreement of my experiments on paraffin with those of Messrs. GIBSON and BARCLAY, seems to indicate the non-existence of any change. The difference between my experiments and Dr. HOPKINSON'S may possibly, by some, be considered an indication that a change takes place. I hope a more or less complete investigation of this point will be the subject of a future memoir.*

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[*] When newly cast. See note to page 438 (September 25, 1879).

[†] Note added June 17, 1879.—After many unsuccessful attempts to obtain plates of Iceland spar of the size required for my induction balance at anything like a reasonable price, I have at last, at the suggestion of Professor CORNU, decided to have a minature induction balance constructed, for the determination of the specific inductive capacity of spar and other expensive substances. This instrument is now almost finished; I hope a description of it, and of experiments performed with it, will form the subject of a future paper.

[‡] 'Proc. Roy. Soc.,' 1877–8, vol. xxvii., p. 238.

I must not conclude this paper without expressing my great gratitude to Professor CLERK MAXWELL for his kindness in assisting me with advice and suggestions from time to time during the progress of the work. The original plan of the investigation, and the first sketches of the more important parts of the apparatus, are all due to him.

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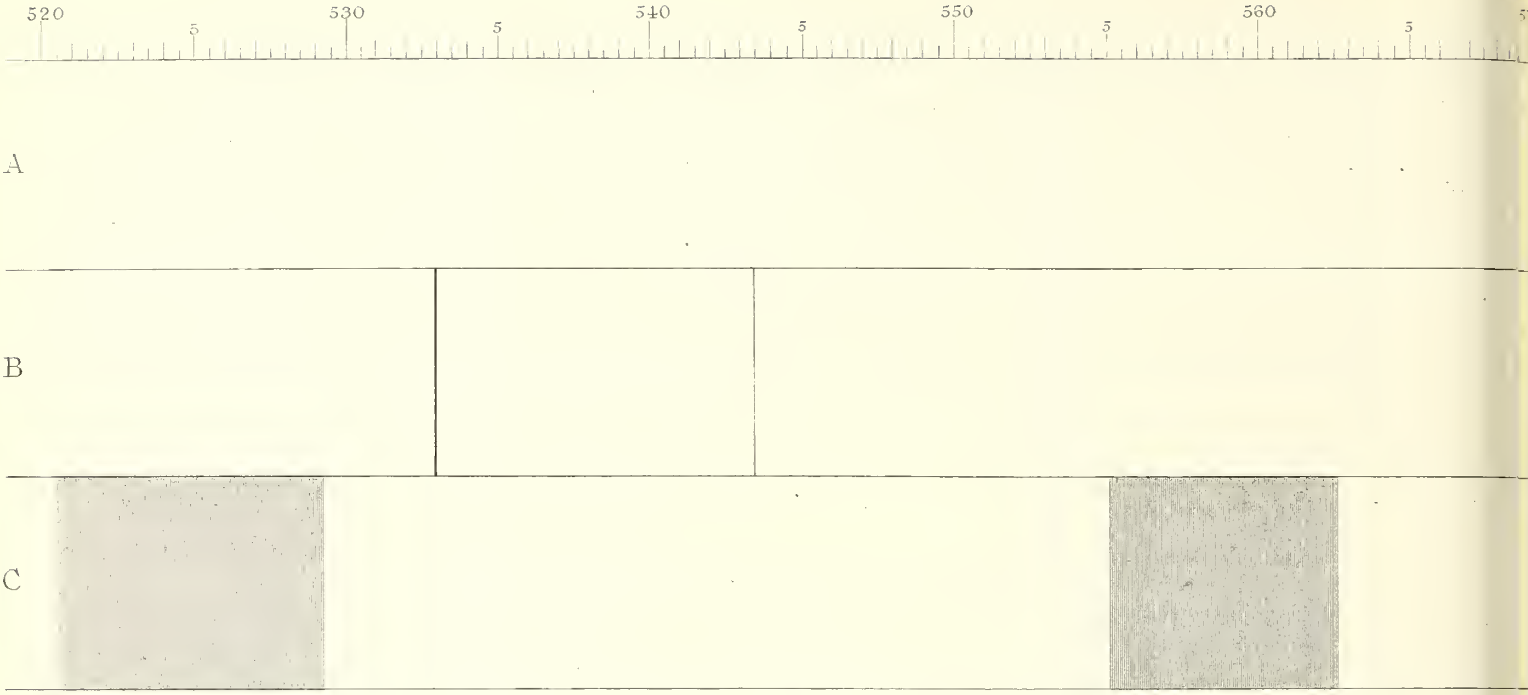
* Note added June 17, 1879.—Until now all attempts to use the battery instead of the coil have proved unsuccessful. No consistent results have been obtained with it. I think the want of success is due to the difficulty of discharging the apparatus, and of properly insulating it from the battery when the latter is not supposed to be connected to it. Four different commutators have already been tried without success. Professor CORNU has, however, had the kindness to devise another one for me, and this is now in course of construction; I have considerable hopes that it will answer its purpose.

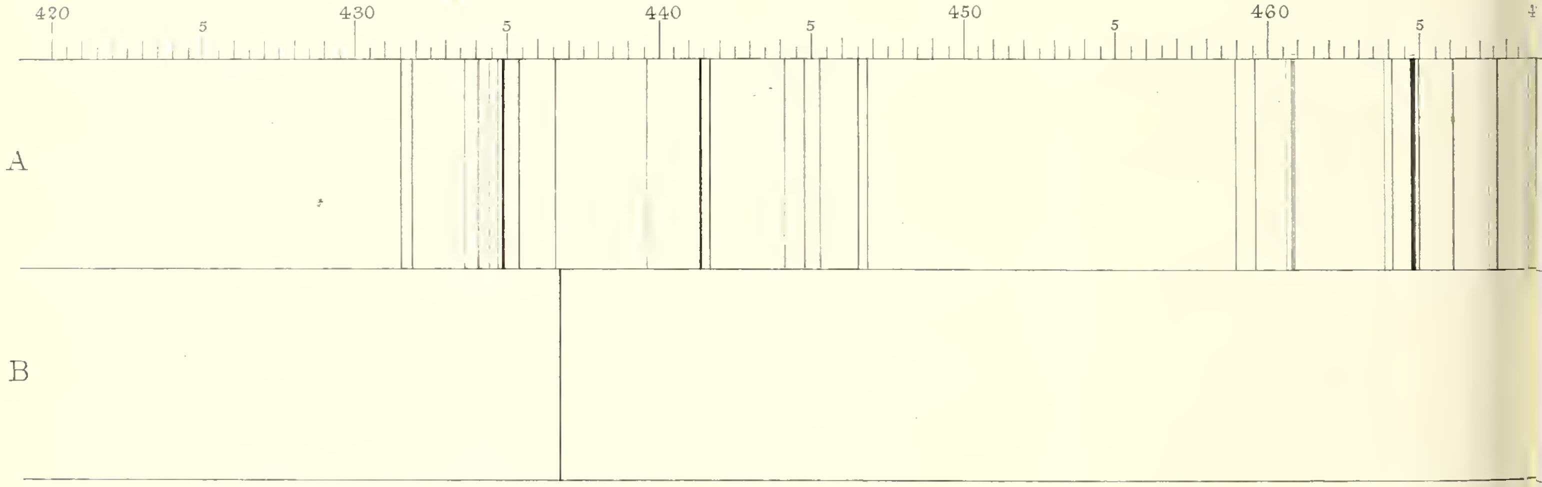
September 25, 1879.—Some preliminary experiments made with the CORNU commutator, and a HOLTZ machine have been successful.

BRITISH MUSEUM NATURAL HISTORY.

Schuster

Spectra





A = Elementary Line Spectrum.

B = Compou

Spectrum of Oxy

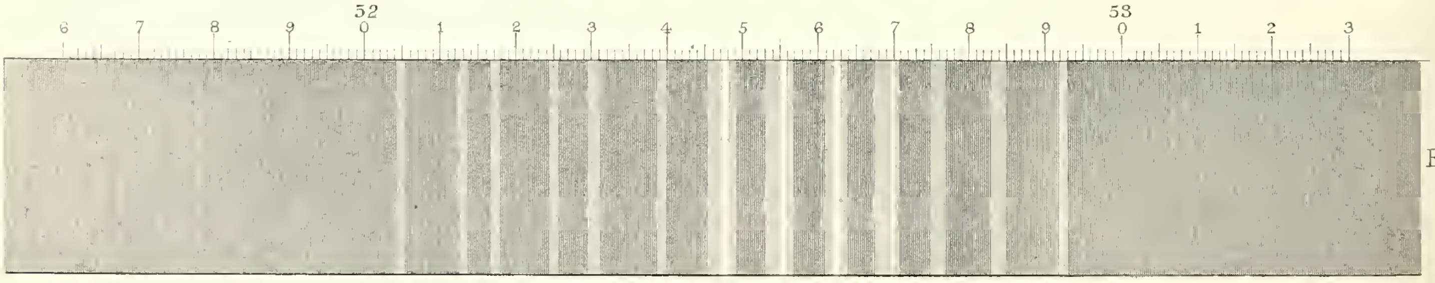
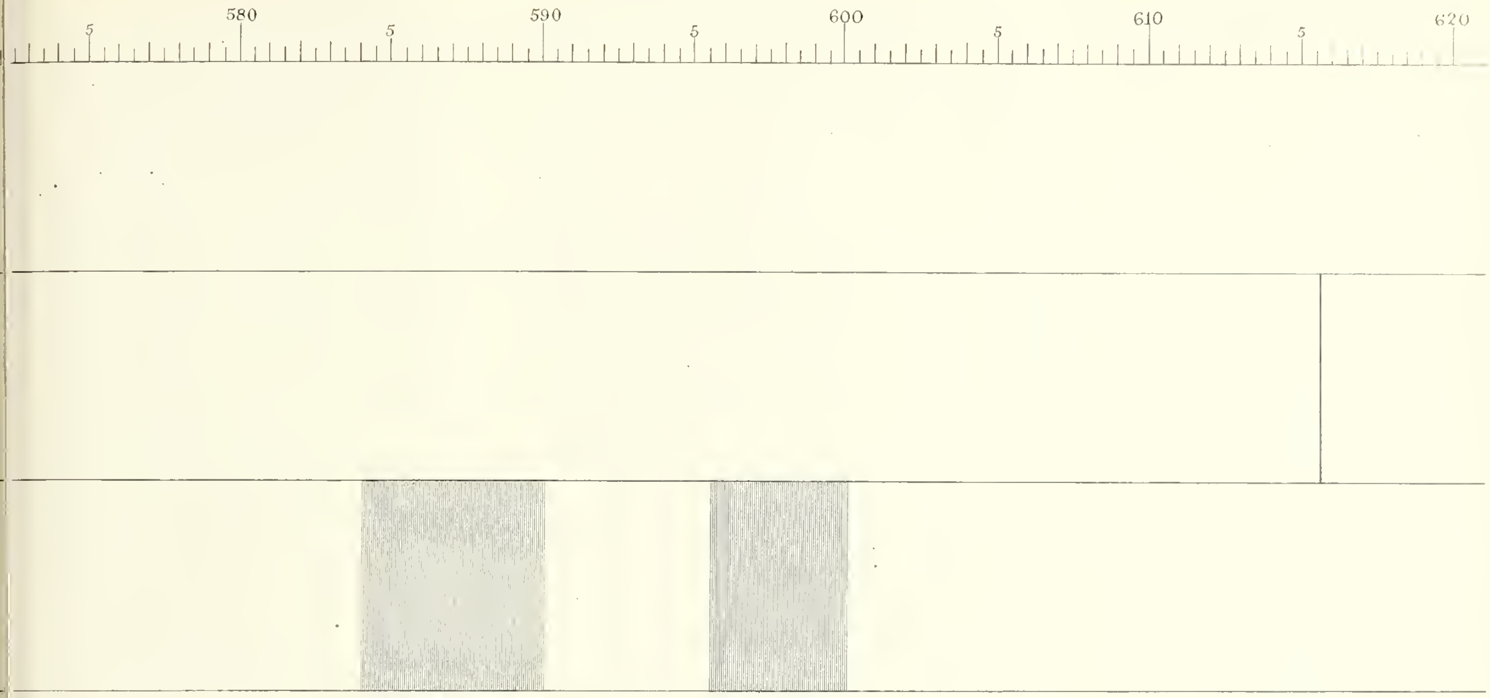
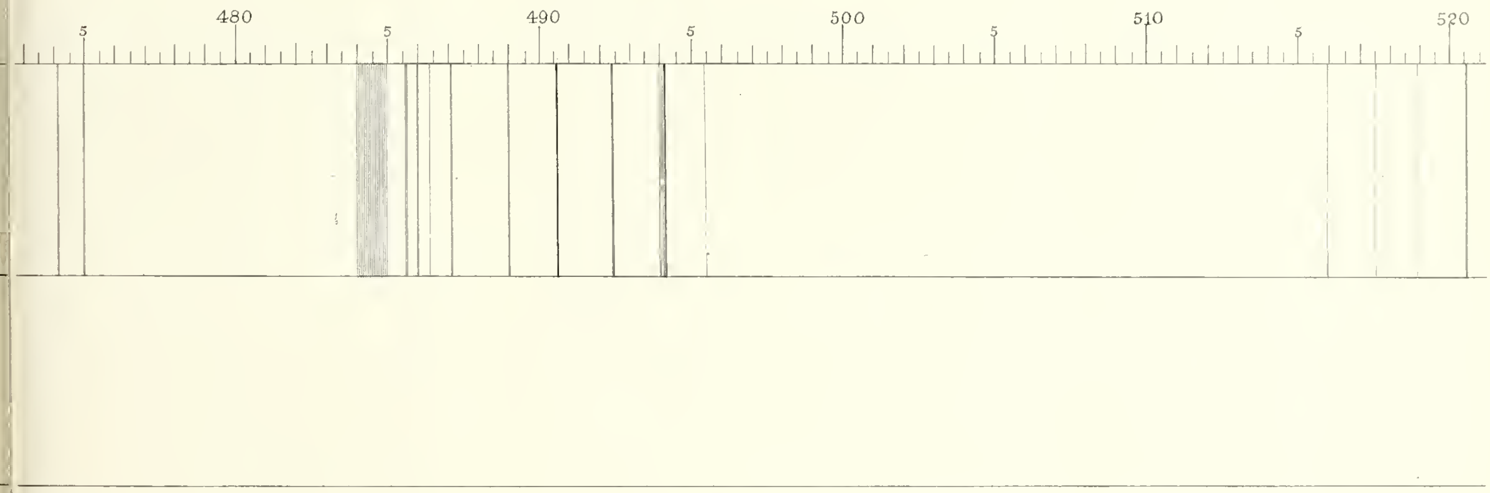


Fig. 1.

Phil. Trans. 1879. Plate 1.

Oxygen.



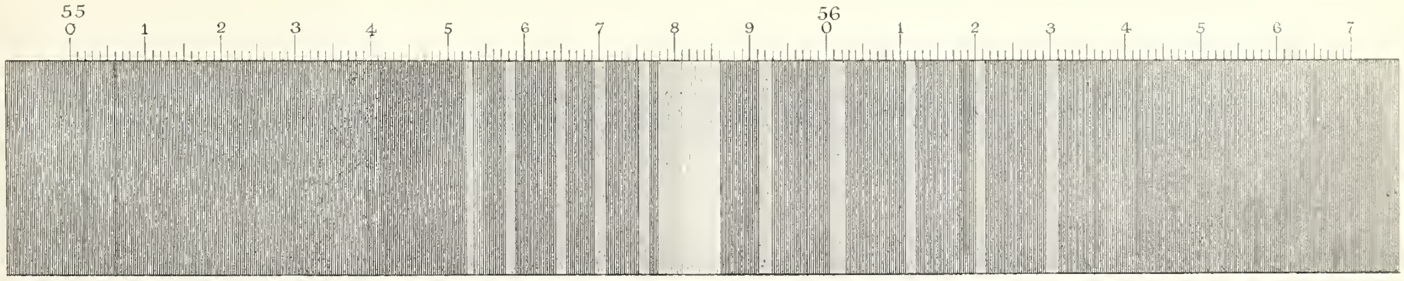


ine pectrum.

C = Spectrum of the Negative Pole.

ut Negative Pole.

ig. 2.



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BRITISH MUSEUM
NATURAL HISTORY

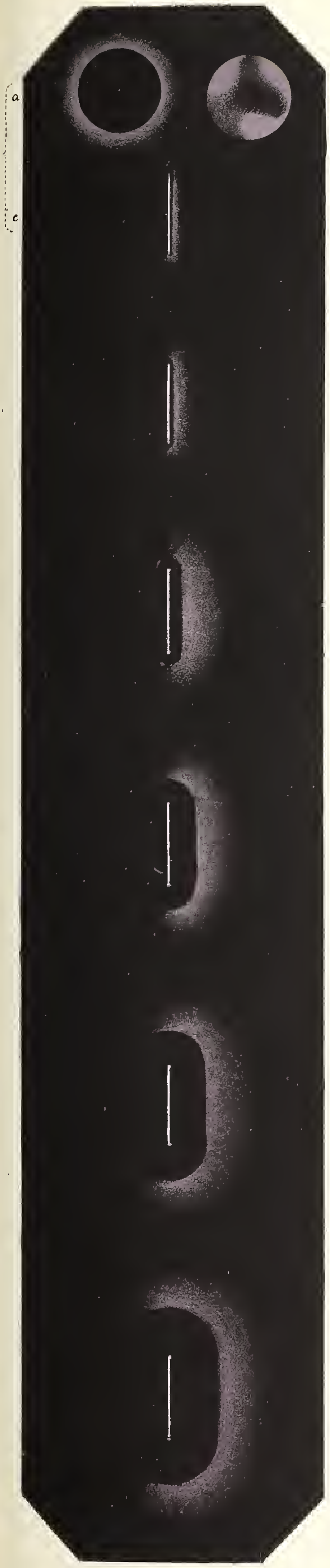


Fig. 2

Fig. 3

ig. 4

ig. 5

ig. 6

g. 7

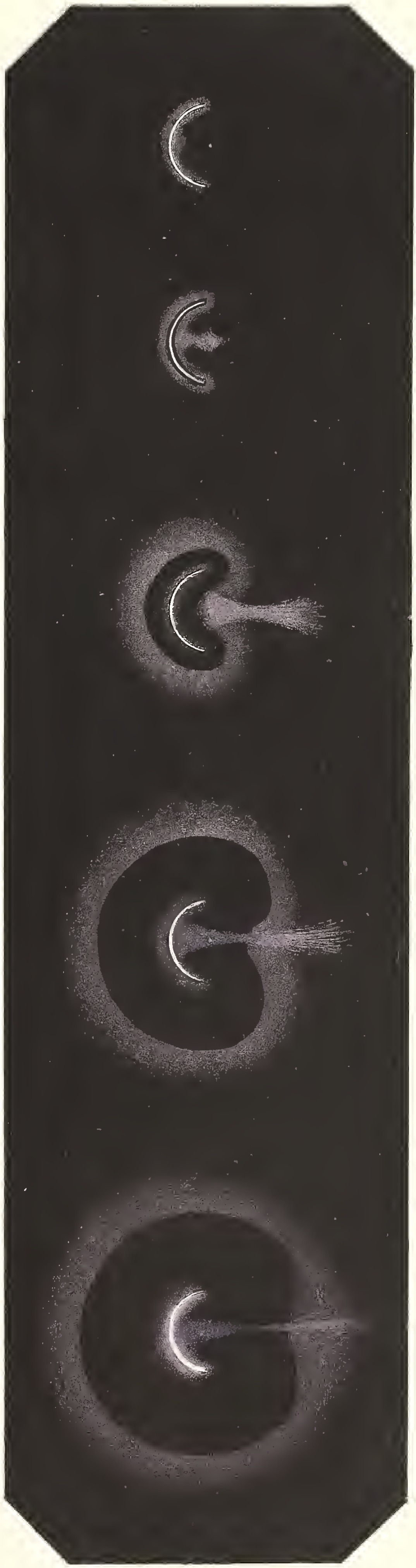
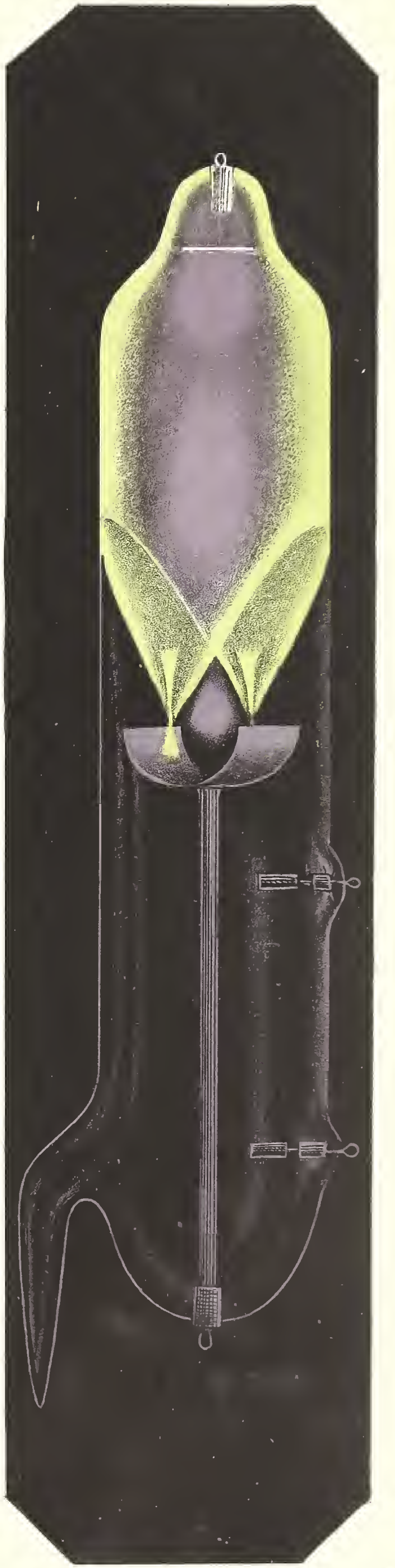


Fig. 11.



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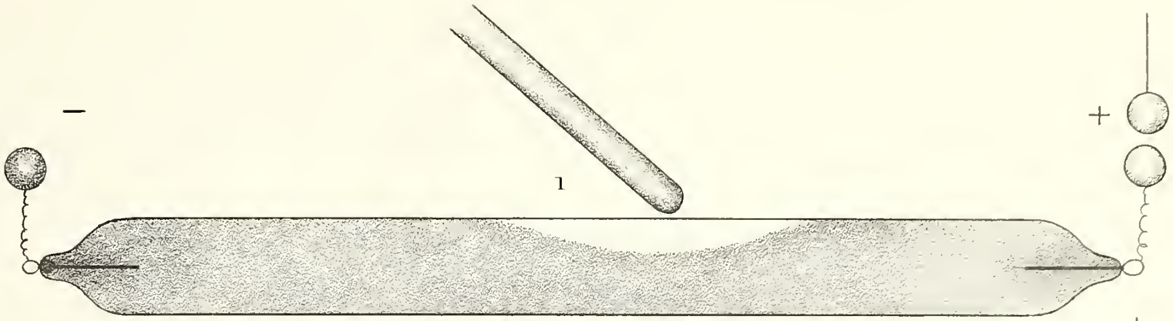

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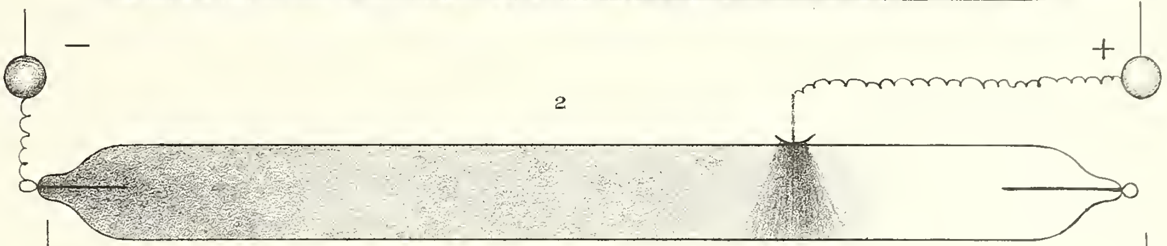

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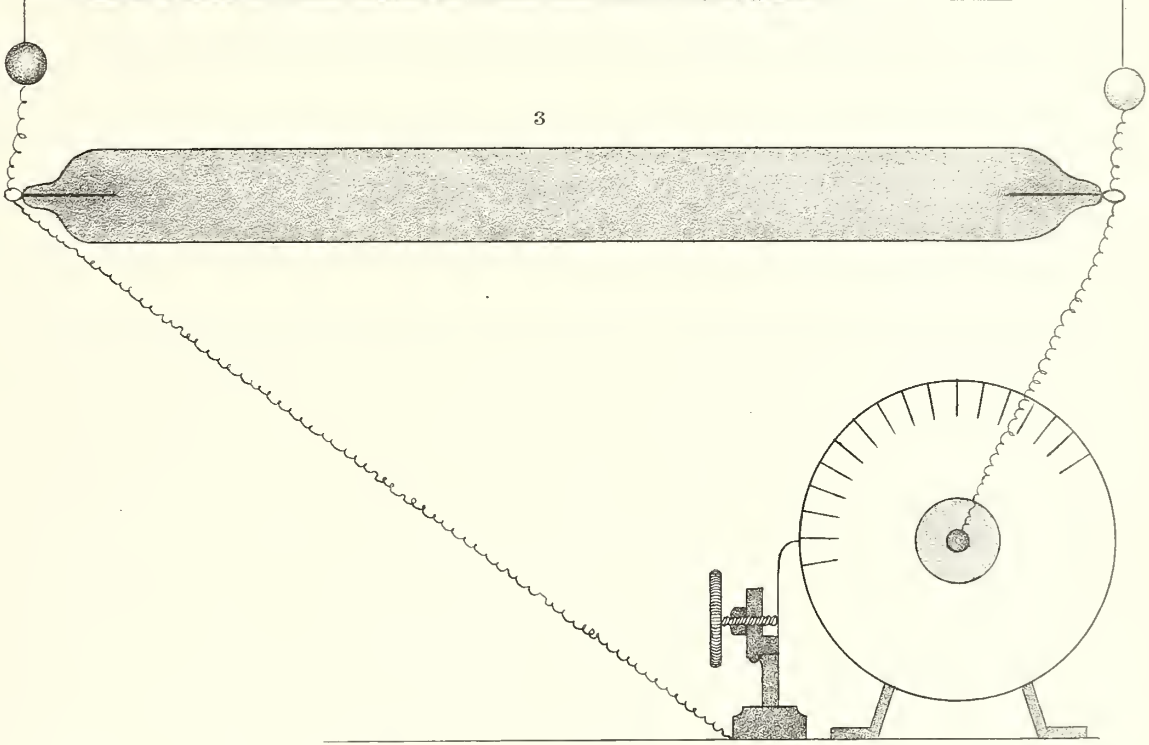

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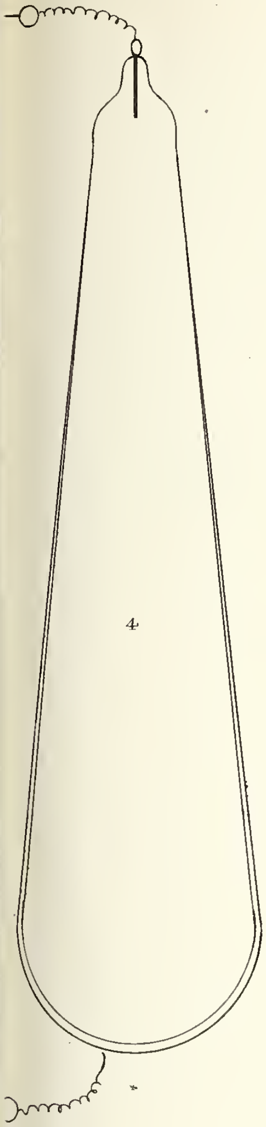

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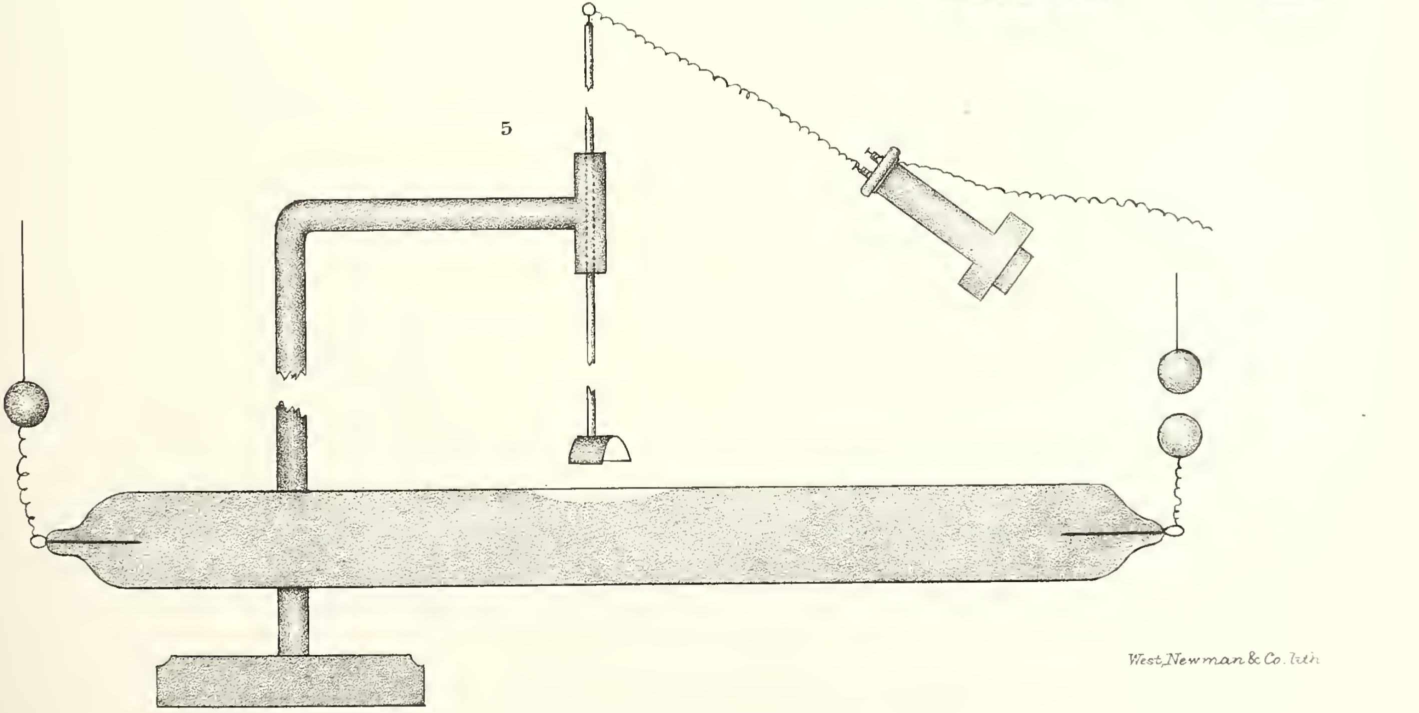

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BRITISH MUSEUM
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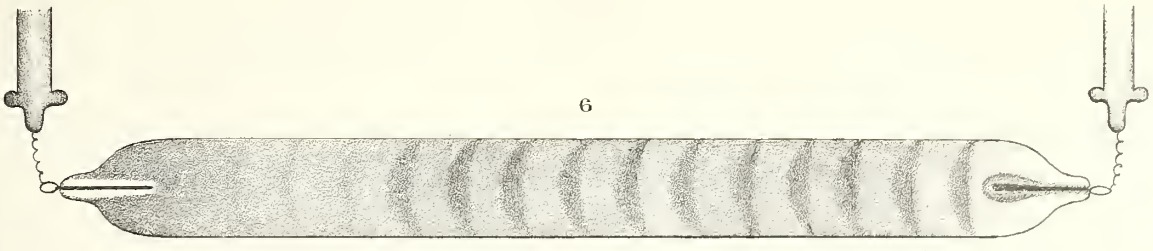
6



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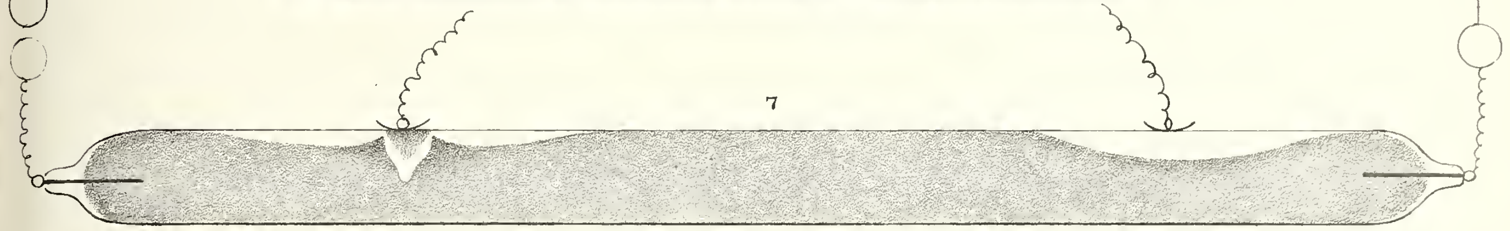

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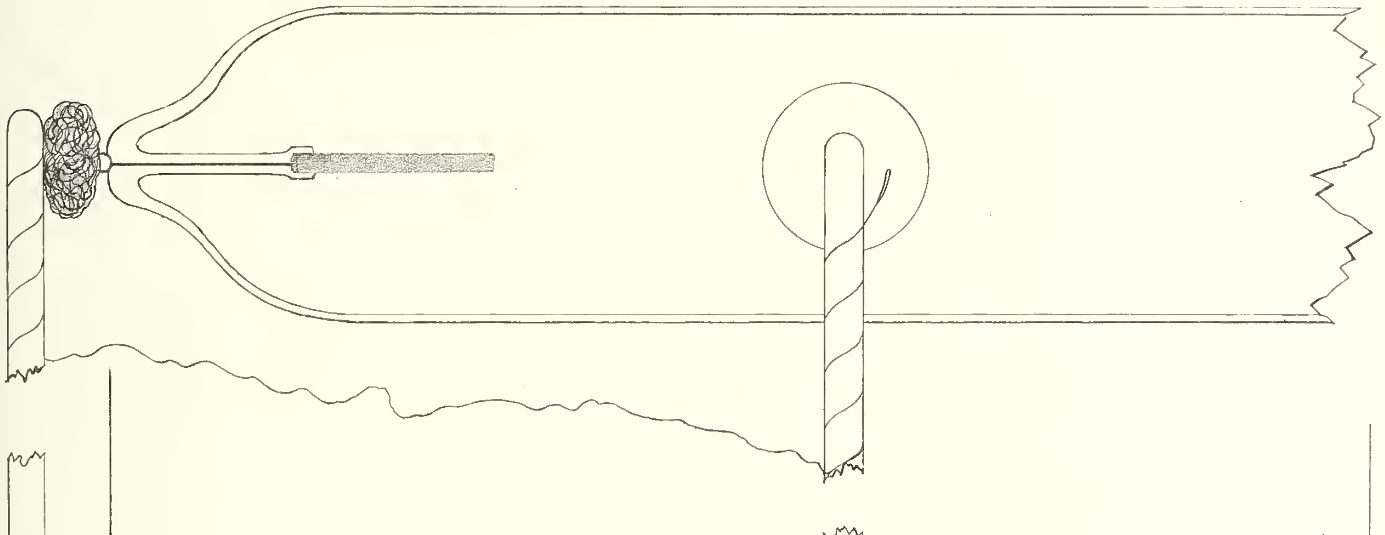

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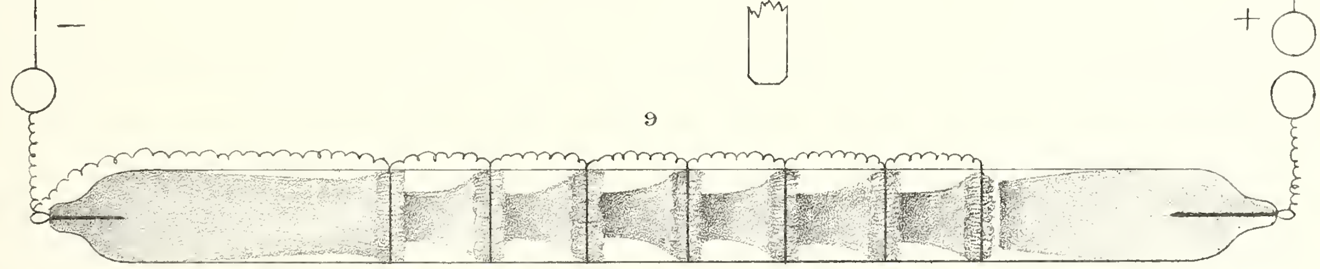

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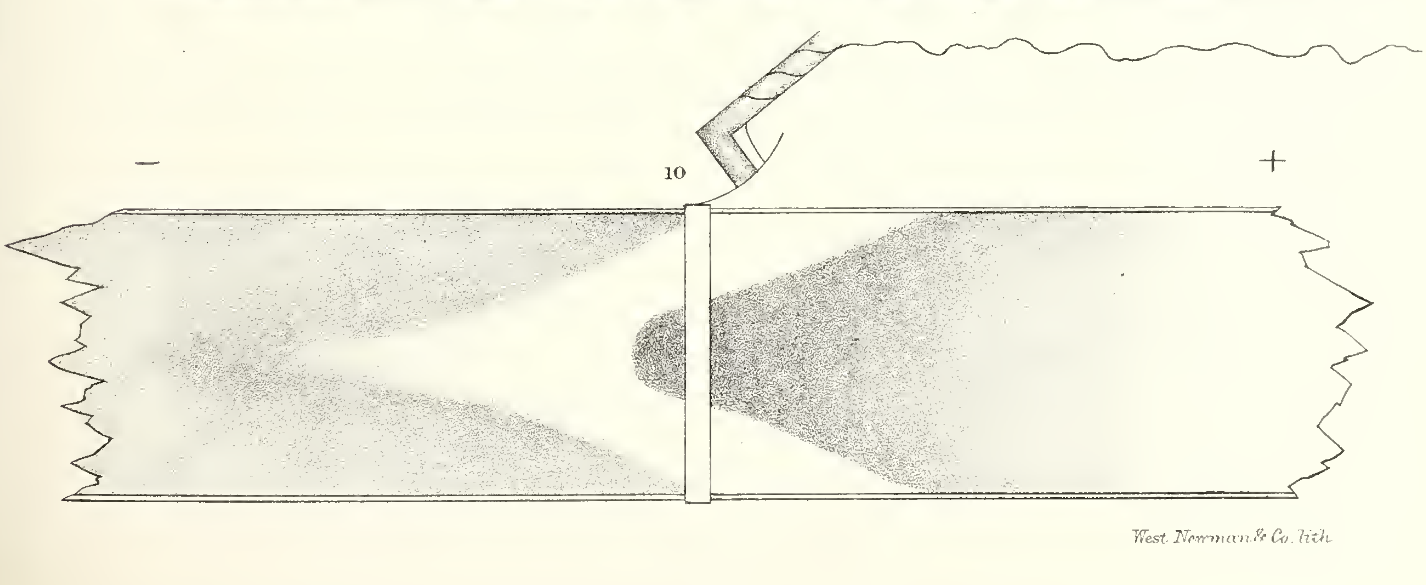

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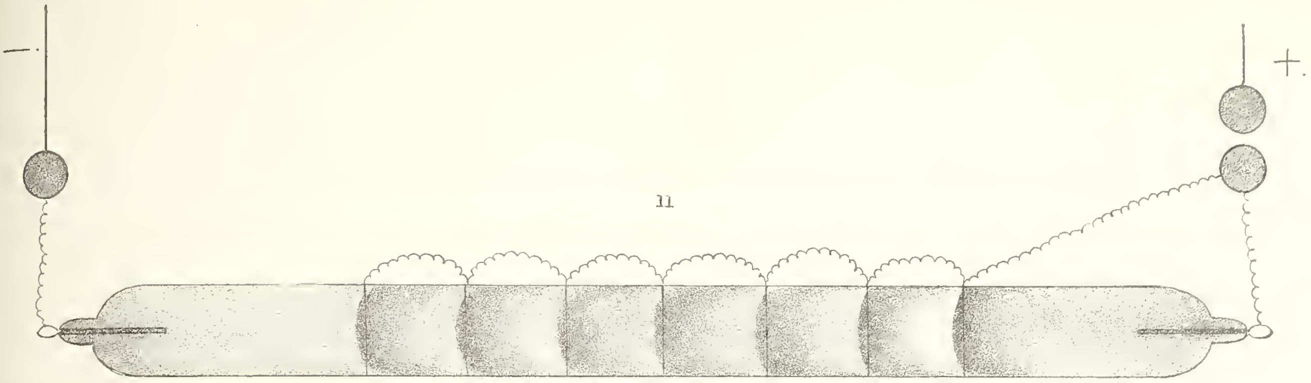
11



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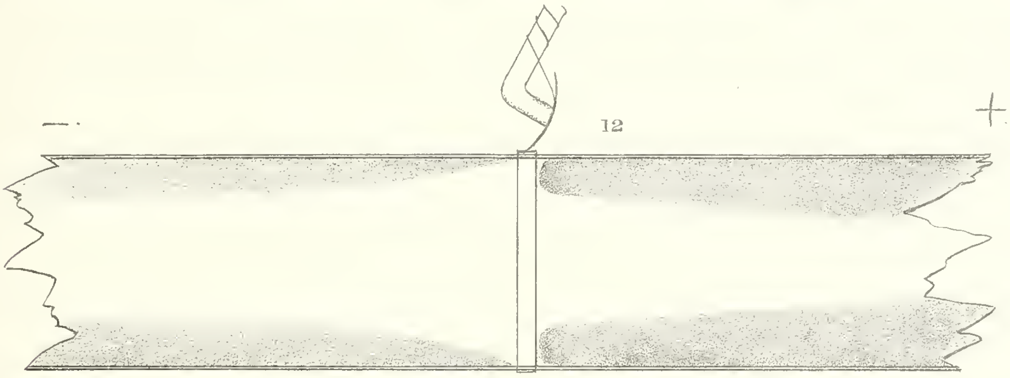

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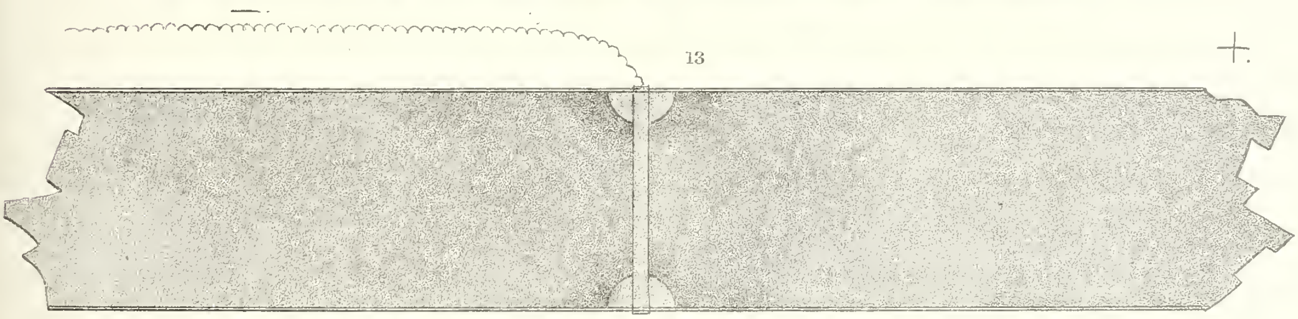

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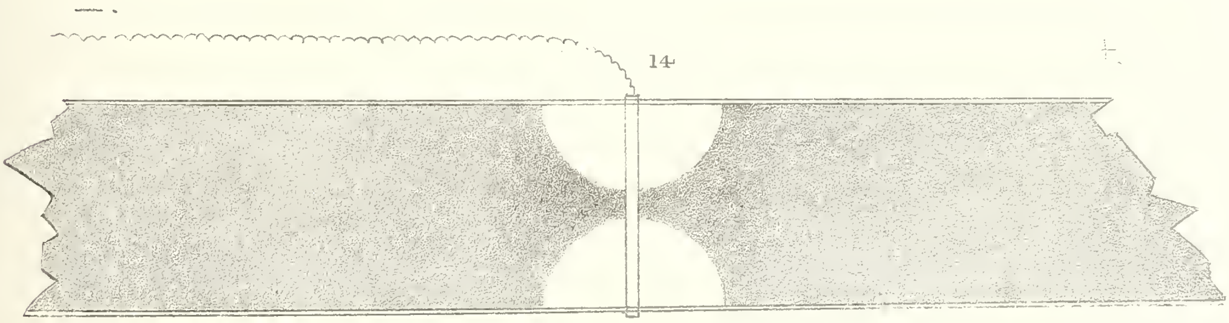

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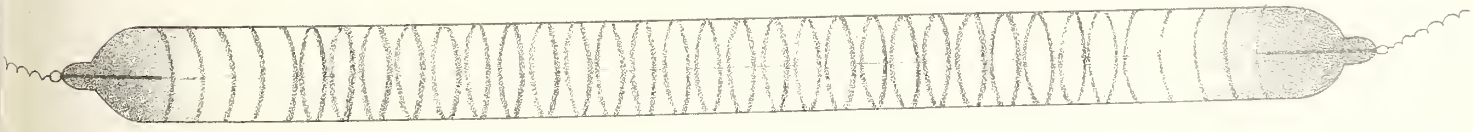

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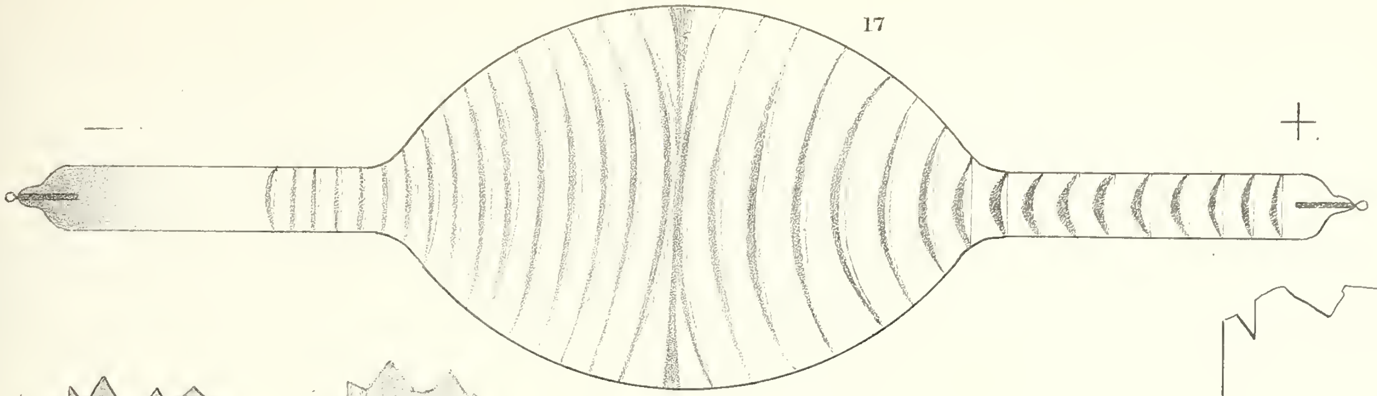
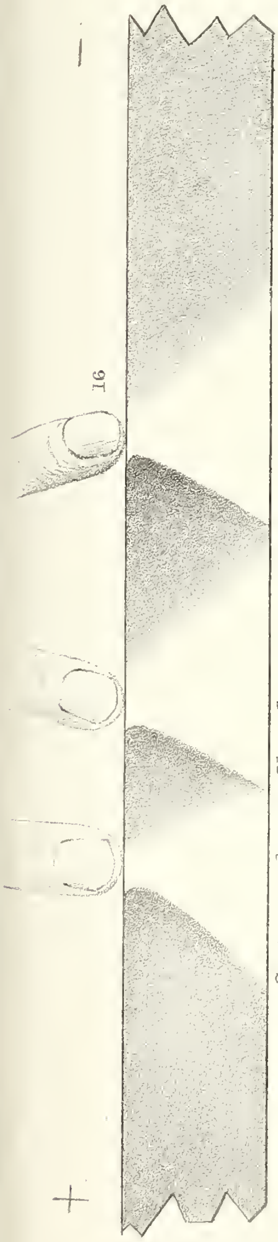
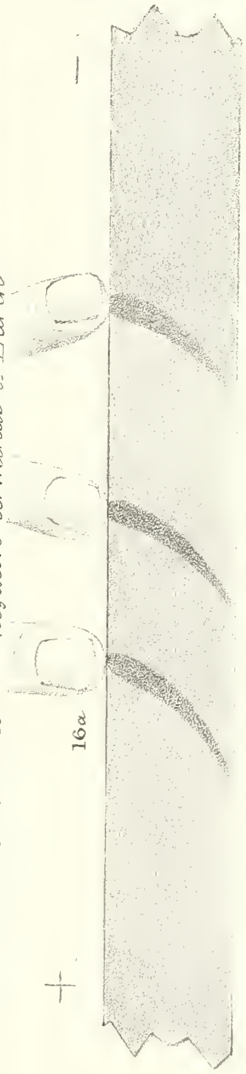




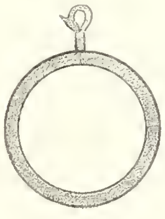
16

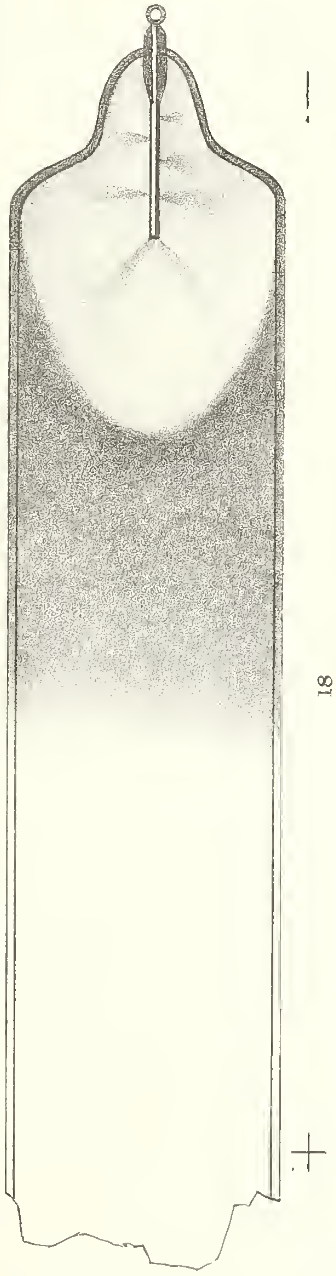
Striæ with small coil — negative terminal to Earth.



16a

Striæ with small coil — positive terminal to Earth.





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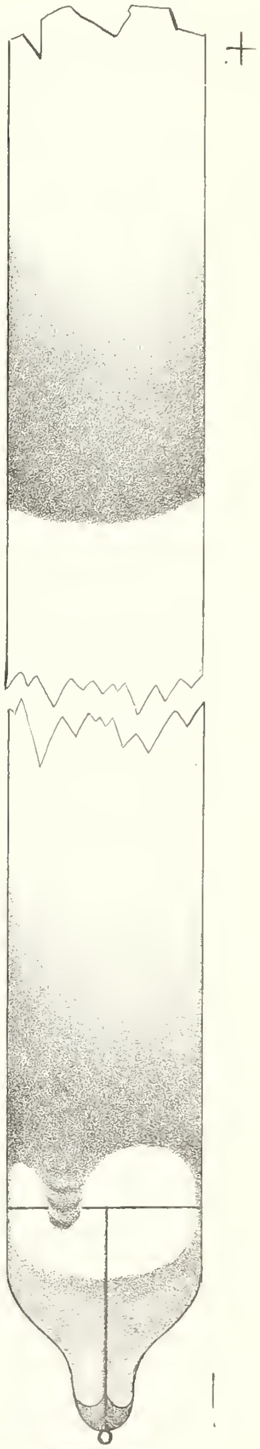

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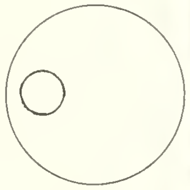

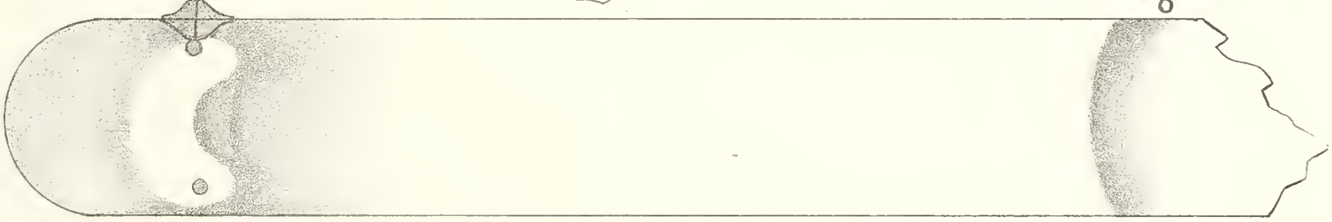

27

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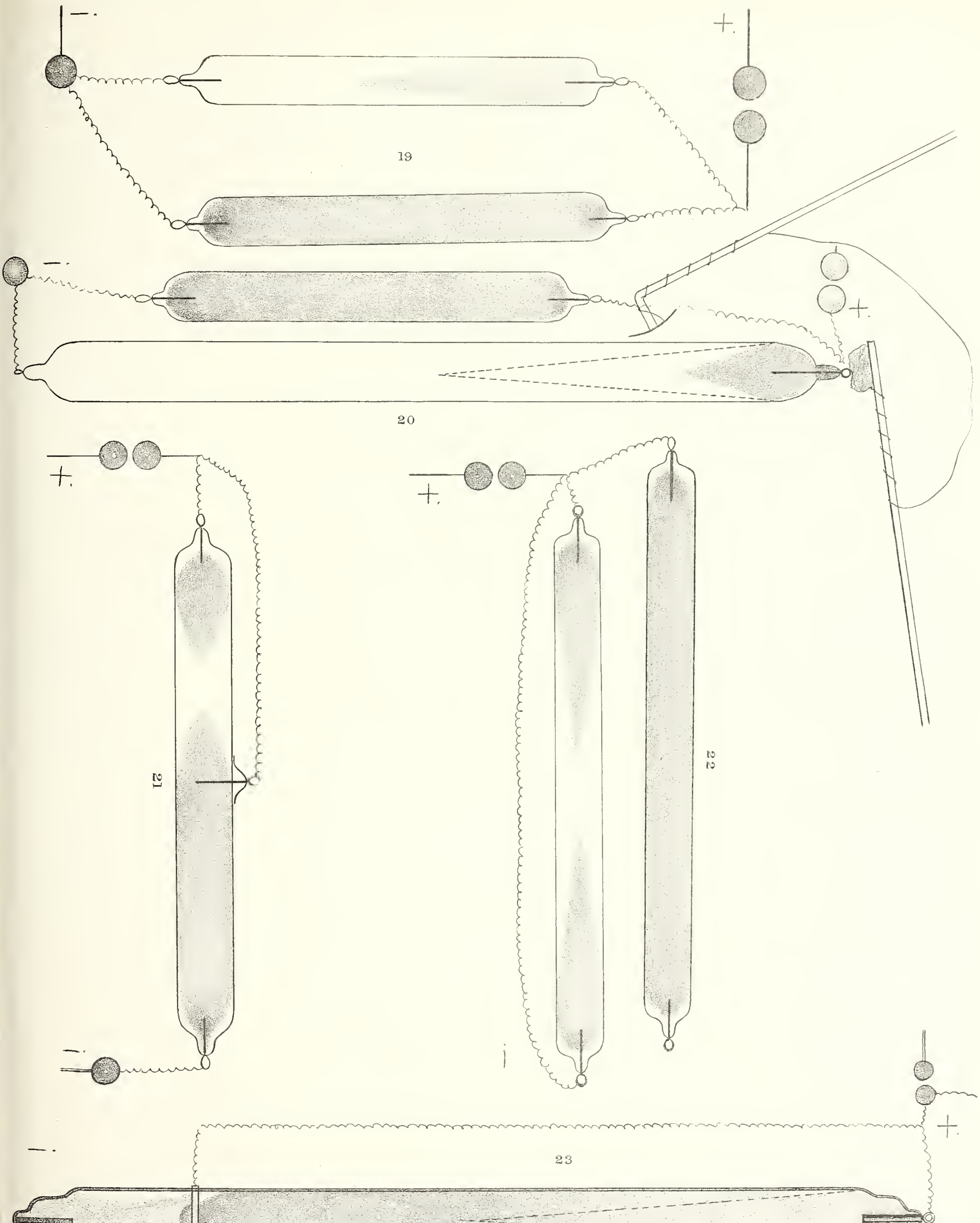
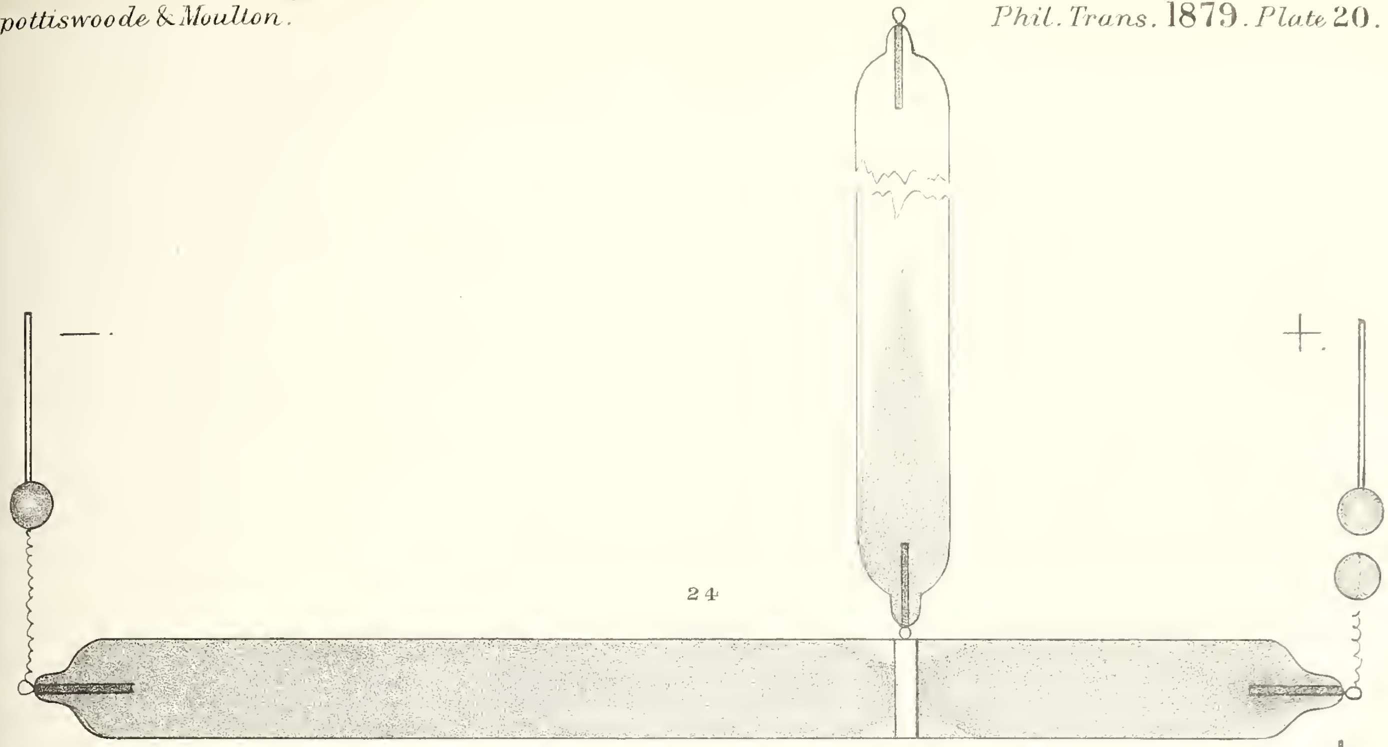
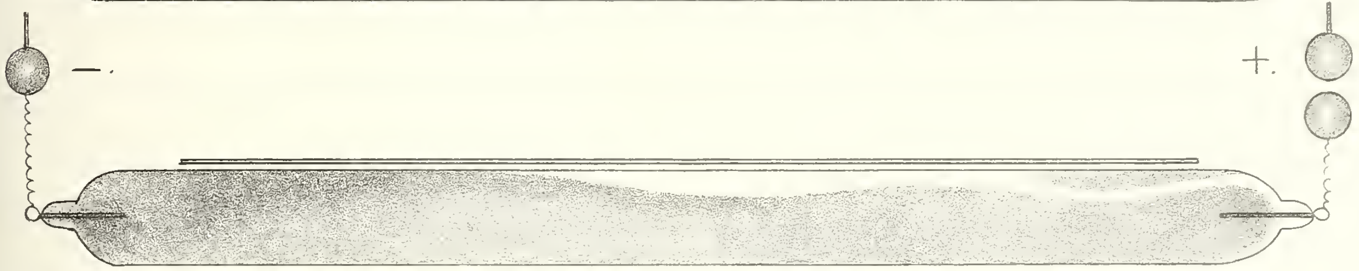


West, Newman & Co. lith.

BRITISH MUSEUM NATURAL HISTORY.





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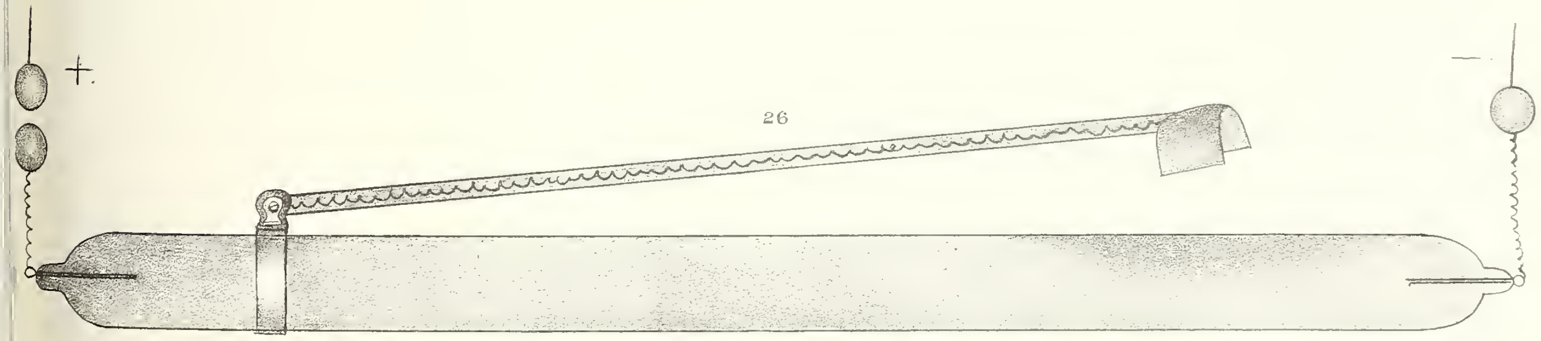
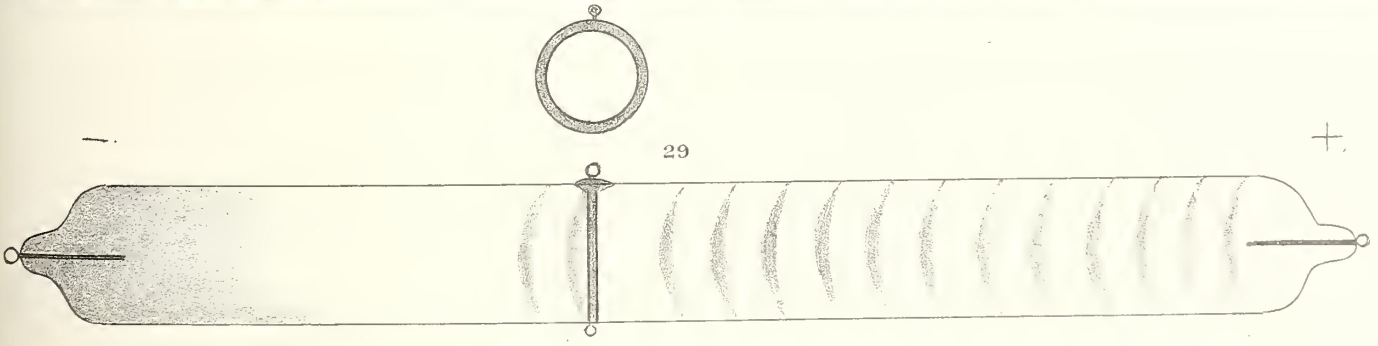
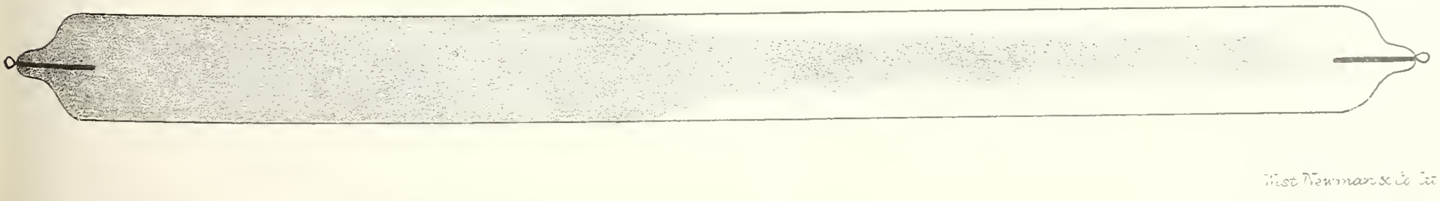






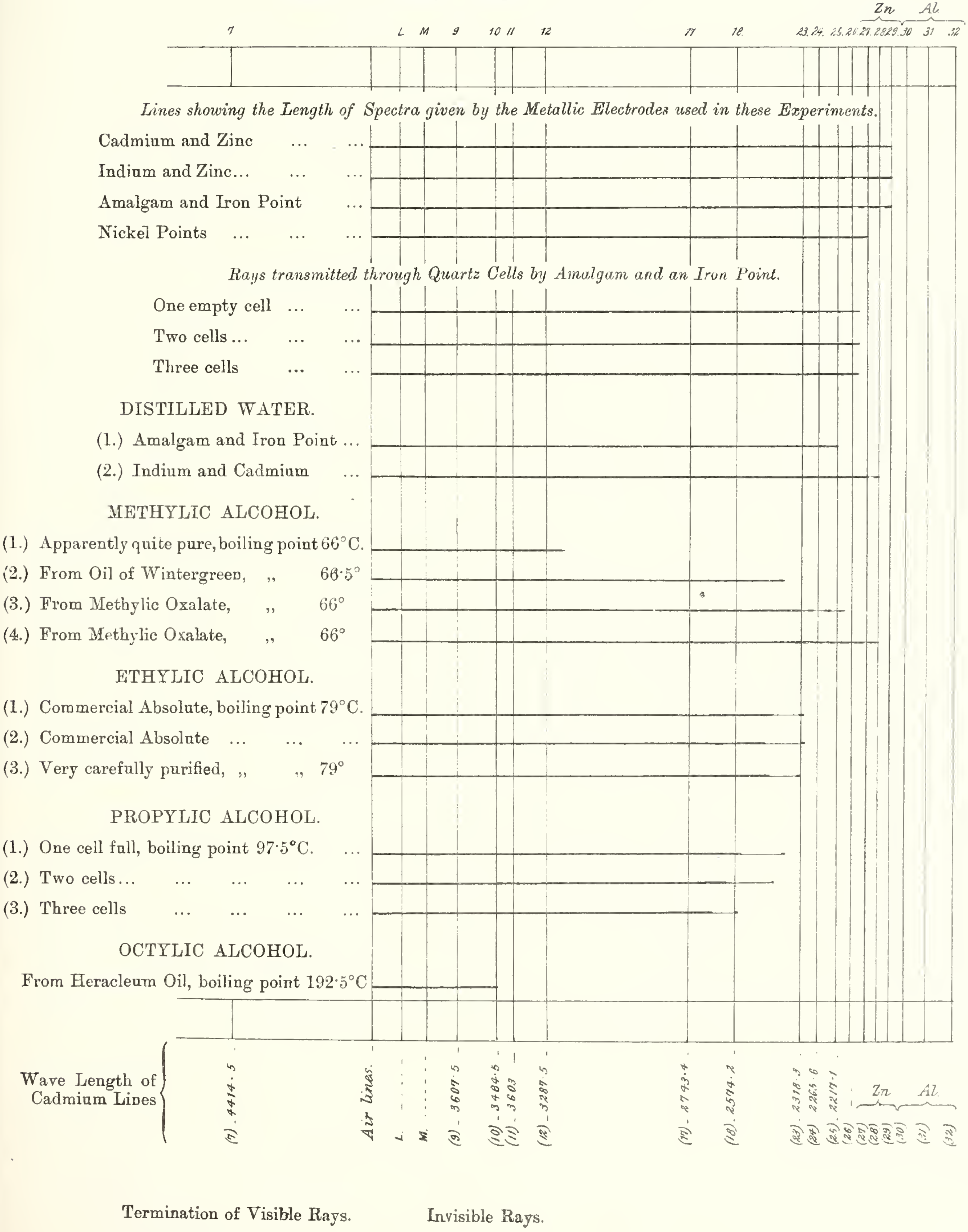
West Newman & Co. lith.

BRITISH MUSEUM NATURAL HISTORY.

No. 1.

Diagram showing the Effect of Water and of the Alcohols on the Ultra-Violet Rays of Metallic Spectra.

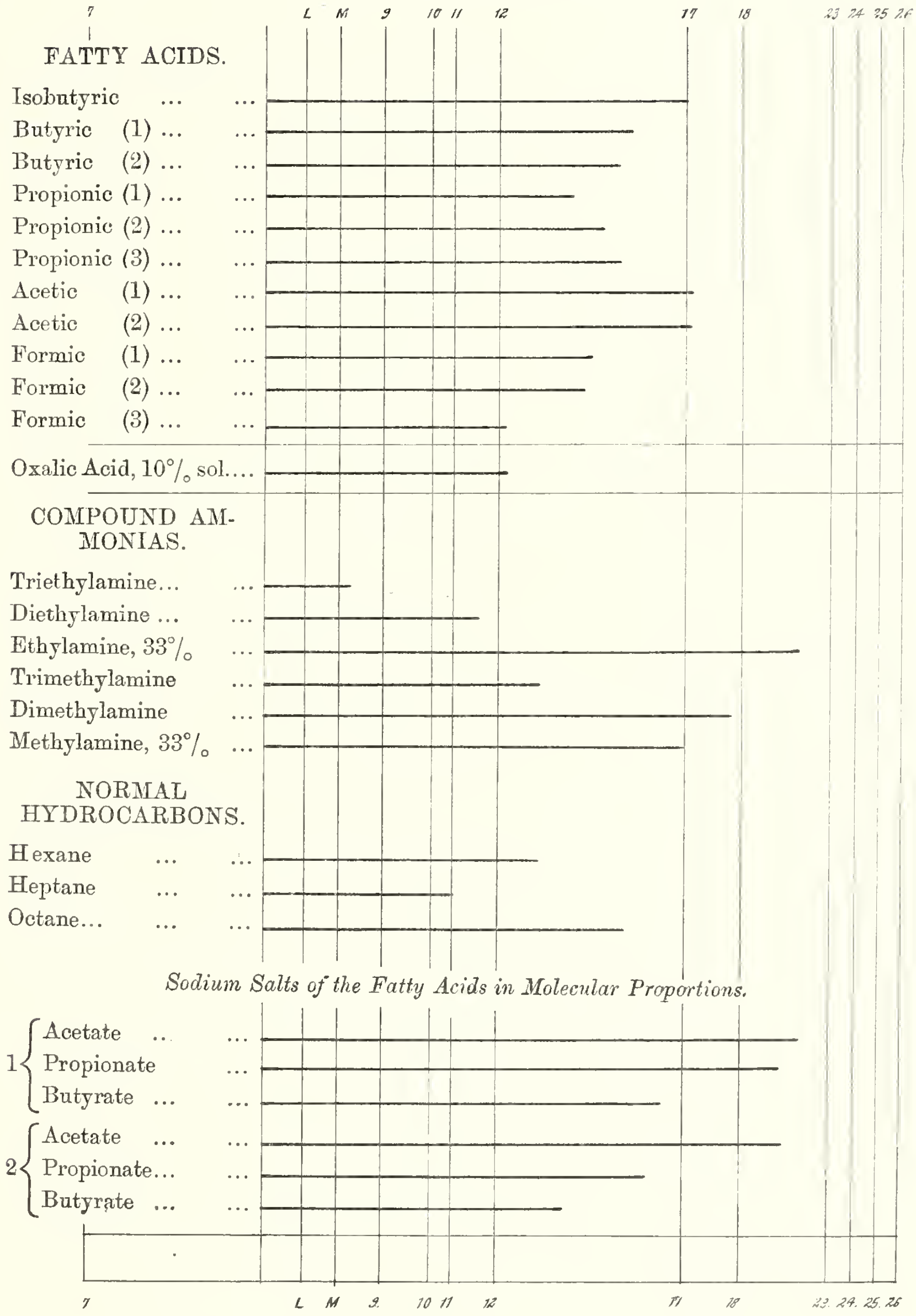
Thickness of the layer of liquid = 0·75 inches.



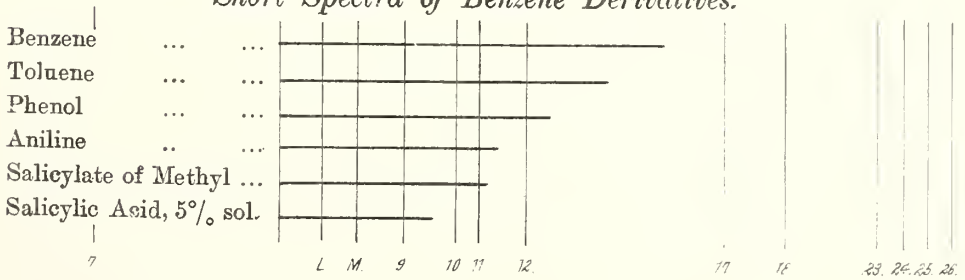
BRITISH MUSEUM
NATURAL HISTORY.

No. 2.

Diagram showing the Absorption of Ultra-Violet Rays by the Fatty Acids and other Organic Substances.



Short Spectra of Benzene Derivatives.



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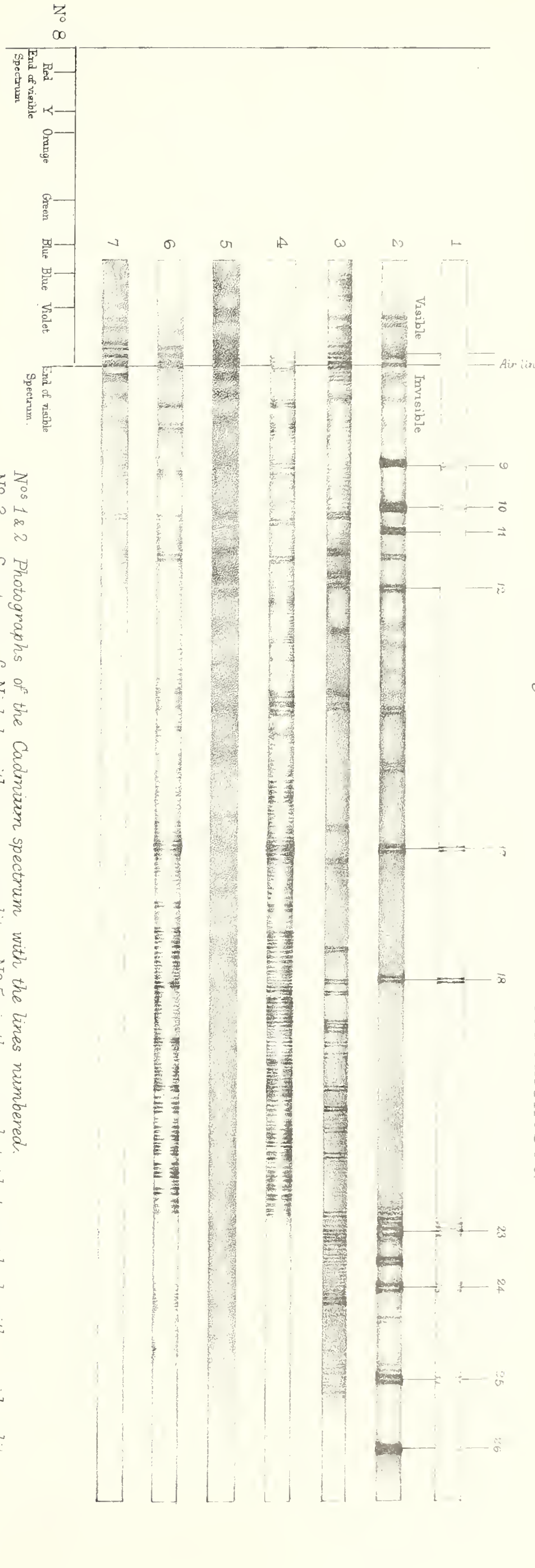
BRITISH MUSEUM NATURAL HISTORY.

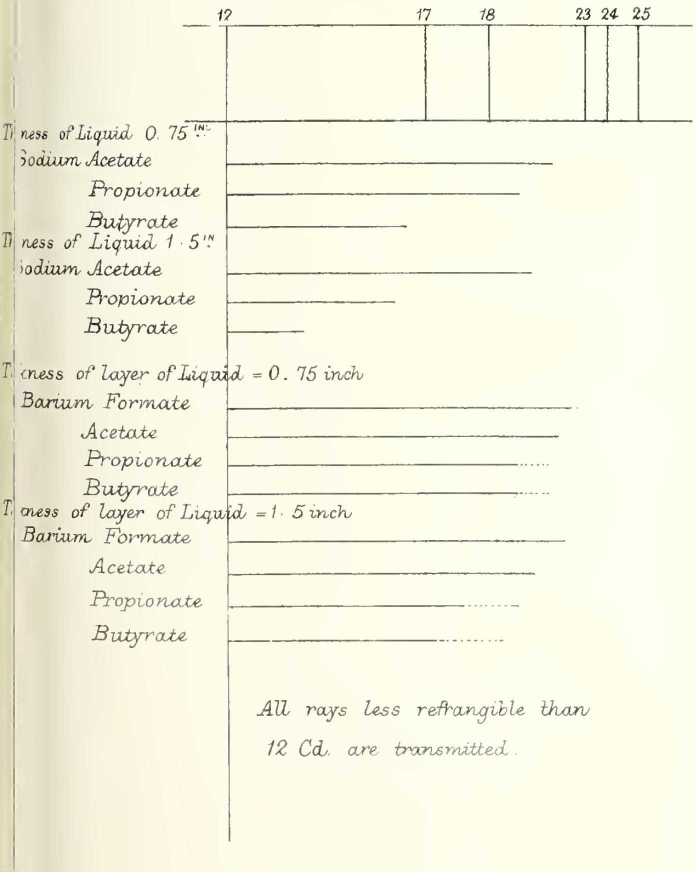
EXAMPLES OF ULTRA-VIOLET SPECTRA,
Showing lines used for measurements &c.



Nos 1 & 2 Photographs of the Cadmium spectrum with the lines numbered.
No 3 Spectrum of Nickel with a narrow slit. No 5 is the same, but photographed with a wide slit.
No 4. Spectrum of Iron on a gelatine plate. No 6. the same on a Beechey plate.
No 7. Spectrum of Nickel photographed through a Nicol's prism, showing the absorption due to the film of Canada balsam.
No 8. The extension of the visible rays beyond the photographic spectrum is here shown.

No. 3.

Diagram showing the absorption of the ultra violet rays by sodium and barium salts of the fatty Acids in molecular proportions.



All rays less refrangible than 12 Cd. are transmitted.

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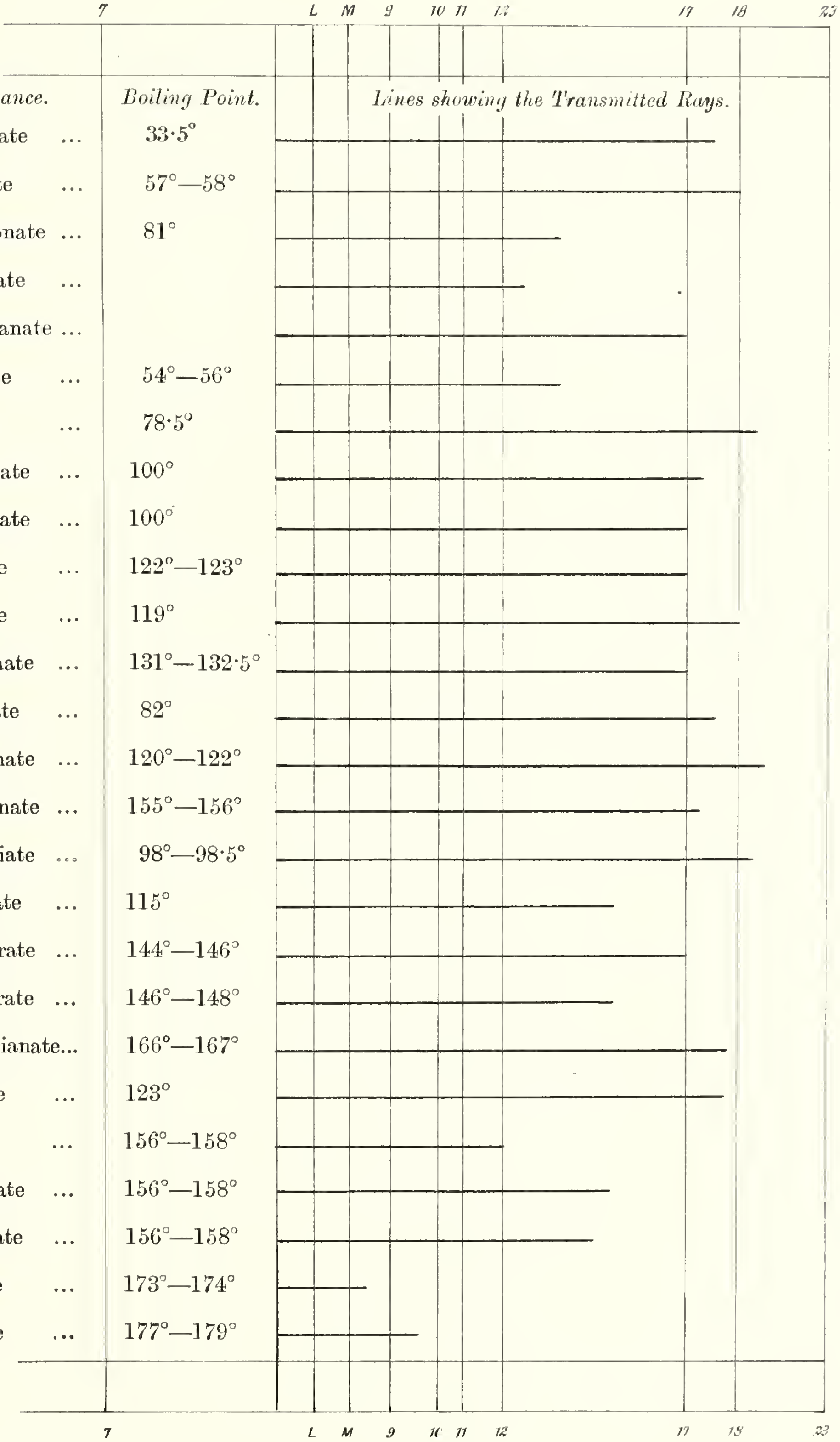
BRITISH MUSEUM
NATURAL HISTORY.

Hartley

No. 4.

Diagram showing the Results of an Examination of a Series of Ethereal Salts.

Thickness of layer of liquid = 0·75 inch.



Name of Substance.	*Boiling Point.*
Methyl Formiate ...	33·5°
„ Acetate ...	57°—58°
„ Propionate ...	81°
„ Butyrate ...	
„ Valerianate ...	
Ethyl Formiate ...	54°—56°
„ Acetate ...	78·5°
„ Propionate ...	100°
„ Propionate ...	100°
„ Butyrate ...	122°—123°
„ Butyrate ...	119°
„ Valerianate ...	131°—132·5°
Propyl Formiate ...	82°
„ Propionate ...	120°—122°
„ Valerianate ...	155°—156°
Isobutyl Formiate ...	98°—98·5°
„ Acetate ...	115°
„ Butyrate ...	144°—146°
„ Butyrate ...	146°—148°
„ Valerianate...	166°—167°
Amyl Formiate ...	123°
„ Acetate ...	156°—158°
„ Propionate ...	156°—158°
„ Propionate ...	156°—158°
„ Butyrate ...	173°—174°
„ Butyrate ...	177°—179°

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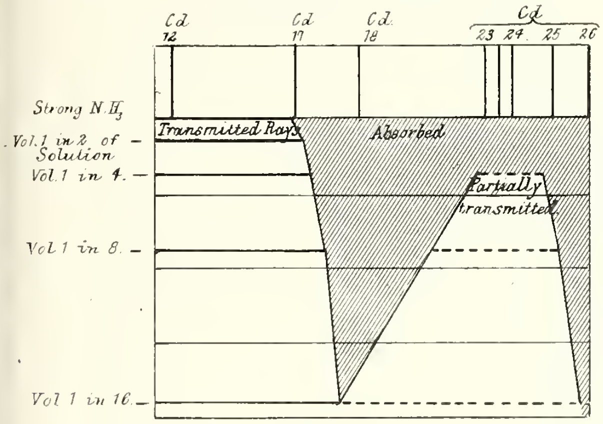
No 5.

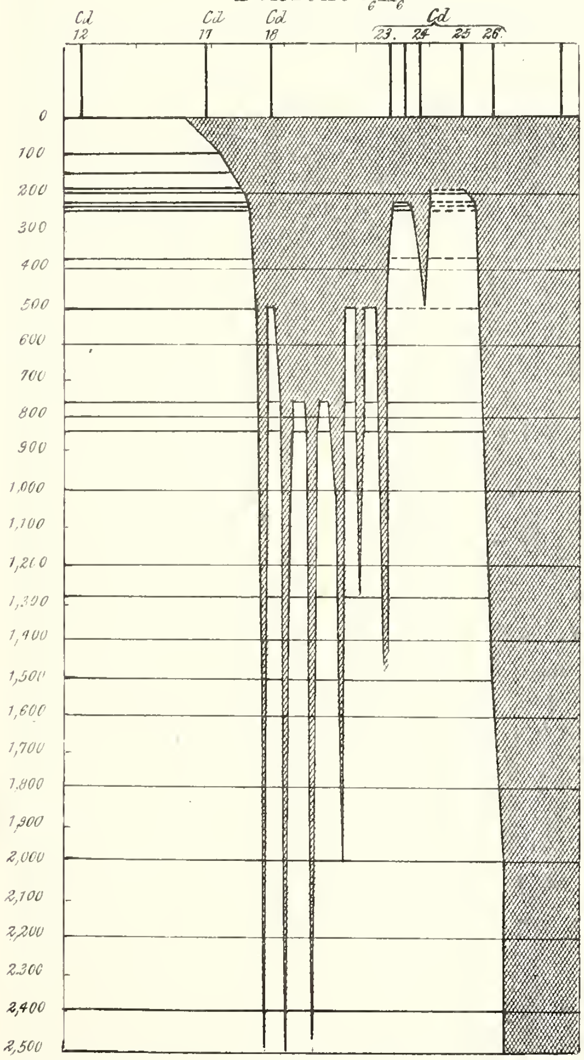
Absorption due to an impurity.
Volcanic Ammonia.



This absorption is not due to nitrous compounds since Mr Soret shows that the absorption due to nitric acid occurs between 12 and 17 Cd.

No 6.

Benzene C_6H_6

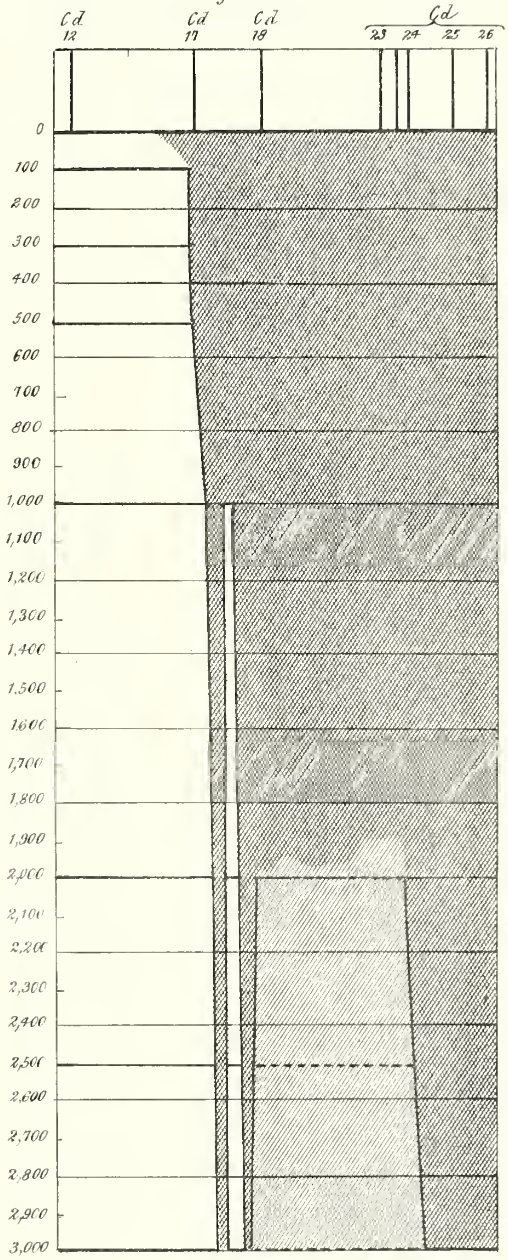


Ordinates = the proportion of liquid containing 1 volume of Benzene.

Thickness of layer of liquid = 0.75 inch

No 7.

Toluene. Methyl-benzene B. P. 111° C.



Ordinates = the porportion of alcohol containing 1 volume of Toluene.

Thickness of layer of liquid = 0·75 inch

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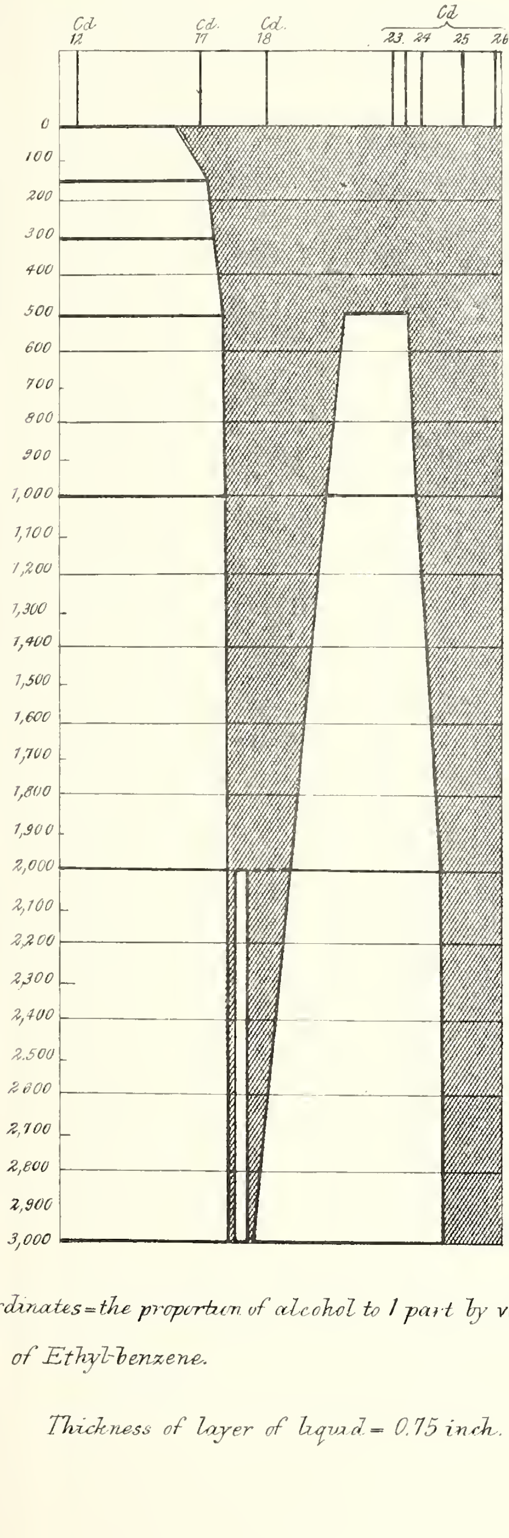
N°8.

Ethyl-benzene $C_6H_5.C_2H_5$ B.P. 134°.

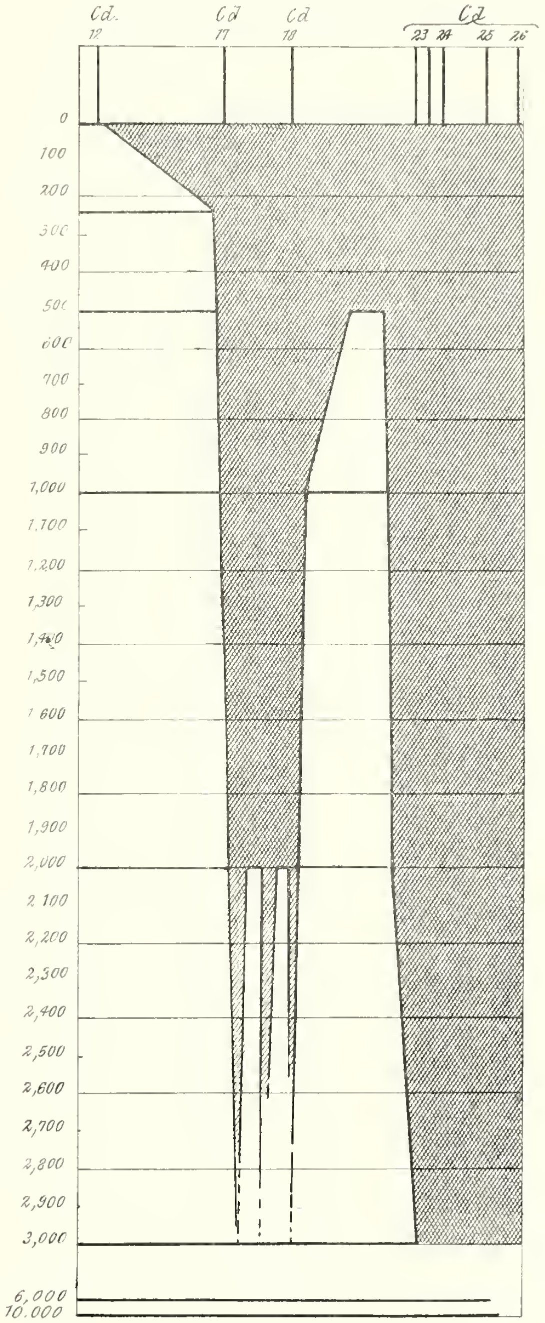


Ordinates = the proportion of alcohol to 1 part by volume of Ethyl-benzene.

Thickness of layer of liquid = 0.75 inch.

N°9.

1.3.5. Trimethyl-benzene $C_6H_3.(CH_3)_3$



Absorption continues till 40,000 parts of water or alcohol have been added.

Ordinates = the proportion of alcohol to 1 part by volume of Trimethyl-benzene

Thickness of layer of liquid = 0·75 inch.

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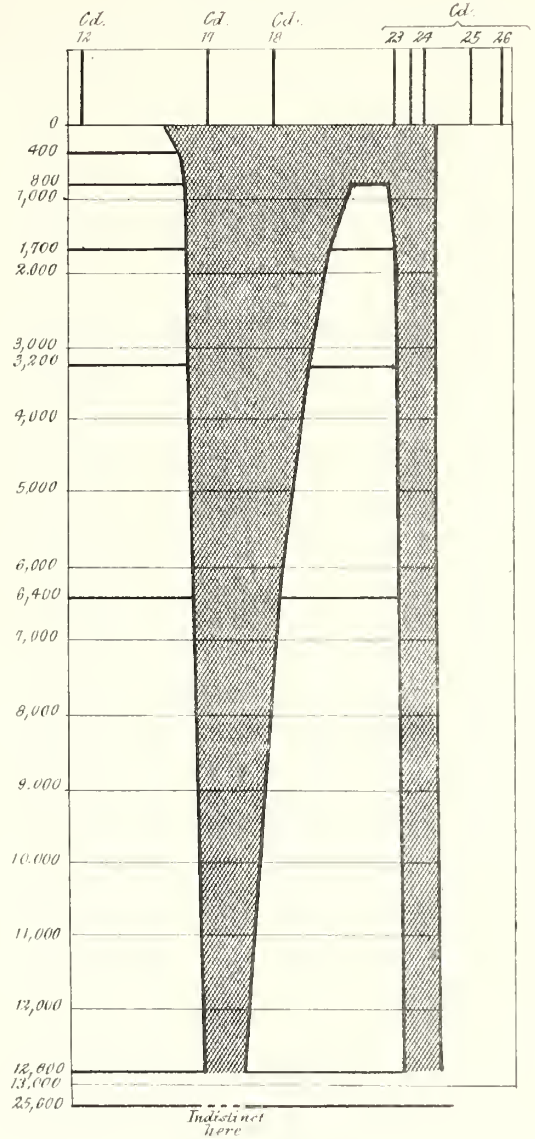
No. 10.

Phenol. $C_6H_5 \cdot OH$.

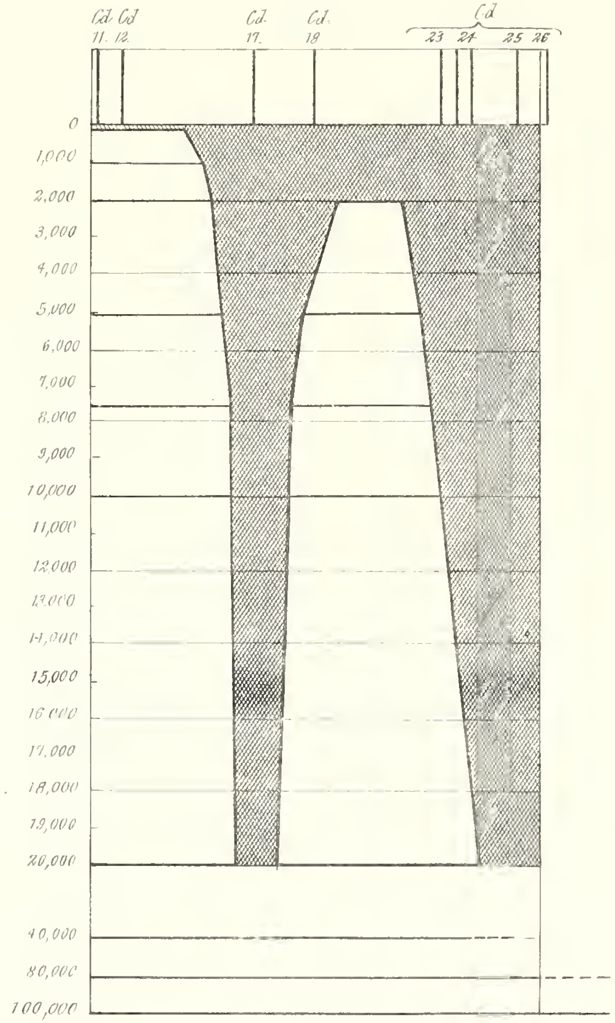


Ordinates = the proportion of water to 1 part by weight of Phenol.

Thickness of layer of liquid = 0·75 inch.

No. 11.

Thymol.

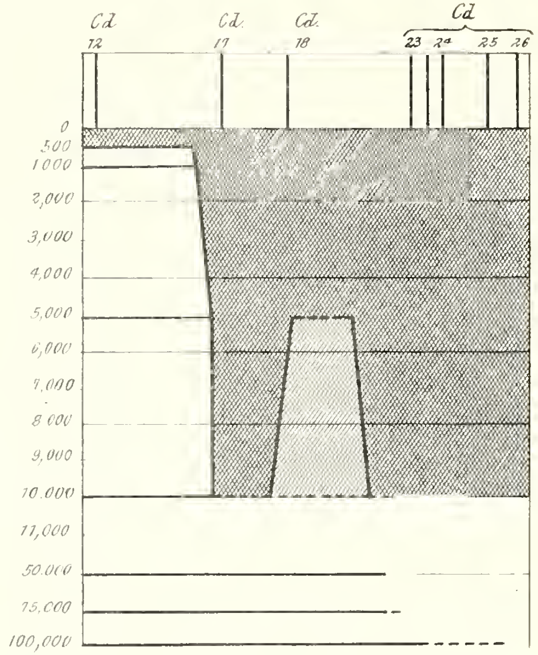


Ordinates = the proportion of water to 1 part by weight of Thymol.

Thickness of layer of liquid = 0·75 inch.

No. 12.

Pyrogallol. $C_6H_3(OH)_3$.



Ordinates = the proportion of solution containing 1 part by weight of substance.

Thickness of layer of liquid = 0·75 inch or 19 m.

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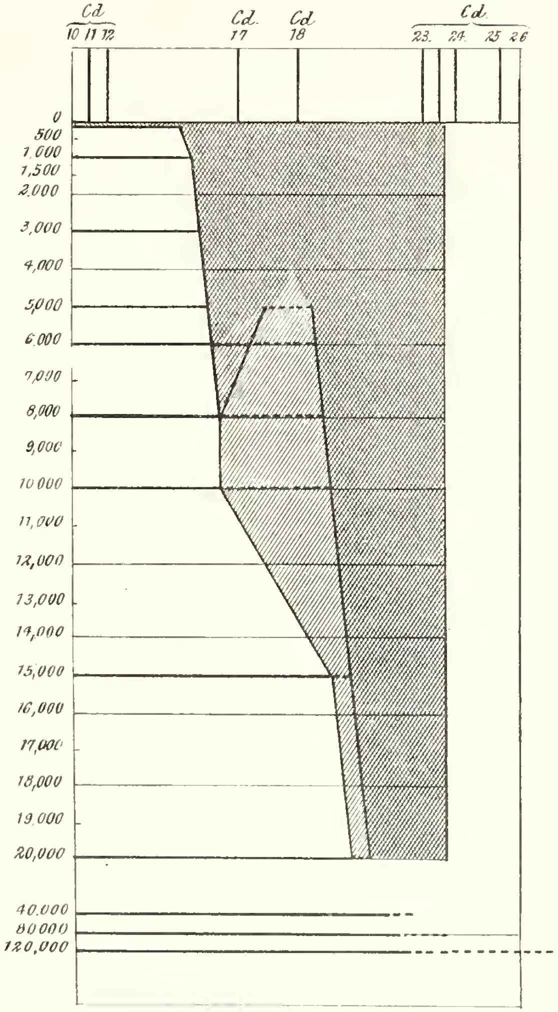
Nº 13.

Benzoic Acid. C_6H_5COOH.

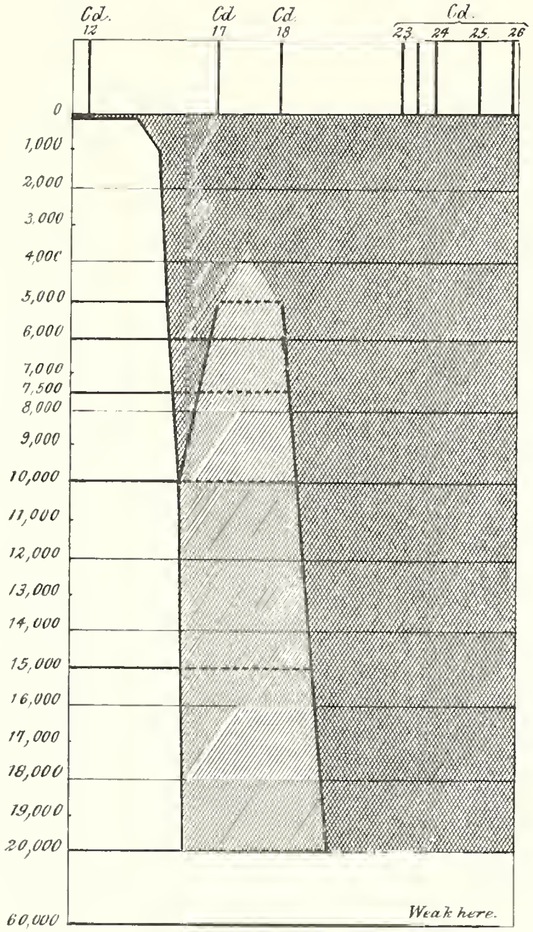


Ordinates = the proportion of water to 1 part by weight of Benzoic Acid.

Thickness of layer of liquid = 0.75 inch.

Nº 14.

Phthalic Acid. $C_6H_4(COOH)_2$



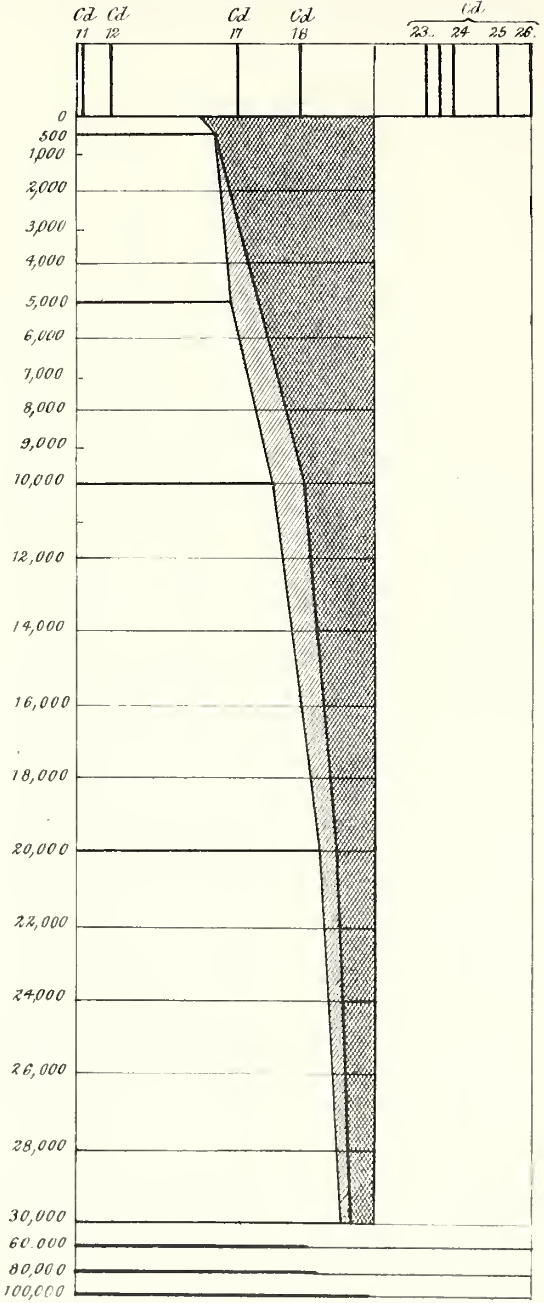
Absorption occurs at the more refrangible part of the spectrum in solutions containing 300,000 parts of water.

Ordinates = the proportion of water to 1 part by weight of Phthalic Acid.

Thickness of layer of liquid = 0.75 inch.

Nº 15.

Hippuric Acid.



Three times the exposure does not effect any alteration in the spectrum. *

Ordinates = The proportion of Hippuric Acid to 1 part by weight of water.

Thickness of layer of liquid = 0.75 inch.

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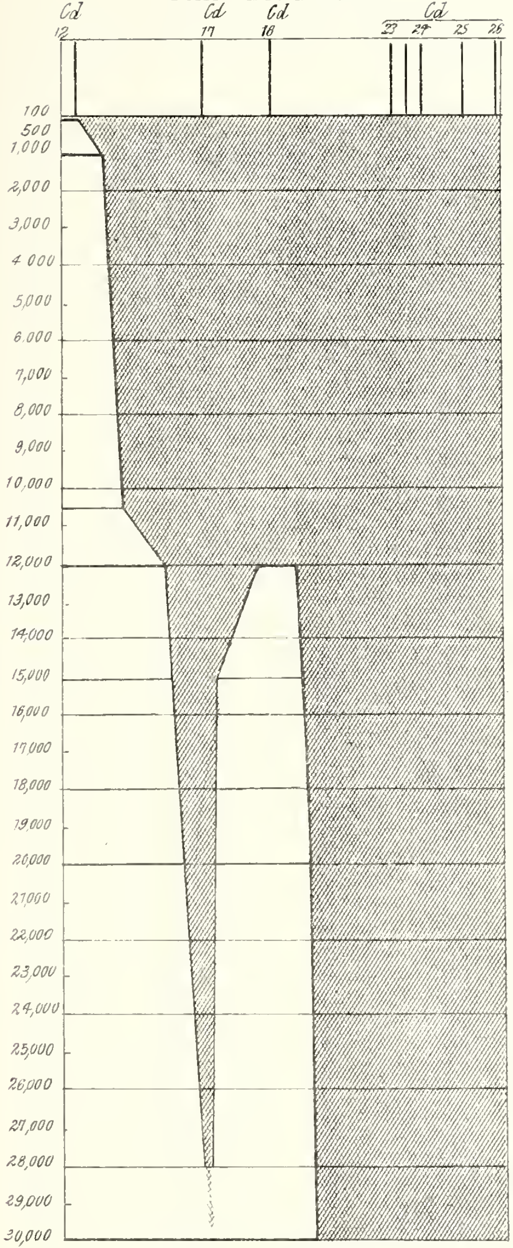
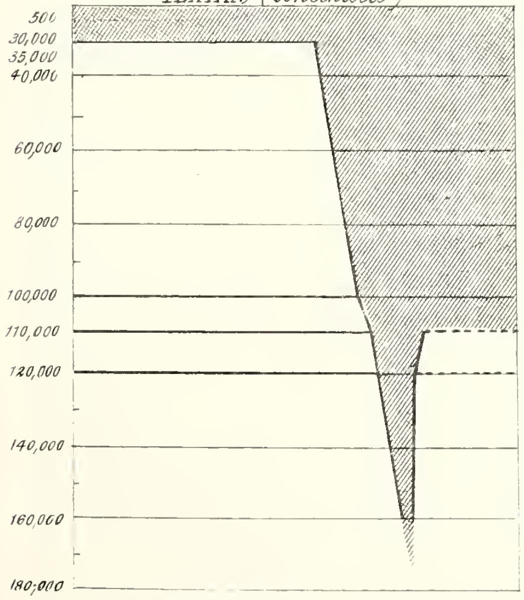
No. 16.

Aniline. B.P. 182°



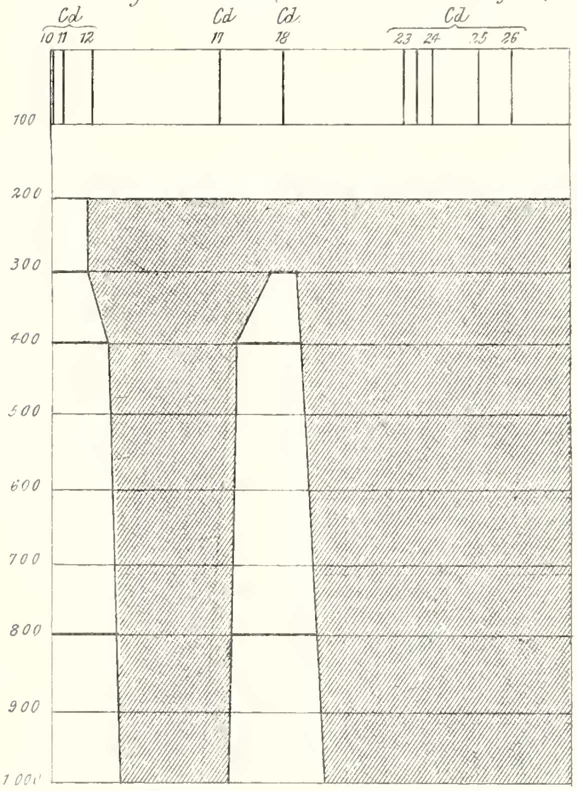
Aniline (continued)



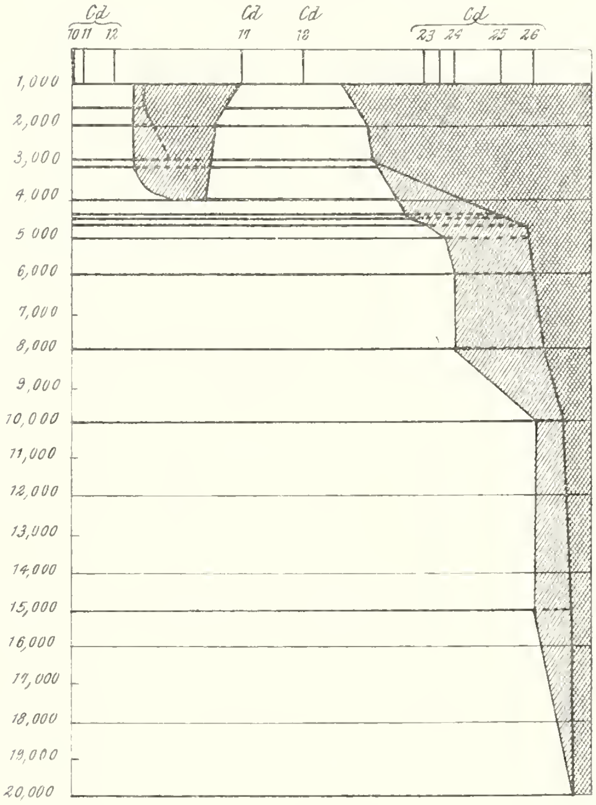
The Aniline was first diluted with Alcohol and then with water.

No. 17.

Salicylic Acid (from Oil of Wintergreen).



Salicylic Acid (continued)



Ordinates = the proportion of water to 1 part by weight of Salicylic Acid

Thickness of layer of liquid = 0.75 inch.

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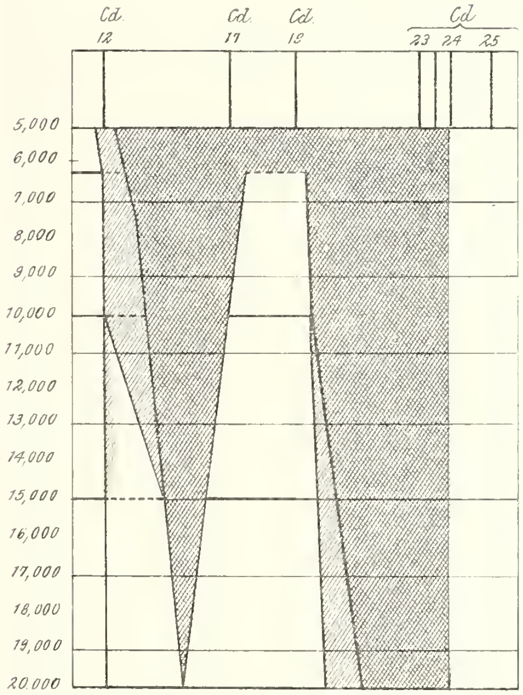
N.º 18.

Oxybenzoic Acid (1.3)

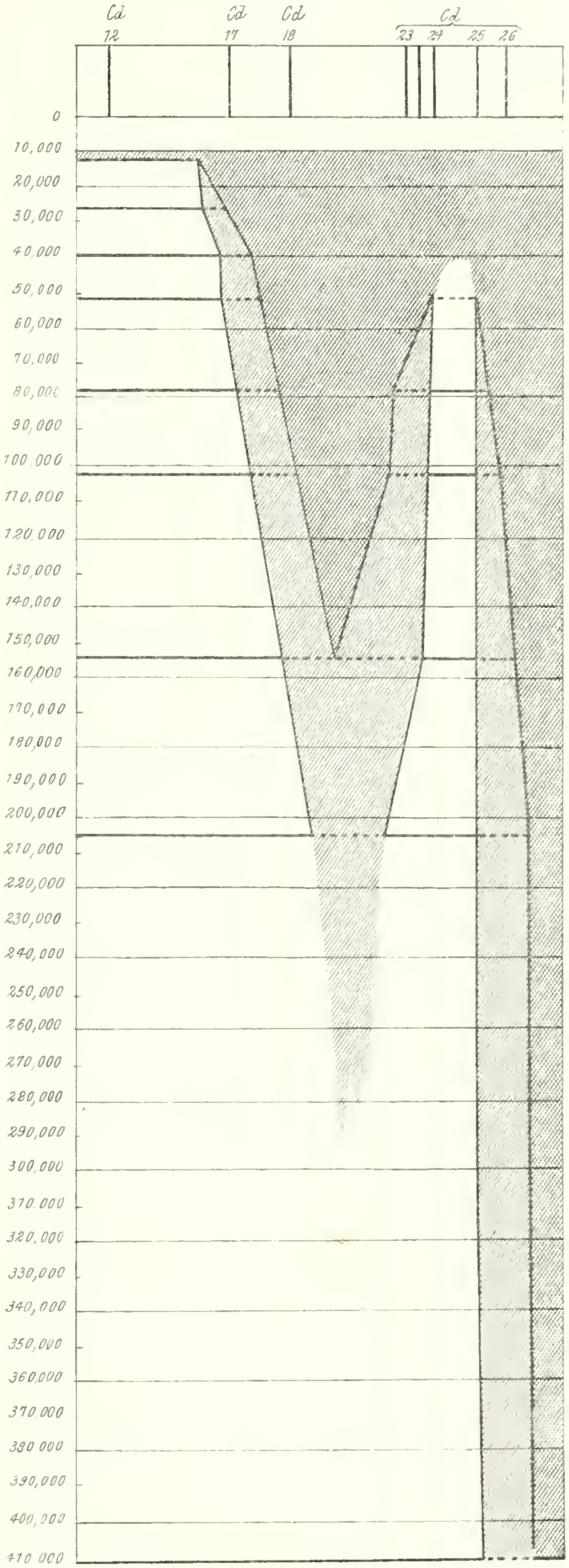


Ordinates = The proportion of water to 1 part by weight of Oxybenzoic Acid.

Thickness of layer of liquid = 0.75 inch.

N.º 19.

Paroxybenzoic Acid.



Ordinates = The proportion of water to 1 part by weight of Paroxybenzoic Acid.

Thickness of layer of liquid = 0.75 inch.

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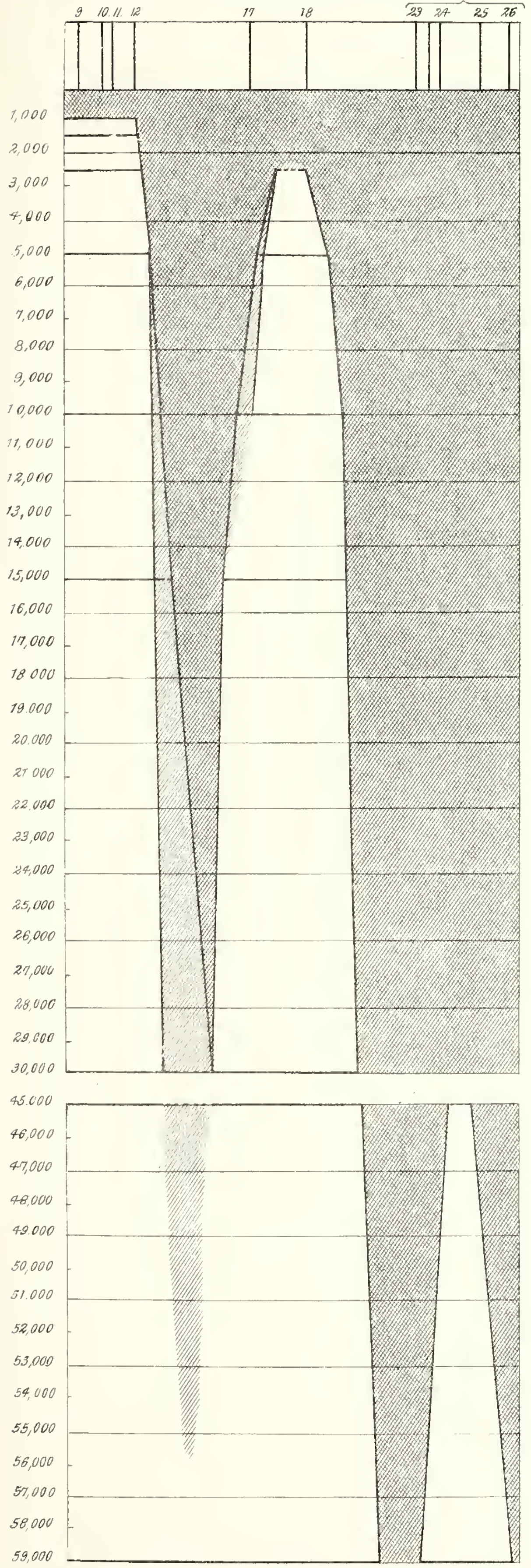
N° 20.

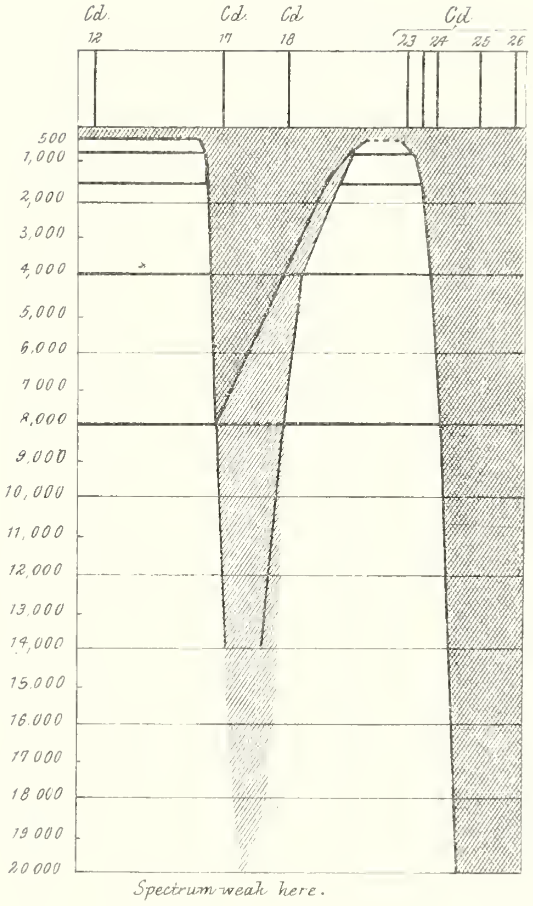
Salicylate of Methyl _ B.P. 220°.7



Ordinates = the proportion of diluent to 1 part by weight of the substance.
Thickness of layer of liquid = 0.75 inch.

N° 21.

Salicine $\left.\begin{matrix}C_6H_{11}O_5\\C_6H_4(OH)CH_2\end{matrix}\right\}O$



The absorption band is traceable in solutions containing $\frac{1}{50000}$ of the substance.

Thickness of layer of liquid = 0.75 inch.

Ordinates = the proportion of solution containing one part by weight of the solid substance.

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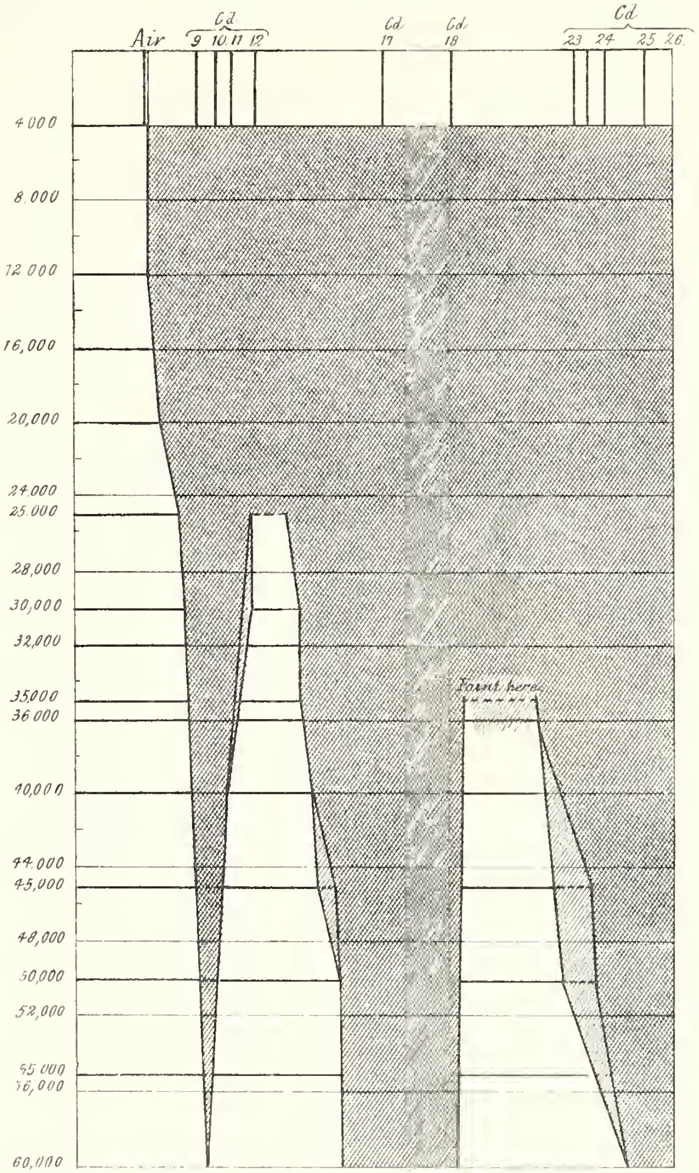
No. 22.

Orthonitrophenol.



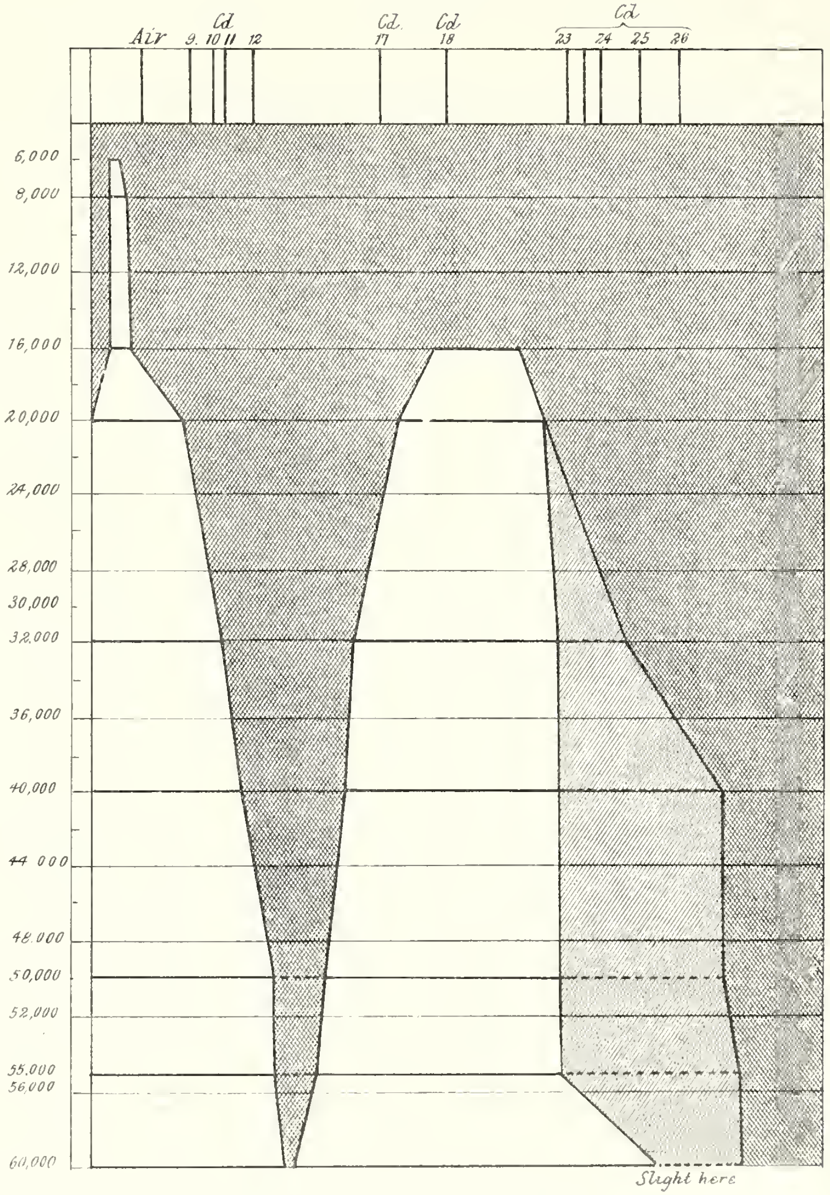
The spectrum can be traced across, but is weak just here and is not free from absorption at $\frac{1}{120,000}$.

Ordinates = the proportion of water to 1 part by weight of Orthonitrophenol

Thickness of layer of liquid = 0.75 inch

No. 23.

Paranitrophenol.



Ordinates = the proportion of water to 1 part by weight of Paranitrophenol.

Thickness of layer of liquid = 0.75 inch.

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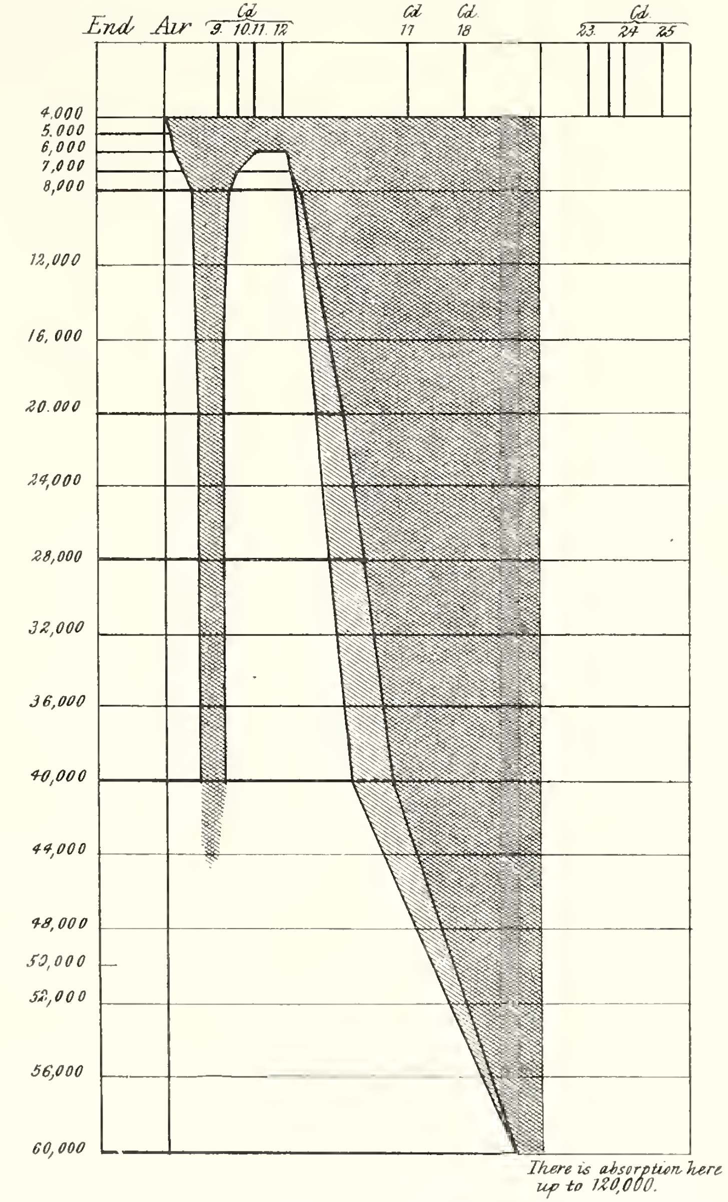
Nº 24.

Meta-nitraniline. 1·3.



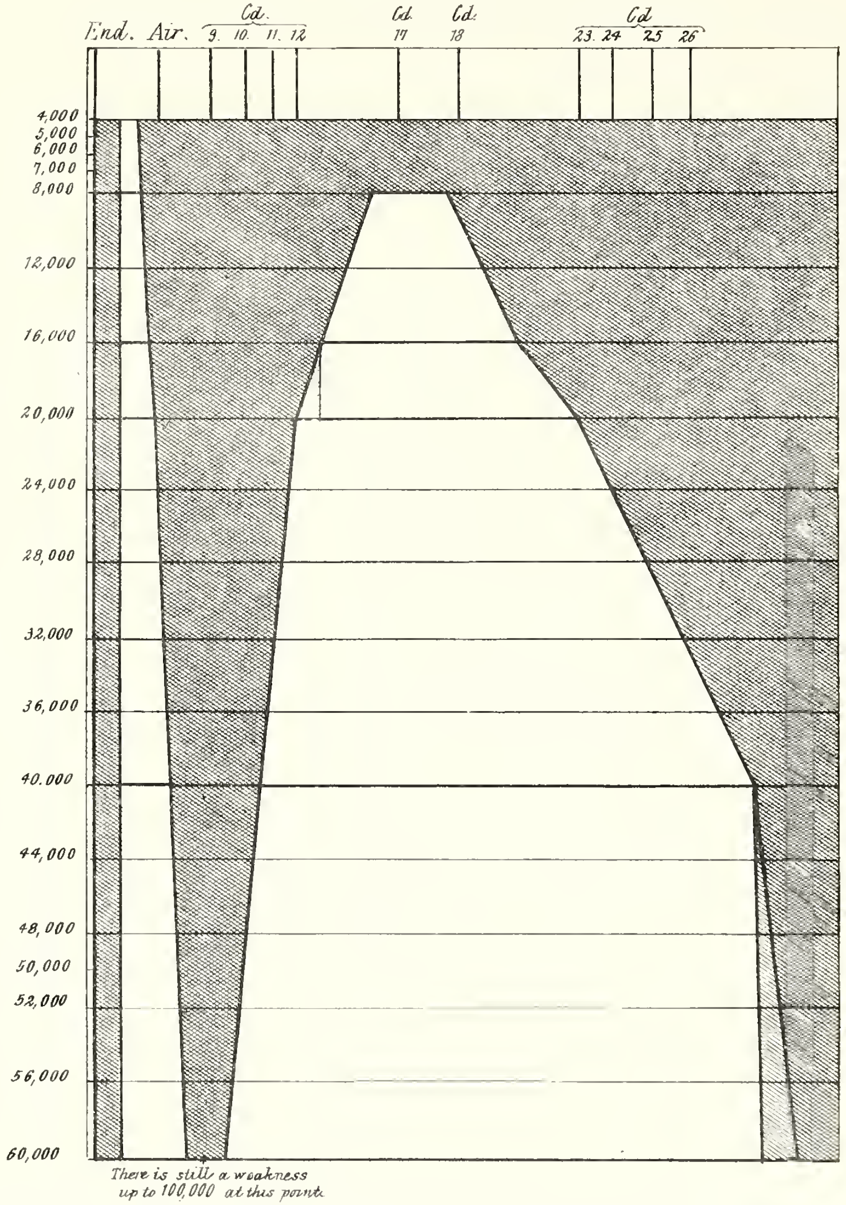
There is absorption here up to 120,000.

Ordinates = the proportion of water to 1 part by weight of Metanitraniline.

Thickness of layer of liquid = 0·75 inch.

Nº 25.

Para-nitraniline. 1·4.



There is still a weakness up to 100,000 at this point.

Ordinates = the proportion of water to 1 part by weight of Paranitraniline.

Thickness of layer of liquid = 0·75 inch

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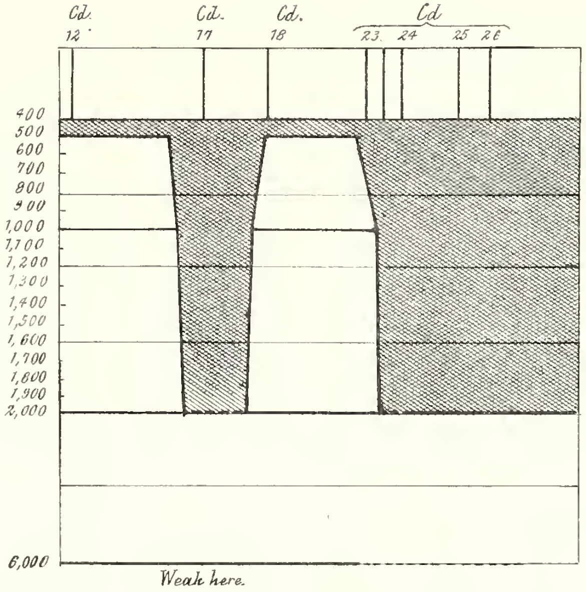
Nº 26.

Tyrosine.

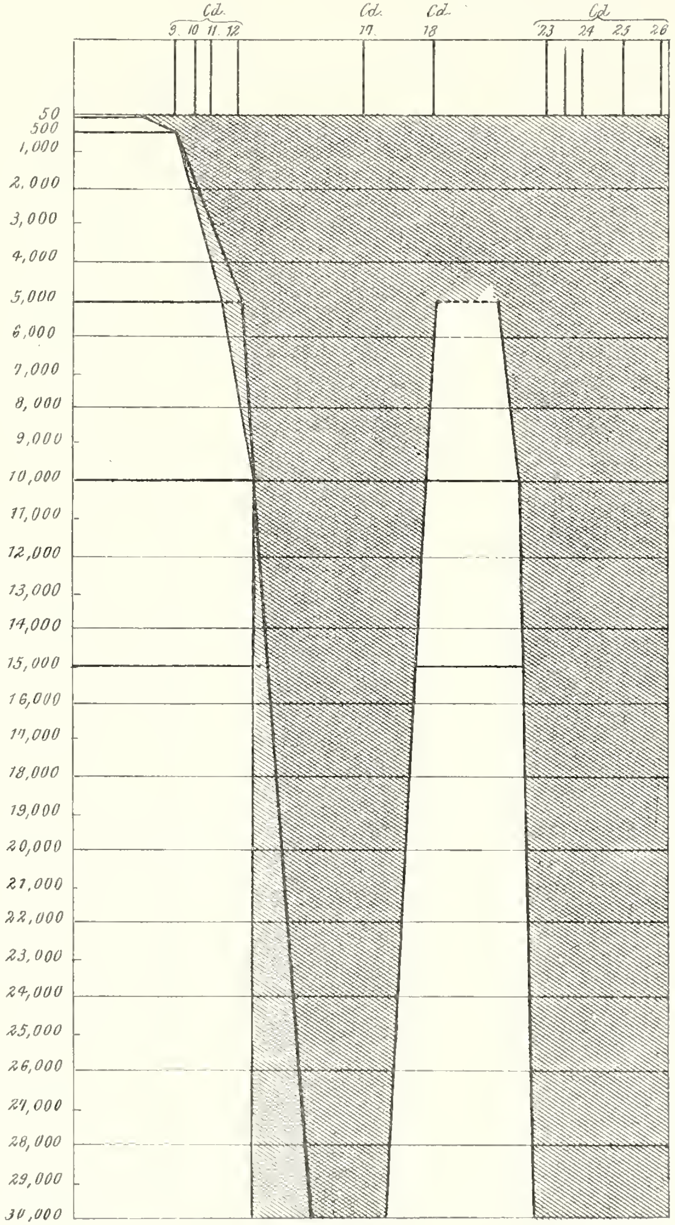


Ordinate = the proportion of water to 1 part by weight of Tyrosine.

Thickness of layer of liquid = 0.75 inch.

Nº 27.

Phlorizine. – Structural formula = $\left.\begin{array}{r} C_6 \cdot H_{11} \cdot O_5 . \\ C_6 \cdot H_3\,(OH) \end{array}\right\} O$ $\left.C_6H_4\,(OH)\;C_2H_4 \cdot CO\right\} O.$



There is very perceptible absorption here in solutions containing only $\frac{1}{50,000}$ of Phlorizine

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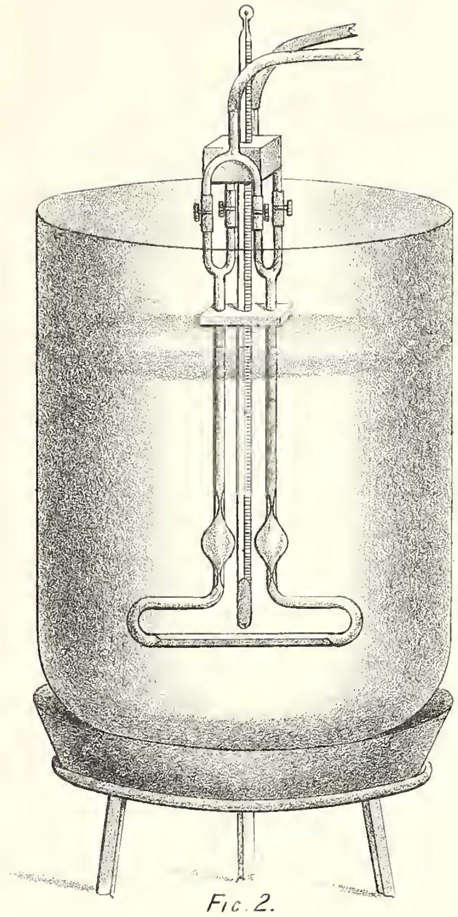

Fig. 2.

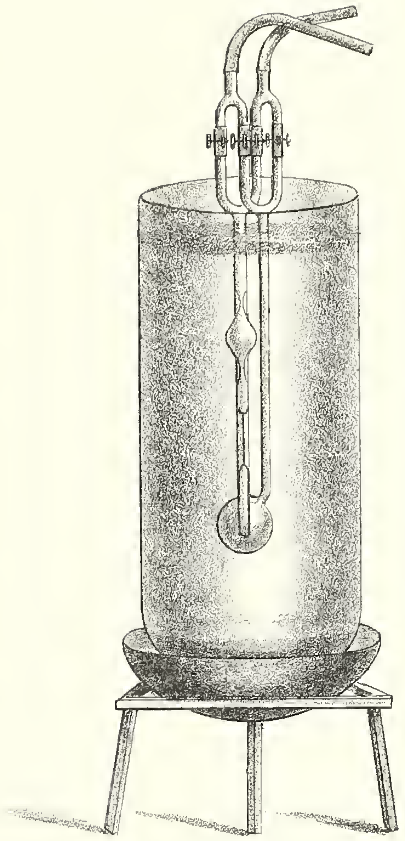

Fig. 3.

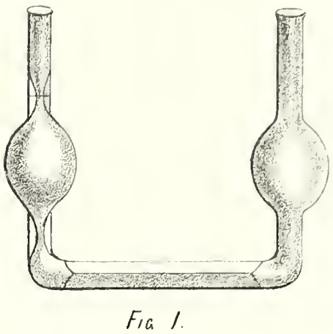

Fig. 1.

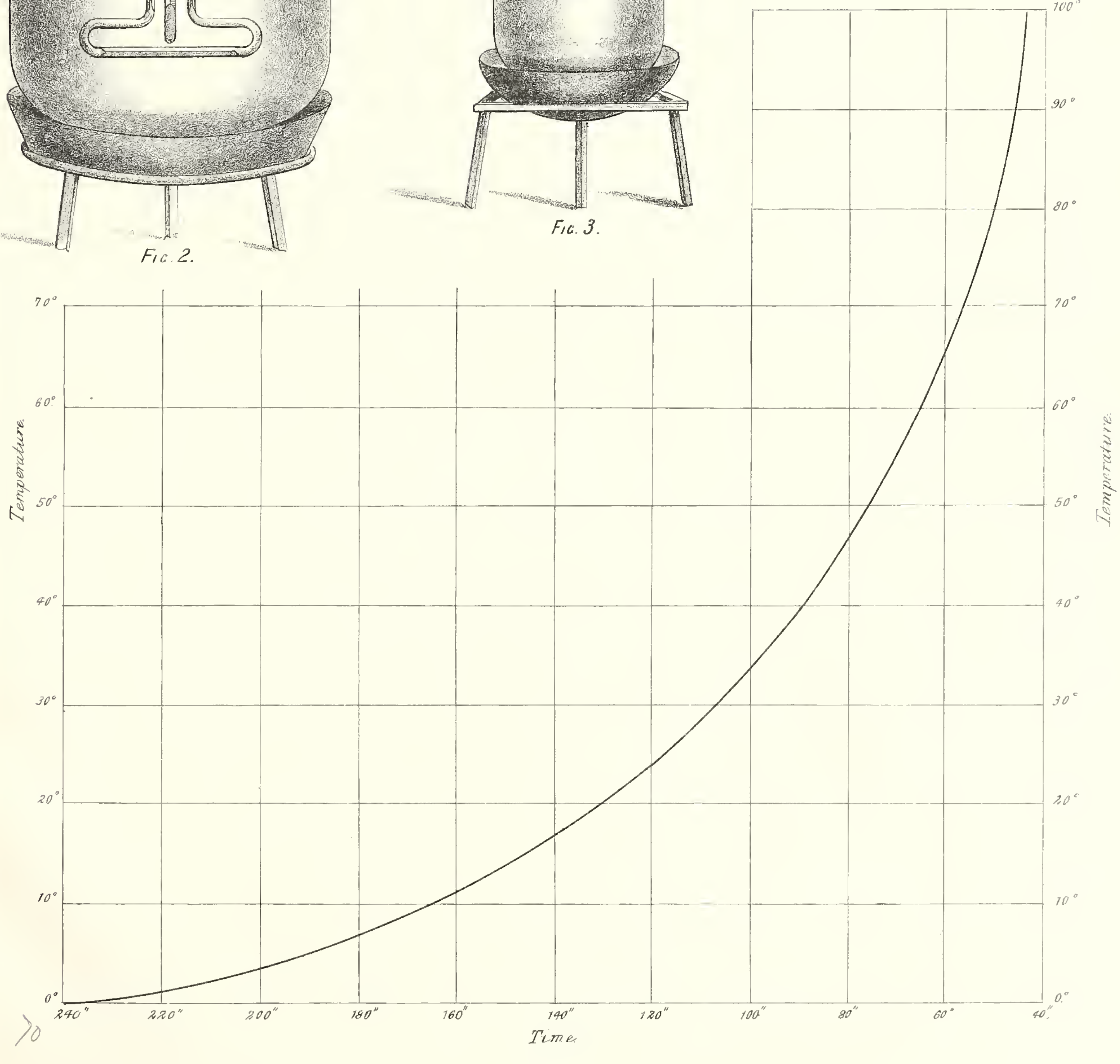

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Fig. 4.

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